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THE CONNECTION BETWEEN THE PROXIMATE ORDER OF AN ENTIRE CHARACTERISTIC FUNCTION AND THE CORRESPONDING DISTRIBUTION FUNCTION

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INTRODUCTION

Let $H(x)$ be a distribution function (d.f.) with an entire characteristic function (e.c.f.) $h(z)$ given by

$$h(z) = \int_{-\infty}^{\infty} e^{izu}\, \mathrm{d}H(u) \quad (z = t + iy)\,. \tag{0,1}$$

ROSSBERG [5] studied the growth of $|h(z)|$ using proximate orders (introduced by VALIRON, see [6]). We want to continue his work and to formulate more precisely the connection which exists between the d.f. and the proximate orders of the corresponding e.c.f.. For instance, we show the existence of the limit (2.2) and we give its value while Rossberg – in the general case – derived only bounds for the lower and upper limits. For the details we refer the reader to [5]. Theorems 2.2 and 2.3 are the main results of our paper.

1. DEFINITIONS AND CERTAIN NECESSARY LEMMAS

The lemmas of this section will serve as tools for the proofs of Section 2. Familiarity with the proofs of this section which are omitted is not required.

Lemma 1.1. *The c.f. $h(t)$ of a d.f. $H(x)$ is an e.c.f. if and only if the relation*

$$H(-x) + 1 - H(x) = o(e^{-rx}) \quad \text{as} \quad x \to \infty \tag{1.1}$$

holds for all positive r.
(See [3], Theorem 7.2.1.)

We denote by

$$M(r, h) = \max_{|z| \leqq r} |h(z)| \tag{1.2}$$

the maximum modulus of $h(z)$ in the circle $|z| \leqq r$. Then it is known for an e.c.f. $h(z)$ that

$$M(r, h) = \max \{h(ir), h(-ir)\} . \tag{1.3}$$

Theorem 1.1. *An e.c.f. which is not identically* 1 *is of an order not less than* 1. *Moreover, there is no e.c.f. of order one and of minimal type. The c.f. of any finite d.f.* $F(x) \not\equiv \varepsilon_0(x)$ *) *is an e.c.f. of order one and of exponential type, and conversely.* (See for instance [3], Theorems 7.1.3, 7.2.9 and 7.2.3.)

Definition 1.1. Let $\varrho(r)$ be a real-valued function defined for every $r > 0$. $\varrho(r)$ is called a *proximate order* (p.o.), if the following three conditions are satisfied:

$$\lim_{r \to \infty} \varrho(r) = \varrho \quad \text{exists and } 0 < \varrho < \infty , \tag{1.4}$$

$$\varrho'(r) \text{ exists for all sufficiently large } r , \tag{1.5}$$

$$\lim_{r \to \infty} r \, \varrho(r) \ln r = 0 . \tag{1.6}$$

Definition 1.2. An e.c.f. $h(z)$ has a p.o. $\varrho(r)$, if

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r, h)}{r^{\varrho(r)}} = \tau \quad (0 < \tau < \infty) . \tag{1.7}$$

τ is called *the type of* $h(z)$ with respect to the p.o. $\varrho(r)$. (If $\tau = 0$ or $\tau = \infty$, the e.c.f. $h(z)$ is said to be of *minimal* or of *maximal* type with respect to the p.o. $\varrho(r)$, respectively.)

Remark 1.1. The p.o. $\varrho(r)$ is not determined uniquely by the e.c.f. $h(z)$. We note that τ depends on the p.o. $\varrho(r)$.

Now, we list some properties of the p.o.:

Lemma 1.2. *The function* $P(r) = r^{\varrho(r)}$ *is increasing for all sufficiently large* r (see [2]).

