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ON THE LIMIT-3 CLASSIFICATION OF THE SQUARE  
OF A SECOND-ORDER, LINEAR DIFFERENTIAL EXPRESSION  

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1. Introduction. In recent years the work of several mathematicians has been  
directed towards a study of the formal powers of the symmetric, second-order differential expression $M$, where, for suitably differentiable complex-valued $f$, $M$ is defined by

$$
M[f] = -(pf')' + qf \text{ on } I \quad (' \equiv d/dx).
$$

Here $I$ is an interval of the real line, and the coefficients $p$ and $q$ are real-valued, with $p > 0$, on $I$. The formal powers $M^n$, where $n = 1, 2, 3, \ldots$, of $M$ are defined by $M^1 = M$ and $M^n = M[M^{n-1}]$ for $n = 2, 3, \ldots$; this definition requires certain differentiability properties of the coefficients $p$ and $q$ if $M^n$ is also to be a differential expression.

The first result on the relationship between $M$ and $M^2$, as symmetric differential expressions, were given by CHAUDHURI and EVERITT in [1]. Since then there have been contributions to the properties of $M^n$, and more general polynomials in $M$, from EVERITT and GIERTZ [3], [4] and [5], KAUFFMAN [6], KUMAR [7], READ [9] and ZETTL [10]. In particular [5] is a survey article on the general powers $M^n$ of $M$.

For the general definition of a real-valued formally symmetric (equivalently formally self-adjoint) differential expression see [2, Ch. XIII, 2.1] or [8, section 15]. When $M$ is given by (1.1) and the power $M^n$ exists then $M^n$ is also formally symmetric. In the particular case of (1.1) with $p = 1$, i.e. $p(x) = 1 \ (x \in I)$, we have

$$
M[f] = -f'' + qf \text{ on } I
$$

and

$$
M^2[f] = f^{(4)} - (2qf')' + (q^2 - q^3)f \text{ on } I.
$$

Here derivatives of order greater than 2 are denoted by $f^{(3)}$ and $f^{(4)}$.  
The results discussed in this paper are concerned with $M$ and $M^2$, as given by (1.2) and (1.3), in the case when the interval $I$ is the half-line $[0, \infty)$. In particular one of the results, see Theorem 1 below, answers a previously unsolved problem posed by Chaudhuri and Everitt in 1969, see [1, section 12].
Since we deal only with the half-line \([0, \infty)\) we use the abbreviations \(L^2\) for the Hilbert function space \(L^2(0, \infty)\), and \(AC_{\text{loc}}\) for \(AC_{\text{loc}}[0, \infty)\), i.e. those complex-valued functions defined on \([0, \infty)\) which are absolutely continuous on all compact sub-intervals of \([0, \infty)\).

Throughout the paper we assume that the coefficient \(q\) in (1.2) and (1.3) satisfies the following basic conditions:

\[
(1.4) \quad \begin{align*}
(i) & \quad q \text{ is real-valued on } [0, \infty) \\
(ii) & \quad q' \in AC_{\text{loc}},
\end{align*}
\]

which ensure that both \(M\) and \(M^2\) exist as formally symmetric differential expressions on \([0, \infty)\).

In these circumstances the minimal closed symmetric operator generated by \(M\) in \(L^2\) has deficiency indices either (1,1) or (2,2), the limit-point and limit-circle classifications at \(\infty\), respectively, of Weyl; see [2, page 1306] or [8, section 17.5]. Similarly the deficiency indices corresponding to \(M^2\) in \(L^2\) are (2,2), (3,3) or (4,4) and we refer to \(M^2\) as limit-\(r\) at \(\infty\) when these are \((r, r)\) for \(r = 2, 3\) or 4, respectively.

The problem raised in [1, section 12] concerned the existence of coefficients \(q\) such that \(M\), given by (1.2), is limit-point and \(M^2\) is limit-3, both at \(\infty\). At the time of writing of [1] it was known that \(M^2\) is limit-4 if and only if \(M\) is limit-circle and that \(M^2\) is frequently limit-2 when \(M\) is limit-point, but the general theory and examples available left the above problem open. Since then, several mathematicians have tried to find an example of such a coefficient \(q\) or, conversely, to prove that \(M^2\) is limit-2 if and only if \(M\) is limit-point at \(\infty\). The situation is further complicated by a recent result in [4] which states that if \(q\) satisfies (1.4) and, additionally, for some non-negative numbers \(k\) and \(X\)

\[
(1.5) \quad q(x) \geq -kx^2 \quad (x \in [X, \infty))
\]

then \(M\) is limit-point at \(\infty\) (previously known, see [8, section 23]) and \(M^2\) is limit-2 at \(\infty\). This shows that if there is a coefficient \(q\) for which \(M\) is limit-point at \(\infty\) and \(M^2\) is limit-3 at \(\infty\), then \(q\) will have to enjoy excursions through the \(-kx^2\) barrier, for every positive number \(k\), and yet do so in a way as to keep \(M\) in the limit-point case at \(\infty\).

