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A MAP ASSOCIATED TO THE LEPAGIAN FORMS ON THE CALCULUS OF VARIATIONS IN FIBRED MANIFOLDS

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1. The role of the so called Lepagian differential forms in the calculus of variations is well known. The simplest form of this kind was introduced by E. CARTAN [1] (see also [3]). LEPAGE (see e.g. [7]) extended the theory to the variational integrals over n-dimensional domains in Euclidean spaces. Since then many authors formulated the foundations of the variational calculus in terms of the Lepagian forms. The concept proved to be useful for a modern, differential-geometric approach to the variational problems in fibred manifolds (see e.g. [2], [4], [5], [8]). Our remark to the theory of the Lepagian forms and Lepagian equivalents is based on a definition given by the author [5], [6], and concerns the first order variational problems, which are mostly used in practice. Unlike the Cartan fundamental form, the Lepagian equivalent we consider is not, in general, 1-horizontal (in the terminology of KOLÁŘ [4]). An example of a Lepagian equivalent for the second order variational problems can be found in [6].

2. Let us briefly recall the main notions of the variational theory used later on. We assume that we are given a smooth finite dimensional *fibred manifold* $\pi : Y \to X$ (a submersion) with an orientable n-dimensional base space X. Put $\mathscr{J}^0 Y = Y$ and denote by $\mathscr{J}^r Y$ the manifold of all r-jets of local sections of π, and by $\pi_r : \mathscr{J}^r Y \to X$ and $\pi_{rs} : \mathscr{J}^r Y \to \mathscr{J}^s Y$ $(0 \leqq s \leqq r)$ the corresponding fibred manifolds defined by the natural projections of jets. We shall denote by R the field of real numbers.

The following spaces of forms, important for many variational considerations, are introduced in [5]: The space $\Omega^{n+1}(\mathscr{J}^1 Y)$ of all $(n + 1)$-forms defined on $\mathscr{J}^1 Y$, the space $\Omega^{n+1}_Y(\mathscr{J}^1 Y)$ of all π_{20}-horizontal $(n + 1)$-forms on $\mathscr{J}^1 Y$ (the *Lagrangians*), the space $\Omega^n(Y)$ of all n-forms on Y, and the space $\Omega_{\text{Lep}}(\mathscr{J}^1 Y)$ of the so called Lepagian forms, a subspace of the real vector space $\Omega^n_Y(\mathscr{J}^1 Y)$ of all π_{10}-horizontal n-forms on $\mathscr{J}^1 Y$. With these spaces we associate the maps $h_1 : \Omega_{\text{Lep}}(\mathscr{J}^1 Y) \to \Omega^n_X(\mathscr{J}^1 Y)$

(a linear surjection), $\tilde{h} : \Omega^{n+1}(\mathscr{J}^1 Y) \to \Omega^{n+1}(\mathscr{J}^2 Y)$, and the Euler map of the calculus of variations [5], $E : \Omega^n_X(\mathscr{J}^1 Y) \to \Omega^n_Y(J^2 Y)$ with the diagram

$$\begin{array}{ccc} \Omega_{\mathrm{Lep}}(\mathscr{J}^1 Y) & \xrightarrow{h_1} & \Omega^n_X(\mathscr{J}^1 Y) \\ \Big\downarrow d & & \Big\downarrow E \\ \Omega^{n+1}(\mathscr{J}^1 Y) & \xrightarrow{\tilde{h}} & \Omega^{n+1}_Y(\mathscr{J}^2 Y) \end{array}$$

being commutative. We note that the left arrow in the diagram means the exterior differentiation of forms. Moreover, it is known that $E(\lambda) = 0$ if and only if there is a (uniquely determined) n-form $\varrho_0 \in \Omega^n(Y)$ such that $h_1(\pi^*_{10}\varrho_0) = \lambda$ and $d\varrho_0 = 0$. If $\lambda \in \Omega^n_X(\mathscr{J}^1 Y)$ is an n-form then each $\varrho \in \Omega_{\mathrm{Lep}}(\mathscr{J}^1 Y)$ such that $h_1(\varrho) = \lambda$ is called a *Lepagian equivalent* of λ. The map h_1 being a surjection, to each λ there exists a Lepagian equivalent.