Lemma 1.3. *Let* $\varrho(x)$ *be a p.o. and denote by* $x = \varphi(r)$ *the inverse function to* $r = x^{\varrho(x)}$ (see also Lemma 1.2). *Then*

$$\bar{\varrho}(r) = \frac{1}{\varrho(\varphi(r))} \quad \left(\to \frac{1}{\varrho} \text{ as } x \to \infty \right) \tag{1.8}$$

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*) $\varepsilon_0(x) = \begin{cases} 0 & \text{for } x \leqq 0 , \\ 1 & \text{for } x < 0 . \end{cases}$

is also a p.o. and

$$r^{\bar{\varrho}(r)} = \varphi(r) . \tag{1.9}$$

Proof. The first assertion is proved in [5]. Since $\varphi(r)$ is the inverse function to $x^{\varrho(x)}$, we have $\varphi(r)^{\varrho(\varphi(r))} = r$ and so

$$r^{\bar{\varrho}(r)} = [\varphi(r)^{\varrho(\varphi(r))}]^{1/\varrho(\varphi(r))} = \varphi(r) .$$

Definition 1.3. The p.o. defined by (1.8) is called *the dual proximate order* (d.p.o.) to the p.o. $\varrho(r)$ and is denoted by $\bar{\varrho}(r)$.

It is convenient to introduce the following terminology.

Definition 1.4. We say that two p.o. $\varrho_1(r)$ and $\varrho_2(r)$ are *equal* if $\varrho_1(r) = \varrho_2(r)$ for all sufficiently large r.

It is immediately seen that

$$\bar{\bar{\varrho}}(r) = \varrho(r) . \tag{1.10}$$

Remark 1.2. Let $\varrho(x)$ be a p.o.. The function

$$H(x) = \begin{cases} 0 & \text{for} \quad x \leqq x_0 , \\ 1 - \exp\left(-x^{1+\varrho(x)}\right) & \text{for} \quad x > x_0 \end{cases} \tag{1.11}$$

can be considered to be a d.f., if x_0 is chosen sufficiently large. By virtue of Lemma 1.2 this is easily seen.

Lemma 1.4. *The relation*

$$\lim_{r \to \infty} \frac{P(kr)}{P(r)} = k^{\varrho} \tag{1.12}$$

holds uniformly in every interval $0 < a \leqq k \leqq b < \infty$ (see [2]).

Lemma 1.5. *Let* $x(y)$ *and* $x_0(y) : (0, \infty) \to (0, \infty)$ *be two unbounded nondecreasing functions and let* $f(x) : (0, \infty) \to (0, \infty)$ *be a non-decreasing function such that for all* $K > 0$

$$\lim_{x \to \infty} \frac{f(Kx)}{f(x)} = K^{\alpha} \tag{1.13}$$

for a certain $\alpha > 0$.

If the relation

$$\lim_{y \to \infty} \frac{f(x(y))}{f(x_0(y))} = A \quad (\textit{finite or infinite}) \tag{1.14}$$

holds, then we have

$$\lim_{y \to \infty} \frac{x(y)}{x_0(y)} = \begin{cases} 0 & \text{for} \quad A = 0 , \\ A^{1/\alpha} & \text{for} \quad 0 < A < \infty , \\ \infty & \text{for} \quad A = \infty . \end{cases} \tag{1.15}$$

Proof. We shall distinguish three cases.

(a) $A = 0$. Let be $\bar{B} = \overline{\lim}_{y \to \infty} (x(y)/x_0(y)) > 0$. Then there exists a sequence $y_j \to \infty$ such that for any ε, $0 < \varepsilon < \bar{B}$, one has $x(y_j)/x_0(y_j) > \bar{B} - \varepsilon$ provided that j is taken sufficiently large. We obtain

$$\lim_{j \to \infty} \frac{f(x(y_j))}{f(x_0(y_j))} \geqq \lim_{j \to \infty} \frac{f((\bar{B} - \varepsilon)\, x_0(y_j))}{f(x_0(y_j))} = (\bar{B} - \varepsilon)^\alpha > 0$$

which contradicts our assumption $A = 0$.

(b) $A = \infty$. The proof is similar, we have only to replace x by x_0.

(c) $0 < A < \infty$. Let be $\bar{B} = \infty$. Then there exists a sequence $y_j \to \infty$ such that for all sufficiently large j, $x(y_j)/x_0(y_j) \geqq M$. Therefore we have

$$A = \lim_{j \to \infty} \frac{f(x(y_j))}{f(x_0(y_j))} \geqq \lim_{j \to \infty} \frac{f(M\, f(x_j))}{f(x_0(y_j))} = M^\alpha$$