An answer to this problem has now been obtained and is given in

**Theorem 1.** There exist coefficients \(q\) which satisfy the basic conditions (1.4) such that when the differential expressions \(M\) and \(M^2\) on \([0, \infty)\) are defined by (1.2) and (1.3) then

(a) \(M\) is limit-point at \(\infty\)

(b) \(M^2\) is limit-3 at \(\infty\).
Proof. This is given in sections 2 and 3 below.

In the proof of Theorem 1 two particular results are used which are themselves of interest and these are stated here separately since they throw some light on the nature of the integrable-square solutions of the differential equations associated with $M$ and $M^2$. These equations are

$$M[y] = 0 \quad \text{on} \quad [0, \infty) \quad \text{and} \quad M^2[y] = 0 \quad \text{on} \quad [0, \infty);$$

both regular at all points of $[0, \infty)$ but with singular points at $\infty$.

**Theorem 2.** Assume that (1.4) holds and that the equation $M^2[y] = 0$ on $[0, \infty)$ has exactly 3 linearly independent solutions which are of integrable-square on $[0, \infty)$, i.e. solutions in $L^2(0, \infty)$; then $M^2$ is limit-3 at $\infty$, $M$ is limit-point at $\infty$ and the equation $M[y] = 0$ on $[0, \infty)$ has exactly one linearly independent solution in $L^2(0, \infty)$.

**Theorem 3.** Assume that (1.4) holds; then the following two statements are equivalent:

1. The equation $M^2[y] = 0$ on $[0, \infty)$ has exactly three linearly independent solutions in $L^2(0, \infty)$.

2. The equation $M[y] = 0$ on $[0, \infty)$ has real-valued solutions $\varphi$ and $\psi \neq 0$ such that

$$\varphi \notin L^2(0, \infty), \quad \psi F \in L^2(0, \infty)$$

where $F(x) = \int_0^x \varphi^2 \quad (x \in [0, \infty))$.

We outline the contents of the paper. Section 2 contains proofs of Theorems 2 and 3 and associated results. The proof of Theorem 1 is given in section 3. In section 4 there are some remarks about the extent of the oscillations in such coefficients $q$ as determined by the construction in the proof of Theorem 1. There is a list of references.

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2. Proof of Theorem 2 and Theorem 3. From now on we assume that $q$ satisfies the basic conditions (1.4). It is then a standard result that the number of linearly independent solutions in $L^2$ of the eigenvalue problem

$$M^2[y] = \lambda y \quad \text{on} \quad [0, \infty)$$
does not depend on $\lambda$ as long as $\lambda$ is a complex but non-real number, and also that this number is $r$ if and only if $M^2$ is limit-$r$. The situation is more complicated when $\lambda$ is real. However, the following statements are known to hold true, see e.g. [5]:

(a) When $M^2$ is limit-$r$ and $\lambda$ is real, then (2.1) has at most $r$ linearly independent solutions which are of integrable square on $[0, \infty)$, that is in $L^2$.

(b) $M^2$ is limit-4 if and only if $M$ is limit-2 (that is, in the limit-circle condition) in which case all solutions of (2.1) are in $L^2$, also for every real $\lambda$.

Theorem 2 is an almost direct consequence of (a) and (b). In fact assume, as in Theorem 2, that the equation $M^2[y] = 0$ on $[0, \infty)$ has exactly 3 linearly independent solutions in $L^2$. Then $M^2$ can not be limit-2 according to (a), and it can not be limit-4 in view of (b). Thus $M^2$ must be limit-3 and, again from (b), $M$ must be limit-1 so that the equation $M[y] = 0$ may have at most one linearly independent solution in $L^2$ according to (a). But if it has no such solution, then $M^2[y] = 0$ can have at most two linearly independent solutions in $L^2$, since every solution of $M[y] = 0$ is also a solution of $M^2[y] = 0$. This completes the proof of Theorem 2.

