

Kim Ki-Hang Butler; Štefan Schwarz
The semigroup of circulant Boolean matrices

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 4, 632–635

Persistent URL: <http://dml.cz/dmlcz/101434>

Terms of use:

© Institute of Mathematics AS CR, 1976

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

THE SEMIGROUP OF CIRCULANT BOOLEAN MATRICES

KIM KI-HANG BUTLER*), Montgomery, and ŠTEFAN SCHWARZ, Bratislava

(Received February 25, 1975, in revised form August 4, 1975)

Let B_n be the semigroup of binary relations on a finite set X , with $\text{card } X = |X| = n$ represented as $n \times n$ matrices over the Boolean algebra $\{0, 1\}$.

A circulant is a Boolean matrix of the form

$$C = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-2} \\ \dots & \dots & \dots & \dots \\ c_1 & c_2 & \dots & c_0 \end{bmatrix}.$$

Denote

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and E the unit matrix of order n . Any circulant $\in B_n$ can be written in the form

$$(1) \quad C = c_0E + c_1P + c_2P^2 + \dots + c_{n-1}P^{n-1}, \quad c_i \in \{0, 1\}.$$

Hereby $P^n = E$. (See [1], [3].) We define also $P^0 = E$.

The set of all circulants (of order n) under multiplication forms a semigroup C_n with $|C_n| = 2^n$ (including the zero circulant Z). C_n contains the cyclic group $G_n = \{E, P, P^2, \dots, P^{n-1}\}$ of order n and we have $G_n \subset C_n \subset B_n$. Clearly every element $\in C_n$ has a unique representation in the form (1). (This will turn out to be essential.) Note that C_n is closed also under addition.

The purpose of this note is to give an explicit description of all idempotents $\in C_n$ and all maximal subgroups of C_n .

It will turn out that the set of all maximal subgroups of C_n is exactly the set of all cyclic groups of order d , d being a divisor of n . A remarkable feature of the results

*) The first named author was supported by NSF Grant No. HES 74-24502.

obtained below is the fact that an explicit description of maximal subgroups in a semigroup is only very rarely available.

Since C_n is finite and commutative, the relations $\mathfrak{Q}, \mathfrak{R}, \mathfrak{D}, \mathfrak{S}$ in C_n coincide and the maximal subgroup $G(E^{(i)})$ belonging to the idempotent $E^{(i)}$ is the \mathfrak{Q} -class containing $E^{(i)}$, i.e. the set $\{A \in C_n : SE^{(i)} = SA\}$. (See [2].) Clearly $G(E) = G_n$.

Lemma 1. *If $A, B \in C_n$, then $A \mathfrak{Q} B$ iff there is an element $P^m \in G_n$ such that $A = P^m B$.*

Proof. (i) For any P^m ($m = 0, 1, 2, \dots, n-1$) we have $C_n P^m = C_n$. Hence $C_n A = C_n P^m B = C_n B$. Therefore $A = P^m B$ implies $A \mathfrak{Q} B$.

(ii) Conversely $A \mathfrak{Q} B, A \neq Z, B \neq Z$ implies that there are two elements $R, S \in C_n$ such that $A = RB, B = SA$. Write

$$R = P^{m_1} + P^{m_2} + \dots + P^{m_r}, \quad S = P^{k_1} + P^{k_2} + \dots + P^{k_s}.$$

Then

$$A = (P^{m_1} + P^{m_2} + \dots + P^{m_r})B, \quad B = (P^{k_1} + P^{k_2} + \dots + P^{k_s})A.$$

Hence (with \subset denoting the usual ordering of Boolean matrices) $P^{m_1}B \subset A, P^{k_1}A \subset B$. This implies $B \supset P^{k_1}A \supset P^{k_1+m_1}B$ and $B \supset P^{k_1+m_1}B \supset P^{2(k_1+m_1)}B \supset \dots \supset P^{n(m_1+k_1)}B = B$, whence $P^{k_1}A = B$. Analogously

$$A \supset P^{m_1}B \supset P^{m_1+k_1}A \supset P^{2(m_1+k_1)}A \supset \dots \supset P^{n(m_1+k_1)}A = A,$$

whence $A = P^{m_1}B$. This proves our lemma.

Remark. The element $P^m \in C_n$ in Lemma 1 is "in general" not uniquely determined.

If $A \neq Z$ and $A = P^{m_1} + P^{m_2} + \dots + P^{m_r}$ we shall call the subset $\{P^{m_1}, P^{m_2}, \dots, P^{m_r}\}$ of G_n the *support* of A . Given A the support is uniquely determined.

Lemma 2. *An element $A \in C_n, A \neq Z$, is an idempotent iff the support of A is a subgroup of G_n .*

Proof. (i) If $A^2 = A$, i.e.

$$(P^{m_1} + \dots + P^{m_r})(P^{m_1} + \dots + P^{m_r}) = (P^{m_1} + \dots + P^{m_r}),$$

then the subset $K = \{P^{m_1}, \dots, P^{m_r}\}$ of G_n is closed under multiplication, i.e. $K^2 \subset K$. A subset K of a finite group G_n closed under multiplication is a subgroup of G_n .