in contradiction to $A < \infty$, since we can choose M arbitrarily large. In the same way we obtain that $\underline{B} = 0$ cannot hold. In order to complete the proof we must show that $\underline{B} = \bar{B}$. Indeed, there exists a sequence $y_j \to \infty$ such that, given any ε, $0 < \varepsilon < \bar{B}$, for all sufficiently large j one has $\bar{B} - \varepsilon < x(y_j)/x_0(y_j)$ and for all sufficiently large y one has $x(y)/x_0(y) < \bar{B} + \varepsilon$. Analogously we obtain $(\bar{B} - \varepsilon)^\alpha \leqq A \leqq (\bar{B} + \varepsilon)^\alpha$. Since ε can be chosen arbitrarily small we find $\bar{B}^\alpha = A$. Similarly we show $\underline{B}^\alpha = A$. Thus the lemma is proved.

For the sake of brevity we now introduce the following function:

$$f(r, u) = e^{ru}(1 - H(u)) \quad \text{for} \quad u \geqq 0\,, \quad r \geqq 0\,. \tag{1.16}$$

We assume that the d.f. considered are left-continuous. Then we have

$$f(r, u) = f(r, u - 0) > f(r, u + 0)\,. \tag{1.17}$$

at the discontinuity points of $f(r, u)$. But this means that the function $f(r, u)$ is upper semicontinuous for all u and for any fixed r. Hence $f(r, u)$ has a maximum on any closed interval for any fixed r.

From now on let us assume that the d.f. $H(x)$ has an e.c.f.. In view of Lemma 0.3 we obtain

$$\lim_{u \to \infty} f(r, u) = 0 \quad \text{for all} \quad r \geqq 0\,. \tag{1.18}$$

Therefore there exists

$$m(r) = \max_{0 \leqq u < \infty} f(r, u)\,. \tag{1.19}$$

Let be

$$u(r) = \sup\,\{u \mid f(r, u) = m(r)\}\,. \tag{1.20}$$

Then we easily see that

$$(1.21) \qquad f(r, u(r)) = m(r).$$

We formulate this result in the following

Lemma 1.6. *Let be $f(r, u) = e^{ru}(1 - H(u))$. For all $r \geqq 0$ there exists a value $u(r)$ so that the following two conditions are satisfied:*

$$(1.22) \qquad f(r, u(r)) \geqq f(r, u) \quad \textit{for} \quad u \leqq u(r)$$

and

$$(1.23) \qquad f(r, u(r)) > f(r, u) \quad \textit{for} \quad u > u(r).$$

It is easily seen that $u(r)$ has the following properties:

Lemma 1.7. *$u(r)$ is a right-continuous, non-decreasing function satisfying the relation*

$$(1.24) \qquad \lim_{r\to\infty} u(r) = \infty.$$

2. THE GROWTH OF ENTIRE CHARACTERISTIC FUNCTIONS WHICH ARE OF FINITE ORDER GREATER THAN 1

In this section we confine ourselves to e.c.f. of non-negative random variables.

Theorem 2.1. *Let $H(x)$ be a d.f. which admits the representation*

$$(2.1) \qquad 1 - H(x) = \exp\{-kx^{1+\varrho(x)}\}, \quad x > 0,$$

where $\varrho(x)$ is a p.o. and $k > 0$ a constant.

Then $H(x)$ has an e.c.f. $h(z)$ which satisfies the relation

$$(2.2) \qquad \lim_{r\to\infty} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} = k^{-1/\varrho} c,$$

where

$$(2.3) \qquad c = \frac{\varrho}{(1 + \varrho)^{1+1/\varrho}}$$

and $\bar{\varrho}(r)$ is the d.p.o. to $\varrho(x)$.

Remark 2.1. This theorem generalizes a part of Theorem 2 of [5]. In a different way Rossberg proved (2.2) assuming that

$$(2.4) \qquad \lim_{x\to\infty} x^{\varrho(x)-\varrho} = \varkappa \quad (0 < \varkappa < \infty)$$

is satisfied (see Theorem 3 of [5]). Moreover, Theorem 2.1 generalizes Theorem 7.1 of [3] in a manner.

Remark 2.2. In view of later applications of this theorem it is convenient to choose the representation (2.1), but we remark that $\varrho(x) + \ln k/\ln x$ is also a p.o..