To prove Theorem 3 we begin by considering certain solutions of $M^2[y] = 0$ given in terms of solutions of $M[y] = 0$. Let $f_1$ and $f_2$ be any two linearly independent real-valued functions which satisfy $M[y] = 0$, and are normalised so that $f_1 f_2^2 \neq f_1^2 f_2 = 1$. Define $f_3$ and $f_4$ by

$$f_3(x) = f_1(x) \int_0^x f_1 f_2 - f_2(x) \int_0^x f_1^2 \quad (x \in [0, \infty)),$$

and

$$f_4(x) = f_1(x) \int_0^x f_2^2 - f_2(x) \int_0^x f_1 f_2 \quad (x \in [0, \infty)).$$

A direct calculation verifies that

$$M[f_3] = f_1 \quad \text{and} \quad M[f_4] = f_2,$$

and it follows that $f_i (i = 1, 2, 3, 4)$ are all solutions of $M^2[y] = 0$. They are linearly independent, since if

$$f = a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 = 0$$

then

$$M[f] = a_3 f_1 + a_4 f_2 = 0,$$

which shows that $a_3 = a_4 = 0$ and thus also $a_1 = a_2 = 0$.

In one direction, the proof is now essentially contained in the following lemma concerning properties of certain pairs of functions in $L^2_{\text{loc}}$. 
Lemma 1. Let \( f \) and \( g \) be two functions which are defined and locally of integrable-square on \([0, \infty)\). Assume that

\[
f \notin L^2 \quad \text{and} \quad gF \in L^2 \quad \text{where} \quad F(x) = \int_0^x |f|^2 \quad (x \in [0, \infty)) ;
\]

then

\[
g \in L^2 \quad \text{and} \quad fG \in L^2 \quad \text{where} \quad G(x) = \int_x^\infty |g|^2 \quad (x \in [0, \infty)) ,
\]

\[
fg \in L \quad \text{and} \quad fH \in L^2 \quad \text{where} \quad H(x) = \int_x^\infty |fg| \quad (x \in [0, \infty)) .
\]

Proof. Let \( f \) and \( g \) satisfy the conditions of the lemma. The assumption that \( f \) is not in \( L^2 \) implies that \( F(x) = \int_0^x |f|^2 \) tends to infinity with \( x \) and thus that there exists a \( y \) in \((0, \infty)\) for which \( F(y) = 1 \) and \( F(x) \geq 1 \) \((x > y)\). The assumption \( gF \in L^2 \) then implies \( g \in L^2 \) and \( g \sqrt{F} \in L^2 \), and it follows that also \( f \sqrt{G} \) is in \( L^2 \), where \( G(x) = \int_x^\infty |g|^2 \), since a partial integration and the fact that \( F \) is increasing give

\[
\int_0^x |f^2G| = F(x) \int_x^\infty |g|^2 + \int_0^x |g^2F| \leq \int_0^\infty |g \sqrt{F}|^2 < \infty .
\]

Since \( G \) is continuous and bounded this proves that \( fG \) is in \( L^2 \).

The inequality, valid for \( 0 \leq y < x < \infty \),

\[
\left[ \int_y^x |fg| \right]^2 = \left[ \int_y^x |gF/f| \right]^2 \leq \int_y^x |gF|^2 + \int_y^x \{F'/F^2\} = [1 - 1/F(x)] \int_y^x |gF|^2
\]

proves that \( fg \) is in \( L \).

The analogous inequality

\[
H^2(x) = \left[ \int_x^\infty |fg| \right]^2 \leq [1/F(x)] \int_x^\infty |gF|^2
\]

shows that \( FH^2(x) \to 0 \) \((x \to \infty)\) and since

\[
\int_0^x |fH|^2 = (FH^2)(x) + 2 \int_0^x |fgFH| \leq (FH^2)(x) + 2 \left\{ \int_0^x |gF|^2 \right\}^{1/2} \left\{ \int_0^x |fH|^2 \right\}^{1/2}
\]

also that \( fH \in L^2 \). This completes the proof of Lemma 1.