An example of a Lepagian equivalent, often used in practice, the *Cartan fundamental form* [2], [3], [4], [8], is provided with the following. Let (x_i, y_σ) be some fibre coordinates on Y, $(x_i, y_\sigma, z_{i\sigma}, z_{ij\sigma})$ the corresponding fibre coordinates on $\mathscr{J}^2 Y$ $(1 \leqq i \leqq j \leqq n,\ \ n = \dim X,\ \ 1 \leqq \sigma \leqq m,\ \ m = \dim Y - \dim X)$. Each n-form $\lambda \in \Omega^n_X(\mathscr{J}^1 Y)$ is expressed as

$$\lambda = \mathscr{L}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n\,,$$

where $\mathscr{L}$ is a function of x_i, y_σ, $z_{i\sigma}$. The Cartan fundamental form is then defined by

$$\varrho = \mathscr{L}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n + \sum_{i,\sigma} \frac{\partial \mathscr{L}}{\partial z_{i\sigma}}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{i-1} \wedge$$

$$\wedge \left(\mathrm{d}y_\sigma - \sum_k z_{k\sigma}\, \mathrm{d}x_k\right) \wedge \mathrm{d}x_{i+1} \wedge \ldots \wedge \mathrm{d}x_n\,.$$

We shall examine another type of Lepagian equivalents better adopted to the conditions for the Euler form $E(\lambda)$ of the Lagrangian λ to vanish.

3. The purpose of this paper is to prove the following

Theorem. *There exists an R-linear map* $l : \Omega^n_X(\mathscr{J}^1 Y) \to \Omega_{\mathrm{Lep}}(\mathscr{J}^1 Y)$ *satisfying the following conditions*:

1) *For each* $\lambda \in \Omega^n_X(\mathscr{J}^1 Y)$,

$$h_1(l(\lambda)) = \lambda\,.$$

2) *If* $\varrho \in \Omega^n_Y(\mathscr{J}^1 Y)$ *is of the form* $\pi^*_{10}\varrho_0$ *for some* $\varrho_0 \in \Omega^n(Y)$ *then*

$$l(h_1(\varrho)) = \varrho\,.$$

3) *If* $\lambda \in \Omega^n_X(\mathscr{J}^1 Y)$ *is a* π_1*-horizontal Lagrangian form then the corresponding Euler form is given by*

$$E(\lambda) = \tilde{h}(\mathrm{d}l(\lambda)) .$$

The equalities

$$\mathrm{d}l(\lambda) = 0 , \quad E(\lambda) = 0$$

are either both true or both wrong.

Proof. Let us first suppose that we have an n-form $\varrho = \pi^*_{10}\varrho_0$, where $\varrho_0 \in \Omega^n(Y)$. In some fibre coordinates (x_i, y_σ) on Y and the corresponding fibre coordinates $(x_i, y_\sigma, z_{i\sigma})$ on $\mathscr{J}^1 Y$, we have

$$(1) \qquad \varrho = g_0 \,\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n + \sum \frac{1}{r!} g^{s_1 \ldots s_r}_{\sigma_1 \ldots \sigma_r} \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{s_1 - 1} \wedge$$

$$\mathrm{d}y_{\sigma_1} \wedge \mathrm{d}x_{s_1+1} \wedge \ldots \wedge \mathrm{d}x_{s_r - 1} \wedge \mathrm{d}y_{\sigma_r} \wedge \mathrm{d}x_{s_r+1} \wedge \ldots \wedge \mathrm{d}x_n ,$$

where g_0, $g^{s_1 \ldots s_r}_{\sigma_1 \ldots \sigma_r}$ are functions of x_i and y_σ, and we sum over all sequences $r, s_1, \ldots$ $\ldots, s_r, \sigma_1, \ldots, \sigma_r$ such that $1 \leqq r \leqq n$, $1 \leqq s_1 < \ldots < s_r \leqq n$, $1 \leqq \sigma_1, \ldots, \sigma_r \leqq m$. Then

$$h_1(\varrho) = \mathscr{K} \,\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n ,$$

where

$$\mathscr{K} = g_0 + \sum g^{s_1 \ldots s_r}_{\sigma_1 \ldots \sigma_r} \cdot z_{s_1 \sigma_1} \ldots z_{s_r \sigma_r} ,$$

the range of the summation being the same as above. The following identities can be obtained by differentiating with respect to $z_{k\sigma}$ (see [5]):