Proof. By means of (1.3), the representation

$$M(r, h) = 1 + r\int_0^\infty e^{ru}(1 - H(u))\,du$$

is valid for the e.c.f. $h(z)$ (integrating by parts). In view of (1.16) we write

$$M(r, h) = 1 + r\int_0^\infty e^{-u} f(r + 1, u)\,du\,. \tag{2.5}$$

We see from Lemma 1.6 that

$$M(r, h) \leqq 1 + r f(r + 1, u(r + 1))\,. \tag{2.6}$$

For the sake of brevity we write $u_1(r)$ instead of $u(r + 1)$. We have

$$\left.\frac{\partial f(r + 1, u)}{\partial u}\right|_{u = u_1(r)} = 0$$

for all sufficiently large r. After an elementary computation we obtain

$$u_1(r)^{\varrho(u_1(r))} = \frac{1}{k}\,\frac{r + 1}{1 + \varrho(u_1(r)) + \varrho'(u_1(r))\,u_1(r)\ln u_1(r)}\,. \tag{2.7}$$

This yields

$$f(r + 1, u_1(r)) = \tag{2.8}$$

$$= \exp\left\{(r + 1)\,u_1(r)\,\frac{\varrho(u_1(r)) + \varrho'(u_1(r))\,u_1(r)\ln u_1(r)}{1 + \varrho(u_1(r)) + \varrho'(u_1(r))\,u_1(r)\ln u_1(r)}\right\}.$$

Remember that $\varphi(r)$ is the inverse function to $x^{\varrho(x)}$, i.e. $\varphi(r)^{\varrho(\varphi(r))} = r$. So we have according to (2.7)

$$\lim_{r\to\infty}\frac{u_1(r)^{\varrho(u_1(r))}}{\varphi(r)^{\varrho(\varphi(r))}} = \frac{1}{k}\,\frac{1}{1 + \varrho}\,.$$

It follows from Lemmas 1.2, 1.4, 1.7 and 1.5 (we substitute $P(x) = x^{\varrho(x)}$ by $f(x)$) that

$$\lim_{r\to\infty}\frac{u_1(r)}{\varphi(r)} = k^{-1/\varrho}\,\frac{1}{(1 + \varrho)^{1/\varrho}}\,. \tag{2.9}$$

Now we see from (2.6) and (2.8) that

$$\frac{\ln M(r,h)}{r^{\bar{\varrho}(r)+1}} \leqq \frac{r+1}{r}\frac{u_1(r)}{r^{\bar{\varrho}(r)}}\frac{\varrho(u_1(r)) + \varrho'(u_1(r))\,u_1(r)\ln u_1(r)}{1+\varrho(u_1(r)) + \varrho'(u_1(r))\,u_1(r)\ln u_1(r)} + o(1)$$

holds.

From (1.9) and (2.9) we conclude

$$\limsup_{r\to\infty}\frac{\ln M(r,h)}{r^{\bar{\varrho}(r)+1}} \leqq \frac{\varrho}{(1+\varrho)^{1+1/\varrho}}\,k^{-1/\varrho}\,. \tag{2.10}$$

In [5] it was already shown that

$$\liminf_{r\to\infty}\frac{\ln M(r,h)}{r^{\bar{\varrho}(r)+1}} \geqq ck^{-1/\varrho}\,, \tag{2.11}$$

where c is given by (2.3).

The statement of the theorem follows from (2.10) and (2.11).

Corollary 2.1.1. *For all p.o. $\bar{\varrho}(r)$ there exists a d.f. $H(x)$ whose c.f. $h(z)$ is entire and has a p.o. $1+\bar{\varrho}(r)$, where the relation (2.2) is satisfied even for $k=1$.*

Proof. We construct the function $H(x)$ by (1.11) (where $\varrho(r)$ is the d.p.o. to $\bar{\varrho}(r)$) which is a d.f. (according to Remark 1.2) and apply Theorem 2.1.