Now assume that the statement (2) in Theorem 3 holds true, with \( \phi \) and \( \psi \) normalised so that \( \phi \psi' - \phi' \psi = 1 \). Then \( \phi \) and \( \psi \) satisfy the conditions of Lemma 1 with \( f = \phi \) and \( g = \psi \). Thus \( \psi \in L^2 \), \( \phi \psi \in L \) and the functions defined by

\[
\phi(x) \int_x^\infty \psi^2 , \quad \phi(x) \int_x^\infty \phi \psi \quad \text{and} \quad \psi(x) \int_0^x \phi \psi \quad (x \in [0, \infty))
\]
are all in $L^2$. With

\begin{align*}
(2.2) \quad f_3(x) &= \varphi(x) \int_0^x \varphi \psi - \psi(x) \int_0^x \varphi^2 \\
(2.3) \quad f_4(x) &= \varphi(x) \int_0^x \psi^2 - \psi(x) \int_0^x \varphi \psi
\end{align*}

it follows that $\psi, f_3 - \varphi \int_0^\infty \varphi \psi$ and $f_4 - \varphi \int_0^\infty \psi^2$ are three linearly independent $L^2$-solutions of $M^2[y] = 0$. Since $\varphi$ is a fourth solution which is not in $L^2$, it is clear that (1) is satisfied.

Conversely, assume that the statement (1) holds true. Then, from Theorem 2, $M$ must be in the limit-point condition at $\infty$ and $M[y] = 0$ must have exactly one linearly independent $L^2$-solution. Let $\varphi$ and $\psi$ be real-valued solutions of $M[y] = 0$ which satisfy $\varphi \notin L^2$, $\psi \in L^2$ and $\varphi \psi - \psi \int_0^\infty \psi^2 = 1$, and let $f_3$ and $f_4$ be defined by (2.2) and (2.3) so that $\{\varphi, \psi, f_3, f_4\}$ is a basis for the solutions of $M^2[y] = 0$. According to (1) there exist linearly independent vectors $(a_2, a_3, a_4)$ and $(b_2, b_3, b_4)$ in $\mathbb{R}^3$ for which $a_2 \varphi + a_3 f_3 + a_4 f_4$ and $b_2 \varphi + b_3 f_3 + b_4 f_4$ are both in $L^2$. It follows that $f_3 + a \varphi \in L^2$ for some unique real number $a$ (eliminating $f_4$ above in case both $a_4 \neq 0$ and $b_4 \neq 0$). Put

$$F(x) = \int_0^x \varphi^2 \quad \text{and} \quad H(x) = a + \int_0^x \varphi \psi \quad (x \in [0, \infty)).$$

Then $\varphi H - \psi F = f_3 + a \varphi \in L^2$, and the identity $(f_3 + a \varphi)^2 = (\varphi H)^2 + (\psi F)^2 - (FH^2)'$ gives

\begin{equation}
(2.4) \quad \int_0^x (f_3 + a \varphi)^2 = 2 \int_0^x (\varphi H)^2 + \int_0^x (\psi F)^2 - (FH^2)(x) \quad (x \in [0, \infty)),
\end{equation}

after a partial integration of the last term.

We shall show that $\psi F \in L^2$, so that the statement (2) follows, by obtaining a contradiction from (2.4) in case $\psi F$ is not in $L^2$.

If $\psi F \notin L^2$ it is clear from (2.4) that

\begin{equation}
(2.5) \quad U(x) = (FH^2)(x) - 2 \int_0^x (\varphi H)^2 \rightarrow +\infty \quad (x \to \infty),
\end{equation}

and since the function $V$ defined by

$$V(x) = F^{-2}(x) \int_0^x (\varphi H)^2 \quad (x \in (0, \infty))$$

satisfies $V' = \varphi^2 F^{-3}U$ it follows from (2.5) and the definition of $F$ that $V'$ (as well
as \( V \) is non-negative for sufficiently large \( x \). Thus \( V(x) > \frac{1}{2}C^2 \), say, that is

\[
\int_0^x (\varphi H)^2 \geq \frac{1}{2}C^2 F^2(x),
\]

for some constant \( C > 0 \) and large \( x \). Returning to (2.5) we obtain

\[ 0 < U(x) < (FH^2)(x) - C^2 F^2(x), \]

that is, \( H(x) > C \sqrt{F(x)} \) for all large \( x \). Now this inequality gives us the required contradiction. In fact, let \( X \) be so large that

\[
\int_0^\infty \psi^2 < C^2/4.
\]

Then for \( x > X \),

\[
H(x) = a + \int_0^x \varphi \psi \leq a + \int_0^x |\varphi \psi| + \left[ \int_0^x \int_0^x \varphi^2 \right]^{1/2} \leq \frac{C}{2} + \left( a + \int_0^x |\varphi \psi| \right)/\sqrt{F(x)} \sqrt{F(x)}
\]

where the term in ( ) is \(< C \) when \( x \) is large enough, since \( F(x) \to \infty \ (x \to \infty) \).