$$(2) \qquad g^{1 \ldots n}_{\nu_1 \ldots \nu_n} = \frac{\partial^n \mathscr{K}}{\partial z_{1\nu_1} \ldots \partial z_{n\nu_n}} ,$$

$$\ldots ,$$

$$g^{s_1 \ldots s_p}_{\nu_1 \ldots \nu_p} = \frac{\partial^p \mathscr{K}}{\partial z_{s_1 \nu_1} \ldots \partial z_{s_p \nu_p}} - \sum g^{k_1 \ldots k_j}_{\sigma_1 \ldots \sigma_j} \frac{\partial^p}{\partial z_{s_1 \nu_1} \ldots \partial z_{s_r \nu_p}} (z_{k_1 \sigma_1} \ldots z_{k_j \sigma_j}) ,$$

$$\ldots ,$$

$$g_0 = \mathscr{K} - \sum g^{s_1 \ldots s_r}_{\sigma_1 \ldots \sigma_r} \cdot z_{s_1 \sigma_1} \ldots z_{s_r \sigma_r} .$$

In the formula for $g^{s_1 \ldots s_p}_{\nu_1 \ldots \nu_p}$, we sum over all sequences satisfying $p + 1 \leqq j \leqq n$, $1 \leqq k_1 < \ldots < k_j \leqq n$, $1 \leqq \sigma_1, \ldots, \sigma_j \leqq m$. With the help of the formulas (2), we are able to reconstruct the n-form ϱ from the known expression for $\mathscr{K}$.

Let now λ be any n-form from the space $\Omega^n_X(\mathscr{J}^1 Y)$. In our fibre coordinates,

$$\lambda = \mathscr{L} \,\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n ,$$

where $\mathscr{L}$ is a function depending on $x_i, y_\sigma, z_{i\sigma}$. Taking into account the preceding remark we define an n-form by the right-hand side of (1) setting

$$(3)\quad g^{1\ldots n}_{\sigma_1\ldots\sigma_n} = \frac{1}{n!}\,\varepsilon^{1\ldots n}_{k_1\ldots k_n}\frac{\partial^n\mathscr{L}}{\partial z_{k_1\sigma_1}\ldots\partial z_{k_n\sigma_n}},$$

$$\ldots,$$

$$g^{s_1\ldots s_r}_{\sigma_1\ldots\sigma_r} = \frac{1}{r!}\,\varepsilon^{s_1\ldots s_r}_{k_1\ldots k_r}\left(\frac{\partial^r\mathscr{L}}{\partial z_{k_1\sigma_1}\ldots\partial z_{k_r\sigma_r}} - \sum g^{l_1\ldots l_j}_{\nu_1\ldots\nu_j}\frac{\partial^r}{\partial z_{s_1\nu_1}\ldots\partial z_{s_r\nu_r}}\left(z_{k_1\sigma_1}\ldots z_{k_j\sigma_j}\right)\right),$$

$$\ldots,$$

$$g_0 = \mathscr{L} - \sum g^{s_1\ldots s_r}_{\sigma_1\ldots\sigma_r}\cdot z_{s_1\sigma_1}\ldots z_{s_r\sigma_r}\,.$$

In these formulas, $\varepsilon^{ijk\ldots}_{pqr\ldots}$ denotes the totally antisymmetric symbol equal to 1 when $(pqr\ldots)$ is an even permutation of $(ijk\ldots)$, -1 when $(pqr\ldots)$ is an odd permutation of $(ijk\ldots)$, and 0 in all the other cases. It follows from the definition that the coordinate expression for $l(\lambda)$ is invariant under coordinate changes which means that $l(\lambda)\in\Omega^n_Y(\mathscr{J}^1Y)$.