Remark 2.3. If for all $x \geqq X(>0)$ we have

$$1 - H(x) \leqq \exp\{-kx^{1+\varrho(x)}\} \tag{2.12}$$

then, by the estimate (1.1) of Lemma 1.1, $H(x)$ has an e.c.f. $h(z)$. In virtue of the proof of Theorem 2.1, $h(z)$ is either of order $1+1/\varrho$ or of order less than $1+1/\varrho$. In the former case we have

$$\overline{\lim_{r\to\infty}}\frac{\ln M(r,h)}{r^{\bar{\varrho}(r)+1}} \leqq k^{-1/\varrho}c$$

while in the latter case we have, obviously,

$$\lim_{r\to\infty}\frac{\ln M(r,h)}{r^{\bar{\varrho}(r)+1}} = 0\,.$$

The growth of an e.c.f. (of a non-finite d.f.) depends on the "tail behaviour" of the corresponding d.f., and conversely. In order to investigate this "tail behaviour", for the sake of convenience we introduce the following notation. If $H(x)$ is a (non-finite) d.f., then we define the function $\gamma_H(x)$ by

$$\gamma_H(x) = \ln(-\ln[1-H(x)])(\ln x)^{-1} - 1\,. \tag{2.13}$$

Theorem 2.2. *A necessary and sufficient condition (NASC) for $H(x)$ to have an e.c.f. of p.o. $\bar{\varrho}(r) + 1$, i.e.*

$$0 < \overline{\lim_{r \to \infty}} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} = \bar{\sigma} < \infty , \tag{2.14}$$

is that

$$1 - H(x) > 0 \quad \text{for every} \quad x > 0 \tag{2.15}$$

and

$$\underline{\lim}_{x \to \infty} x^{\gamma_H(x) - \varrho(x)} = (c/\bar{\sigma})^{\varrho} \tag{2.16}$$

where c is defined by (2.3).

Remark 2.4. The given result generalizes parts of Theorems 4 and 8 of [5]. The statement of this theorem was already proved in a different way by Rossberg under the condition (2.4) (see Theorem 3 of [5]). Theorem 2.2 may also be considered as a generalization of Theorem 7.1 of [4] with respect to p.o.

Proof. Let $k = (c/\bar{\sigma})^{\varrho}$. We prove the sufficiency. By the assumption (2.16), given any $\varepsilon > 0$ with $\varepsilon < k$, we have for x sufficiently large

$$1 - H(x) < \exp\{-(k - \varepsilon)\, x^{\varrho(x)+1}\} .$$

Hence, by Remark 2.3, $h(z)$ is an e.c.f. and it holds

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} \leqq \bar{\sigma} .$$

By Theorem 4 of [5] it follows that

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} \geqq \bar{\sigma} .$$

From both these inequalities we obtain (2.14).

Now we turn to the necessity. By Theorem 1.1 we have (2.15) and Theorem 8 of [5] implies

$$\underline{\lim}_{x \to \infty} x^{\gamma_H(x) - \bar{\varrho}(x)} \geqq k . \tag{2.17}$$

We have to prove that the inequality sign cannot hold in (2.17). Suppose that it does, then it follows by the first part of the proof that

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} < \bar{\sigma} .$$

This contradiction establishes the necessity of the condition (2.16).

Corollary 2.2.1. *Let $H(x)$ be a non-finite d.f., $\varrho(r)$ a p.o. A NASC for $H(x)$ to have an e.c.f. $h(z)$ and*

$$(2.18) \qquad \lim_{r \to \infty} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} = 0$$

is

$$(2.19) \qquad \lim_{x \to \infty} x^{\gamma_H(x) - \varrho(x)} = \infty .$$

Proof. We start with the necessity of (2.19). In view of our assumption, to any given $\varepsilon > 0$ there exists an $R = R(\varepsilon)$ such that for all $r \geqq R$

$$M(r, h) \leqq \exp\{\varepsilon r^{\bar{\varrho}(r)+1}\}$$

and similarly to the second part of the proof of Theorem 8 of [5] we obtain (2.19).

As to the sufficiency, we note that to any given $k > 0$ there exists an $X = X(k)$ such that for all $x \geqq X$

$$1 - H(x) < \exp\{-kx^{\varrho(x)+1}\} .$$

Since k can be chosen arbitrary large we see from Remark 2.3 that $h(z)$ is an e.c.f. and (2.18) holds.

Corollary 2.2.2. *Let $H(x)$ be a non-finite d.f. and have an e.c.f.. A NASC for $H(x)$ to have a c.f. such that*

$$(2.20) \qquad \overline{\lim_{r \to \infty}} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} = \infty$$

is

$$(2.21) \qquad \underline{\lim}_{x \to \infty} x^{\gamma_H(x) - \varrho(x)} = 0 .$$

Proof follows from Theorem 2.2 and Corollary 2.2.1. (Note that (2.21) does not guarantee that $h(z)$ is an e.c.f..)