Thus \( \psi F \in L^2 \) and the proof of Theorem 3 is complete.

3. Proof of Theorem 1. To prove Theorem 1 it is sufficient, according to Theorem 2, to produce differential expressions \( M \) for which (1) of Theorem 3 holds true. To achieve this it is not only sufficient but also necessary for us to ensure that the equation \( M[y'] = 0 \) on \([0, \infty)\) has solutions \( \varphi \) and \( \psi \) which satisfy (2). As it turns out, such solutions must necessarily be of an oscillatory nature on \([0, \infty)\), with an unbounded sequence of discrete and simple zeros. In the following lemma we give additional conditions on the zeros of the linearly independent \( L^2 \)-solution \( \psi \) which ensure that \( M^2 \) is, indeed, in the limit-3 case.

**Lemma 2.** Let \( \varphi \) and \( \psi \) be two real-valued functions in \( C^4[0, \infty) \) which satisfy \( \varphi \psi' - \varphi' \psi = 1 \) on \([0, \infty)\). Assume that \( \psi \) has a denumerable increasing sequence \((x_n)_{n=0}^\infty \) of zeros, with \( x_0 = 0 \) and \( x_n \to \infty \ (n \to \infty) \), and that

(i)

\[
\sum_{n=1}^\infty \{(x_n - x_{n-1})^3 x_n^2\} \text{ converges},
\]

(ii) there exist positive numbers \( A, B \) and \( C \) such that

\[
\int_{x_{n-1}}^{x_n} \psi^2 < A(x_n - x_{n-1})^3 \quad \text{and} \quad Bx_n < \int_0^{x_n} \varphi^2 < Cx_n
\]

for all positive integers \( n \).
Then \((\psi''/\psi)(x)\) tends to a finite limit as \(x\) tends to a zero of \(\psi\); the coefficient \(q\) defined by \(q(x) = (\psi''/\psi)(x)\) \((x \in [0, \infty))\) is in \(C^2[0, \infty)\) (where, of course, \(q\) is defined by continuity at the zeros of \(\psi\)); and \(\varphi\) and \(\psi\) satisfy (2) of Theorem 3 with \(M\) defined by \(M[f] = -f'' + qf\).

**Proof.** Let \(\varphi\) and \(\psi\) satisfy the conditions of the lemma. The facts that \((\psi''/\psi)(x) = q(x)\) tends to a finite limit as \(x\) tends to a zero of \(\psi\) and that \(q \in C^2[0, \infty)\) follow directly from the assumptions \(\varphi, \psi \in C^4[0, \infty)\) and \(\varphi \psi' - \varphi' \psi = 1\), which imply that \(\varphi(x) \neq 0\) when \(\psi(x) = 0\), and that \(\varphi \psi'' = \varphi' \psi\). It is also clear from the last equality that \(M[\varphi] = M[\psi] = 0\).

The lower bound in the assumption (ii) implies that \(\varphi\) is not in \(L^2\). On the other hand with \(F(x) = \int_0^x \varphi^2 (x \in [0, \infty))\) the upper bounds give, for \(x \in [x_{n-1}, x_n]\),

\[
\int_0^{x_n} (\psi F)^2 < \sum_{n=1}^{N} \left\{ \int_{x_{n-1}}^{x_n} (\psi F)^2 \right\} < AC^2 \sum_{n=1}^{N} (x_n - x_{n-1})^3 x_n^2.
\]

In view of the assumption (ii) it follows that \(\psi F \in L^2\), that is, \(\varphi\) and \(\psi\) satisfy (2) of Theorem 3. This completes the proof of Lemma 2.

We now prove Theorem 1 by displaying functions \(\varphi\) and \(\psi\) which satisfy the conditions of Lemma 2. We shall construct such functions in terms of a real-valued function \(f\) in \(C^\infty[0, 1]\) with the properties that

(i) \(f\) is infinitely differentiable, positive and convex upwards with

\[
f(0) = 0, \quad f'(0) = k > 0, \quad f(1) = B > 0 \quad \text{and} \quad f'(1) = 0,
\]

and that, for some number \(r \in (0, \frac{1}{2})\),

(ii) \(f(x) = kx \quad (x \in [0, r])\) and \(f(x) = B \quad (x \in [1 - r, 1])\),

(iii) \(\int_1^r (1/f)^2 = 1/(k^2 r)\).