Recall the following elementary result. All subgroups of G_n are obtained in the following manner. Let t be a divisor of n and $tn^* = n$. Then G_n contains a subgroup of order t which can be explicitly given in the form

$$G^* = \{P^{n^*}, P^{2n^*}, \dots, P^{(t-1)n^*}, P^{tn^*} = E\}.$$

Though P^{n^*} , as a generator of G^* , is not uniquely determined, G^* as a whole is uniquely determined. (For $t = 1$ we have $G^* = \{E\}$, for $t = n$ we have $G^* = G_n$.)

(ii) Let conversely

$$G^* = \{P^{n^*}, P^{2n^*}, \dots, P^{tn^*}\}, \quad n^*t = n,$$

be a subgroup of G_n . Direct computation shows that

$$A = P^{n^*} + P^{2n^*} + \dots + E$$

is an idempotent $\in C_n$.

Let now $t_i \mid n$ and $t_i n_i = n$. To find the maximal subgroup $G(E^{(i)})$ belonging to the idempotent

$$E^{(i)} = P^{n_i} + P^{2n_i} + \dots + P^{t_i n_i}$$

we use Lemma 1 and the fact that

$$G(E^{(i)}) = \{Y \in C_n : Y \Omega E^{(i)}\}.$$

We have $Y \Omega E^{(i)}$ iff there is an element $P^m \in G_n$ such that $Y = P^m E^{(i)}$. Hence

$$G(E^{(i)}) = \{E, P, P^2, \dots, P^{n-1}\} E^{(i)}.$$

This set contains exactly n_i different elements, namely

$$E^{(i)}, PE^{(i)}, P^2 E^{(i)}, \dots, P^{n_i-1} E^{(i)}.$$

These are exactly those elements $\in C_n$ whose supports are the cosets of G_n modulo the cyclic group

$$\{P^{n_i}, P^{2n_i}, \dots, P^{t_i n_i}\}.$$

We have proved:

Theorem 3. *Let t_i be a divisor of n and $t_i n_i = n$. Then*

$$E^{(i)} = P^{n_i} + P^{2n_i} + \dots + P^{t_i n_i}$$

is an idempotent $\in C_n$. The maximal subgroup of C_n belonging to $E^{(i)}$ is the cyclic group

$$\{E^{(i)}, PE^{(i)}, \dots, P^{n_i-1} E^{(i)}\}.$$

All idempotents and all maximal subgroups of C_n are obtained in this manner.

Denote by $d(n)$ the number of divisors of n (including 1 and n) and by $\sigma(n)$ the sum of all divisors of n .

We have:

Corollary 4. *C_n contains $d(n)$ different idempotents $\neq Z$ and $\sigma(n)$ distinct regular elements $\neq Z$.*

Example. The semigroup C_6 contains (besides Z) the following four idempotents:

$$E^{(1)} = P^6 = E, \quad E^{(3)} = P^2 + P^4 + E,$$

$$E^{(2)} = P^3 + E, \quad E^{(4)} = P + P^2 + P^3 + P^4 + P^5 + E.$$

The corresponding maximal subgroups of C_6 are

$$G(E^{(1)}) = \{E, P, P^2, P^3, P^4, P^5\}, \quad G(E^{(3)}) = \{P^2 + P^4 + E, P^3 + P^5 + P\},$$

$$G(E^{(2)}) = \{P^3 + E, P^4 + P, P^5 + P^2\}, \quad G(E^{(4)}) = E.$$

Remark 1. To avoid misunderstanding we note. The semigroup B_n may contain cyclic subgroups of order t which does not divide n and even cyclic subgroups of order $t > n$. To show this denote

$$Q_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and consider the 5×5 matrix

$$A = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}.$$

It is immediately to seen that $\{A, A^2, A^3, A^4, A^5, E\}$ is a cyclic subgroup of B_5 of order $t = 6$. The cyclic subgroup $\{A^2, A^4, E\}$ is of order 3 though 3 does not divide 5.

Remark 2. In a "general" semigroup there may exist different idempotents with the corresponding maximal subgroups isomorphic one to the other. It is worth to mention that in C_n no two maximal subgroups are isomorphic one to the other.

References

- [1] Kim Ki-Hang Butler and J. R. Krabill: Circulant Boolean relation matrices, Czech. Math. J., 24 (99) (1974), 247–251.
- [2] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups, Math. Surveys, No.7, Vol. 1, 1961, Amer. Math. Soc., Providence, R. I.
- [3] Št. Schwarz: Circulant Boolean relation matrices, Czech. Math. J., 24 (99) (1974), 252–253.

Authors' addresses: Kim Ki-Hang Buttler, Alabama State University, Montgomery 36101, U.S.A., Štefan Schwarz, 880 19 Bratislava, Gottwaldovo nám. 19, ČSSR (Slovenská vysoká škola technická).