We shall show that the map $\lambda\to l(\lambda)$ satisfies all conditions of the theorem. First we are to prove that for each λ the n-form $l(\lambda)$ is Lepagian. We use for this purpose a coordinate formula for $\tilde{h}(\mathrm{d}\varrho)$, where $\varrho\in\Omega^n_Y(\mathscr{J}^1Y)$, derived in [6]. If ϱ is expressed by (1), where g_0 and $g^{s_1\ldots s_r}_{\sigma_1\ldots\sigma_r}$ are functions of all variables x_i, y_σ, $z_{i\sigma}$, then

$$\tilde{h}(\mathrm{d}\varrho) = \left(\left(\frac{\partial\mathscr{K}}{\partial y_\sigma} - \mathrm{d}_i\mathscr{B}_{i\sigma}\right)\mathrm{d}y_\sigma + \left(\frac{\partial\mathscr{K}}{\partial z_{i\sigma}} - \mathscr{B}_{i\sigma}\right)\mathrm{d}z_{i\sigma}\right)\wedge\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n\,,$$

where $\mathscr{K}$ is defined by

$$h_1(\varrho) = \mathscr{K}\,\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n$$

and

$$\mathscr{B}_{i\sigma} = \sum g^{s_1\ldots s_r}_{\sigma_1\ldots\sigma_r}\frac{\partial}{\partial z_{i\sigma}}\left(z_{s_1\sigma_1}\ldots z_{s_r\sigma_r}\right).$$

The symbol d_i stands for the formal derivative operator [5]. By definition, ϱ is Lepagian if and only if

$$(4)\quad \mathscr{B}_{i\sigma} = \frac{\partial\mathscr{K}}{\partial z_{i\sigma}}$$

(see [6]).

To show that this condition is satisfied by $l(\lambda)$ we use the definition of q_0 and g^i_σ (3) obtaining

$$\frac{\partial\mathscr{L}}{\partial z_{i\sigma}} = \frac{\partial g_0}{\partial z_{i\sigma}} + \sum\left(\frac{\partial g^{s_1\ldots s_r}_{\sigma_1\ldots\sigma_r}}{\partial z_{i\sigma}}z_{s_1\sigma_1}\ldots z_{s_r\sigma_r} + g^{s_1\ldots s_r}_{\sigma_1\ldots\sigma_r}\frac{\partial}{\partial z_{i\sigma}}\left(z_{s_1\sigma_1}\ldots z_{s_r\sigma_r}\right)\right) =$$

$$= \frac{\partial g_0}{\partial z_{i\sigma}} + g^i_\sigma + \sum_{r=2}^{n}\sum_{s_1<\ldots<s_r}\sum_{\sigma_1,\ldots,\sigma_r} g^{s_1\ldots s_r}_{\sigma_1\ldots\sigma_r}\frac{\partial}{\partial z_{i\sigma}}\left(z_{s_1\sigma_1}\ldots z_{s_r\sigma_r}\right) + \sum\frac{\partial g^{s_1\ldots s_r}_{\sigma_1\ldots\sigma_r}}{\partial z_{i\sigma}}z_{s_1\sigma_1}\ldots z_{s_r\sigma_r}\,,$$

$$g^i_\sigma = \frac{\partial\mathscr{L}}{\partial z_{i\sigma}} - \sum_{j=2}^{n}\sum_{s_1<\ldots<s_j}\sum_{\sigma_1,\ldots,\sigma_j} g^{l_1\ldots l_j}_{\nu_1\ldots\nu_j}\frac{\partial}{\partial z_{i\sigma}}\left(z_{l_1\nu_1}\ldots z_{l_j\nu_j}\right)$$

which gives

$$\frac{\partial g_0}{\partial z_{i\sigma}} + \sum \frac{\partial g^{s_1 \dots s_r}_{\sigma_1 \dots \sigma_r}}{\partial z_{i\sigma}} z_{s_1\sigma_1} \dots z_{s_r\sigma_r} = 0 ,$$

proving (4). This means that the n-form $l(\lambda)$ is Lepagian.

It remains to prove the equalities 1)–3) of the theorem. The first two of them are easy consequences of the definition of $l(\lambda)$. Since $l(\lambda)$ is Lepagian, the equality $E(\lambda) = \tilde{h}(\mathrm{d}l(\lambda))$ follows from the diagram of section 2. If $\mathrm{d}l(\lambda) = 0$ then this equality immediately implies $E(\lambda) = 0$. To prove the converse let us assume that $E(\lambda) = 0$. Then there is a unique $\varrho_0 \in \Omega^n(Y)$ such that $h_1(\pi^*_{10}\varrho_0) = \lambda$ and $\mathrm{d}\varrho_0 = 0$ (see section 2). According to 2), $l(\lambda) = l(h_1(\pi^*_{10}\varrho_0)) = \pi^*_{10}\varrho_0$ ad we get $\mathrm{d}l(\lambda) = 0$. This completes the proof.

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