Then we shall show that functions \(f\) with the above properties do indeed exist, provided \(k/B \in (1, 2)\) and \(r\) is small enough.

Assuming that \(f\) has the properties (i)—(iii), define \(g\) on \([0, 1]\) by

\[
g(x) = -f(x) \int_x^1 (1/f)^2 \quad (x \in [0, 1]),
\]

where \(g(0)\) is defined by continuity.

Since \(f\) is positive this function \(g\) is negative, and since, by a direct calculation,

\[(3.1) \quad fg' - f'g = 1 \quad \text{and} \quad fg'' = f''g \quad \text{on} \quad [0, 1],\]

we infer from (ii) that

\[(3.2) \quad g \quad \text{is convex downwards with} \quad g(0) = -1/k \quad \text{and} \quad g(1) = 0.\]
Near the left end point of \([0, 1]\) we obtain from (iii)

\[ g(x) = -kx \left( \int_x^1 (kt)^{-2} \, dt + \int_1^r f^{-2} \right) = -1/k \quad (x \in [0, r]). \]

and near the right end point we have

\[ g(x) = -B \int_x^1 B^{-2} = (x - 1)/B \quad (x \in [1 - r, 1]). \]

It is clear from (i) that \(f\) is increasing and satisfies \(Bx \leq f(x) \leq B \quad (x \in [0, 1])\) and it follows from (3.2) and (3.3) that \(|g(x)| \leq 1/k\) on \([0, 1]\). Thus

\[ B^2/3 < \int_0^1 f^2 < B^2 \quad \text{and} \quad \int_0^1 g^2 < 1/k^2. \]

\[ \text{Fig. 1.} \]

In the interval \([x_0, x_1] = [0, 2]\) \(\varphi(x) = -f_1(x)\) and \(\psi(x) = -g_1(x)\). In the intervals \([x_{n-1}, x_n]\)
\(\varphi(x) = (-1)^n f_1(2(x - x_{n-1}))/2(x_{n-1} - x_n - 1)\), \(\psi(x) = (-1)^{n-1} (x_n - x_{n-1}) - g_1(2(x - n))\)
\((x_{n-1} - x_n - 1))\). In this particular example, \(L_n = 2/n\) and \(B = 1, k = 1/2\), and \(r = 1/2\); as we shall see later the set of functions \(f\) satisfying (i)–(iii) is non-empty for these values of \(B, k, \) and \(r\).

Let \((x_n)_{n=0}^\infty\) be a sequence of real numbers tending monotonically to infinity with \(n\), with \(x_0 = 0\) and \(x_1 = 2\), and put \(L_n = x_n - x_{n-1}\). Define, first \(f_1\) and \(g_1\) on \([0, L_1] = [0, 2]\) by \(f_1(x) = -f(1 - x)\), \(g_1(x) = g(1 - x) \quad (x \in [0, 1])\) and then, for each integer \(n > 1, f_n\) and \(g_n\) on \([0, L_n]\) by

\[ f_n(x) = f_1(2x/L_n) \quad \text{and} \quad g_n(x) = (L_n/2) g_1(2x/L_n) \quad (x \in [0, L_n]). \]

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The property (ii) of \( f \) ensures that \( f_1 \) is infinitely differentiable on its interval of definition and takes the constant value \( -B \) near the left end point and the value \( B \) near the right end point of this interval. Similarly, (3.3) shows that \( g_1 \) is also infinitely differentiable and (3.4) that \( g_1 \) vanishes linearly at the end points of its interval of definition, with slope \( -1/B \) near the left end point and slope \( 1/B \) near the right one. It follows from (3.1) that \( f_1 g_1' - f_1' g_1 = 1 \) on this interval. Clearly, from the definition (3.6), the functions \( f_n \) and \( g_n \) inherit these properties for all integers \( n > 1 \). But this means that we may patch the \( f_n : s \) together, and also the \( g_n : s \), to obtain two functions \( \varphi \) and \( \psi \) in \( C^\infty [0, \infty) \) of the form shown in Fig. 1 by defining, for each interval \( I_n = [x_{n-1}, x_n] \),

\[
\varphi(x) = (-1)^n f_n(x - x_{n-1}) \quad \text{and} \quad \psi(x) = (-1)^n g_n(x - x_{n-1}) \quad (x \in I_n).
\]