Now we consider the assumption

$$(2.22) \qquad \overline{\lim_{x \to \infty}} x^{\gamma_H(x) - \varrho(x)} = \overline{\lim_{r \to \infty}} u(r)^{\gamma_H(u(r)) - \varrho(u(r))} .$$

It requires a certain regularity of the function $u(r)$. For instance, it is satisfied if $u(r)$ is continuous (we know that $u(r)$ is always right-continuous by Lemma 1.7) or more generally, if

$$\lim_{r \to \infty} \frac{u(r-)}{u(r)} = 1 .$$

We omit the proof of this result, since it is not required for the following considerations.

Theorem 2.3. *Let $H(x)$ have an e.c.f. and let (2.22) hold. NASC for $H(x)$ to have a c.f. such that*

$$0 < \underline{\lim}_{r \to \infty} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} = \underline{\sigma} < \infty \tag{2.23}$$

are (2.15) *and*

$$0 < \overline{\lim}_{x \to \infty} x^{\gamma_H(x) - \varrho(x)} = (c/\underline{\sigma})^{\varrho}, \tag{2.24}$$

where c is defined by (2.3).

Proof. Let $k = (c/\underline{\sigma})^{\varrho}$. We start with the sufficiency; in view of our assumption (2.24), given any $\varepsilon > 0$, we have for x sufficiently large

$$1 - H(x) > \exp\{-(k + \varepsilon)\, x^{\varrho(x)+1}\}.$$

From Theorem 2.1 we obtain (see also Theorem 2 of [5])

$$\underline{\lim}_{r \to \infty} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} \geqq \underline{\sigma}. \tag{2.25}$$

Now let $H_\varepsilon(x)$ be a d.f. defined by

$$1 - H_\varepsilon(x) = \exp\{-(k - \varepsilon)\, x^{\varrho(x)+1}\}. \tag{2.26}$$

By virtue of (2.24) there exists a sequence $x_i \to \infty$ such that

$$1 - H(x_i) \leqq 1 - H_\varepsilon(x_i).$$

Let $f(r, u)$, $u(r)$ and $f_\varepsilon(r, u)$, $u_\varepsilon(r)$ be defined by means of Lemma 1.6 for $H(x)$ and for $H_\varepsilon(x)$, respectively. By (2.22), there exists a subsequence x_i' of x_i, $x_i' \to \infty$ such that for a certain sequence r_i, $r_i \to \infty$ we have $u(r_i) = x_i'$. By virtue of (2.26) we obtain

$$f(r_i, u(r_i)) \leqq f_\varepsilon(r_i, u(r_i)),$$

and this yields

$$f(r_i, u(r_i)) \leqq f_\varepsilon(r_i, u_\varepsilon(r_i)).$$

Hence, by (2.6) we have

$$\underline{\lim}_{r \to \infty} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} \leqq \underline{\lim}_{r \to \infty} \frac{\ln f(r, u(r))}{r^{\bar{\varrho}(r)+1}} \leqq \overline{\lim}_{r \to \infty} \frac{\ln f_\varepsilon(r, u_\varepsilon(r))}{r^{\bar{\varrho}(r)+1}}.$$

Similarly to the proof of Theorem 2.1 we conclude that

$$\underline{\lim}_{r \to \infty} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} \leqq \frac{c}{(k - \varepsilon)^{1/\varrho}}. \tag{2.27}$$

As ε can be arbitrarily small the statement (2.23) follows from (2.25) and (2.27).

We turn to the necessity. We note that for any $\varepsilon > 0$ there exists a sequence $r_i \to \infty$ such that

$$M(r, h) < \exp\{(\sigma + \varepsilon)\, r_i^{\bar{\varrho}(r_i)+1}\} .$$

Similarly as in [5], pp. 329–330, we obtain now

$$\overline{\lim_{x\to\infty}}\, x^{\gamma_H(x) - \varrho(x)} \geqq k . \tag{2.28}$$

We have to show that the inequality sign cannot hold in (2.28). Suppose that it does. Then it follows from the first part of the proof that

$$\underline{\lim_{r\to\infty}} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} < \underline{\sigma} .$$

This contradiction establishes the necessity of the condition (2.24).