These functions satisfy \( \varphi \varphi' - \varphi' \psi = 1 \) on \([0, \infty)\) and since

\[
\int_{I_n} \varphi^2 = (L_n/2) \int_0^2 f_1^2 = L_n \int_0^1 f^2 \quad \text{and} \quad \int_{I_n} \psi^2 = (L_n^3/4) \int_0^1 g^2
\]

it follows from (3.5) that

\[
\int_{x_{n-1}}^{x_n} \psi^2 < L_n^3/(2k)^2 \quad \text{and} \quad B^2 L_n/3 < \int_{x_{n-1}}^{x_n} \varphi^2 < B^2 L_n.
\]

Thus \( \varphi \) and \( \psi \) satisfy the conditions of Lemma 2 provided we select the sequence \( (L_n)_{n=1}^\infty \) so that \( L_1 = 2 \) and

\[
\sum_{n=1}^\infty \{L_n^3 x_n^2\} = \sum_{n=1}^\infty \{L_n^3 (\sum_{k=1}^n L_k)^2\} \text{ converges.}
\]

To obtain examples of such sequences we may choose \( L_n = 2n^{-x} \) with \( x \in (\frac{3}{4}, 1] \); the fact that these satisfy (3.7) is easily verified on using

\[
\sum_{k=1}^n L_k < 2 \left( 1 + \int_1^n t^{-2} \, dt \right) < \frac{2(1 - x)^{-1} n^{1-x}}{2 + 2 \log n} \quad (x \in (\frac{3}{4}, 1))
\]

It remains to verify that there exist functions \( f \) with the above properties (i)–(iii).

Intuitively, it seems clear that there are functions which have the properties stated in (i) and (ii) when \( 1 < k/B < 2 \), but for lack of a suitable reference we sketch a proof of

**Lemma 3.** Let \( a, b, c \) and \( d \) be positive real numbers satisfying \( b < c/d < a \). Then there are functions in \( C^\infty [0, d] \) which are convex upwards and satisfy

\[
f(0) = 0, f'(0) = a, f(d) = c, f'(d) = b \quad \text{and} \quad f^{(n)}(0) = f^{(n)}(d) = 0 \quad (n \geq 2).
\]
Proof. Define \( P : [0, 1] \to [0, 1] \) by

\[
P(x) = K \int_0^x \exp \left\{ -\frac{1}{s} - \frac{1}{1-s} \right\} \, ds \quad (x \in [0, 1]),
\]

where the constant \( K \) is determined by the requirement \( P(1) = 1 \). It is straightforward to verify that \( P \) is infinitely differentiable and increases monotonically from \( P(0) = 0 \) to \( P(1) = 1 \), with all derivatives vanishing at \( x = 0 \) and at \( x = 1 \), and satisfies \( P(x) + P(1-x) = 1 \) \( (0 \leq x \leq \frac{1}{2}) \) so that \( \int_0^1 P = \frac{1}{2} \).

Now the assumption \( bd < c < ad \) implies that \( l = (c - bd)/(ad - bd) \) satisfies \( 0 < l < 1 \), and thus in turn that in a \((s,t)\)-plane, the line \( t = \frac{1}{2}s \) has a nonempty intersection with the half-square determined by \( 0 \leq t < t + s \leq 1 \). For each \((s, t)\) in this intersection, define \( Q_{st} : [0, d] \to [0, a - b] \) by

\[
Q_{st}(x) = \begin{cases} 
0 & (0 \leq x \leq td) \\
(a - b) P \left( \frac{x - td}{sd} \right) & (td < x < (t + s)d) \\
a - b & ((t + s)d \leq x \leq d).
\end{cases}
\]

Then \( Q_{st} \) is in \( C^\infty[0, d] \) and

\[
\int_0^d Q_{st} = (ad - bd) s \int_0^1 P + (ad - bd) (1 - s - t) = (ad - bd) (1 - t - \frac{1}{2}s) = (ad - bd) (1 - l) = ad - c.
\]

Thus each function \( f : [0, d] \to [0, c] \) defined by

\[
f(x) = ax - \int_0^x Q_{st} \quad (x \in [0, d])
\]

satisfies \( f(d) = c \). A direct computation shows that \( f \) has the other properties stated in the lemma as well.

Now let \( S = S(B, k, r) \) be the subset of \( C^\infty[0, 1] \) containing real-valued functions which satisfy (i) and (ii). According to Lemma 3, with \( a = k, b = 0, c = B \) and \( d = 1 \), this set is nonempty, and it follows from the convexity requirement in (i) that each \( f \) in \( S \) is bounded by \( l(x) \leq f(x) \leq m(x) \) in the interval \([r, 1 - r]\), where the graph of

\[
l(x) = kr + \frac{B - kr}{1 - 2r} (x - r) \quad (x \in [r, 1 - r])
\]

is the line segment connecting the points \((r, kr)\) and \((1 - r, B)\), and

\[
m(x) = \begin{cases} 
kh & (x \in [0, B/k)) \\
B & (x \in [B/k, 1])
\end{cases}
\]
For each $s$ in the open interval $(r, B/k)$, let $l_s$ be the line parallel to $l$ which intersects the graph of $m$ at the points $(s, ks)$ and $(r, B)$, say. On rounding off the corners following the recipe in Lemma 3 we obtain, for any sufficiently small positive number $\varepsilon$ and all $s$ in $(r + \varepsilon, Bk^{-1} - \varepsilon)$, functions in $S$ with graphs coinciding with that of $m$ in $(0, s - \varepsilon) \cup (t + \varepsilon, 1)$ and with $l_s$ in $(s + \varepsilon, t - \varepsilon)$. A continuity argument shows that $S$ contains functions for which $\int_r^1 (1/f)^2$ takes any prescribed value in the open interval

$$\left( \int_r^{1-r} (1/m)^2 + r/B^2, \; \int_r^{1-r} (1/l)^2 + r/B^2 \right).$$

![Fig. 2.](image)

In particular, it contains functions which satisfy (iii) when $1/(k^2 r)$ lies in this interval. Putting $k/B = b$ we find after some elementary calculations that this condition takes the form

$$(1 + b^2 r - 2br)/b^2 r < 1/b^2 r < (b + b^2 r^2 - 2br)/b^2 r,$$

or equivalently

$$b < 2 < b + (1 - br)^2,$$

which is satisfied for $b \in (1, 2)$ provided $r$ is small enough — in the two figures above we have used $B = 1$, $k = \frac{5}{3}$ and $r = \frac{1}{3}$.

4. Comments on the above examples. All examples constructed by the method used in section 3 have the common property that there exist arbitrarily large $x$ for which $q(x) = (\varphi''/\varphi)(x) < -x^3$. In fact, the property (i) for $f$ implies that $f''/f$ must be strictly negative at some points in $(r, 1 - r)$. Let $y$ be such a point with, say, $(f''/f)(y) = -c^2$. Then at the points $y_n = x_{n-1} + (L_n/2)$ we have

$$q(y_n) = (2/L_n)^2 (f''/f_1)(y) = -(2c/L_n)^2.$$

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Thus if $q(y_n) \geq -y_n^3$ for $n > N_0$ then

$$(2c/L_n)^2 \leq (x_{n-1} + (L_n/2)y)^3 < x_n^3 \quad (n > N_0),$$

so that for $N > N_0$

$$\sum_{n=N_0}^{N} \{L_n x_n^3\} > 4c^2 \sum_{n=N_0}^{N} \{L_n|x_n|\} = 4c^2 \sum_{n=N_0}^{N} \{L_n|\sum_{k=1}^{n} L_k|\}.$$ 

Here the last sum diverges since $\sum_{k} L_k$ is divergent, just as $\int_{-\infty}^{\infty} \{h(x)\int_{-\infty}^{x} h\} \, dx$ diverges when $\int_{-\infty}^{\infty} h$ is divergent. This contradicts (3.7) and so $q(y_n) < -y_n^3$ for arbitrarily large $n$. On the other hand, given any positive number $\varepsilon$ the method in section 3 yields coefficients which satisfy $q(x) \geq -x^{3+\varepsilon}$ ($x \in [0, \infty]$). These $q : s$ result from functions $\varphi$ and $\psi$ constructed by means of functions $f$ which hold close to $\sin(\pi x/2)$ on $[0, 1]$, with very small intervals of linearity near the end-points.

Thus the results of this paper still leave open the question whether the condition (1.5) for $M^2$ to be limit-2 at $\infty$ is best possible or not in the case when $M$ is limit-point at $\infty$.

References


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