The condition (2.15) is satisfied, or else $h(z)$ would have the order one (see Theorem 1.1).

Corollary 2.3.1. *Let $H(x)$ be a non-finite d.f. with an e.c.f. $h(z)$ and let (2.22) be satisfied. A NASC for*

$$\underline{\lim_{r\to\infty}} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} = 0 \tag{2.29}$$

is

$$\overline{\lim_{x\to\infty}}\, x^{\gamma_H(x) - \varrho(x)} = \infty . \tag{2.30}$$

Proof. Let (2.29) be satisfied. Similar to the second part of the proof of Theorem 2.3 we obtain (2.30).

The sufficiency of (2.30) follows from the second part of the first part of the proof of Theorem 2.3.

Corollary 2.3.2. *Let $H(x)$ be a non-finite d.f. with an e.c.f. $h(z)$ and let (2.22) be satisfied. A NASC for*

$$\lim_{r\to\infty} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} = \infty \tag{2.31}$$

is

$$\lim_{x\to\infty} x^{\gamma_H(x) - \varrho(x)} = 0 . \tag{2.32}$$

Proof. This is an immediate consequence of Theorem 2.3 and Corollary 2.3.1.

Theorem 2.4. *Let $H(x)$ be a d.f., $\varrho(x)$ a p.o. and let (2.22) hold. NASC for $H(x)$ to have an e.c.f. such that*

$$(2.33) \qquad 0 < \lim_{r \to \infty} \frac{\ln M(r, h)}{r^{\bar{\varrho}(r)+1}} = \sigma < \infty$$

are (2.15) *and*

$$(2.34) \qquad \lim_{x \to \infty} x^{\gamma_H(x) - \varrho(x)} = (c/\sigma)^{\varrho},$$

where c is defined by (2.3).

Proof. This is an immediate consequence of Theorems 2.2 and 2.3.

It is possible to transcribe the preceding theorems to arbitrary d.f.'s. As an example we give only the following theorem.

Theorem 2.2'. *NASC for a d.f. $F(x)$ to have an e.c.f. $f(z)$ of p.o. $\varrho(r) + 1$ such that*

$$(2.14') \qquad 0 < \overline{\lim_{r \to \infty}} \frac{\ln M(r, f)}{r^{\bar{\varrho}(r)+1}} = \bar{\sigma} < \infty$$

are

$$(2.15') \qquad 1 - F(x) + F(-x) > 0 \quad \text{for every} \quad x > 0$$

and

$$(2.16') \qquad \underline{\lim}_{x \to \infty} \frac{-\ln(1 - F(x) + F(-x))}{x^{\bar{\varrho}(x)+1}} = (c/\bar{\sigma})^{\varrho},$$

where c is defined by (2.3).

Remark 2.5. This result generalizes Theorem 6.1 in [4].

Proof. It is easily seen that

$$H(x) = \begin{cases} 0 & \text{for} \quad x \leqq 0, \\ F(x) - F(-x) & \text{for} \quad x > 0, \end{cases}$$

is a d.f. of a non-negative random variable. Using the inequalities

$$M(r, f) \leqq M(r, h) \leqq 2\, M(r, f)$$

we obtain Theorem 2.2' from Theorem 2.2.

Remark 2.6. Let a d.f. $F(x)$ which does not belong to a finite distribution be represented by

$$1 - F(x) = \exp\{-x^{\gamma_{H+}(x)+1}\} \quad \text{for all} \quad x > 0,$$

and

$$F(-x) \quad = \exp\{-x^{\gamma_{H-}(x)}\} \qquad \text{for all} \quad x > 0.$$

We can investigate the behaviour of $f(z)$ more precisely by means of p.o. in the upper half-plane $\{z : z = t + iy, y \geqq 0\}$ or in the lower half-plane $\{z : z = t + iy, y \leqq 0\}$ by studying the function $\gamma_{H_-}(x)$ or $\gamma_{H_+}(x)$, respectively. We do not mention these results since in all cases the method is similar to that used in the proofs of the preceding theorems (see, for instance, Theorem 7 of [5]). It is possible to study the connection of the growth of an e.c.f. and the tail behaviour of the corresponding d.f. in a more general scale of growth than the p.o. For these investigations we refer the reader to [1]. C.f. of both finite and infinite orders considered there.

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