Alois Švec

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ON THE RIGIDITY OF CERTAIN SURFACES IN E^5

ALOIS ŠVEC, Olomouc

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E. BOMPIANI [1] has presented the classes of surfaces in E^n which may admit non-trivial higher order deformations (at least locally). In E^5, the general surfaces of this class are those possessing a conjugate net. In what follows, I just show the global infinitesimal rigidity of a subclass of surfaces with a conjugate net. Of course, Bompiani's results deserve a further study.

1. Let $G \subset \mathcal{R}^2$ be a bounded domain, ∂G its boundary and $M : \bar{G} \to E^5$, $\bar{G} = G \cup \partial G$, a surface in the 5-dimensional Euclidean space. To each point $m \in M \equiv M(\bar{G})$, associate an orthonormal frame $\{v_1, \ldots, v_5\}$ such that $v_1, v_2 \in T(M)$. Then

(1)
$$\begin{aligned}
\mathrm{d}m &= \omega^1 v_1 + \omega^2 v_2 \,,\\
\mathrm{d}v_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 + \omega_1^5 v_5 \,,\\
\mathrm{d}v_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 + \omega_2^5 v_5 \,,\\
\mathrm{d}v_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 + \omega_3^5 v_5 \,,\\
\mathrm{d}v_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 + \omega_4^5 v_5 \,,\\
\mathrm{d}v_5 &= -\omega_1^5 v_1 - \omega_2^5 v_2 - \omega_3^5 v_3 - \omega_4^5 v_4
\end{aligned}$$

with the usual integrability conditions $\mathrm{d}\omega^i = \omega^j \wedge \omega_j^i$, $\mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j$. From

(2)
$$\omega^3 = \omega^4 = \omega^5 = 0 \,,$$

we get

(3)
$$\omega^1 \wedge \omega_1^3 + \omega^3 \wedge \omega_2^3 = 0 \,, \quad \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0 \,,$$
$$\omega^1 \wedge \omega_1^5 + \omega^2 \wedge \omega_2^5 = 0$$

and the existence of functions $a_1, \ldots, c_3$ such that

$$\omega_1^3 = a_1\omega^1 + a_2\omega^2, \quad \omega_2^3 = a_2\omega^1 + a_3\omega^2, \tag{4}$$
$$\omega_1^4 = b_1\omega^1 + b_2\omega^2, \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2,$$
$$\omega_1^5 = c_1\omega^1 + c_2\omega^2, \quad \omega_2^5 = c_2\omega^1 + c_3\omega^2.$$

Let $T_m^2(M)$ denote the 2-osculating space of M at m, and suppose dim $T_m^2(M) = 4$ for each $m \in M$. The frames $\{v_i\}$ be chosen in such a way that $v_3, v_4 \in T^2(M)$, i.e.,

$$\omega_1^5 = \omega_2^5 = 0. \tag{5}$$

Let $\{w_i\}$ be another field of moving frames, and let

$$v_1 = \varepsilon_1(\cos\alpha \,.\, w_1 - \sin\alpha \,.\, w_2), \quad v_2 = \sin\alpha \,.\, w_1 + \cos\alpha \,.\, w_2, \tag{6}$$
$$v_3 = \varepsilon_2(\cos\beta \,.\, w_3 - \sin\beta \,.\, w_4), \quad v_4 = \sin\beta \,.\, w_3 + \cos\beta \,.\, w_4,$$
$$v_5 = \varepsilon_3 w_5; \quad \varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 1.$$

The equations (1) being now $\mathrm{d}m = \Omega^i w_i$, $\mathrm{d}w_i = \Omega_i^j w_j$, we get

$$\omega^1 = \varepsilon_1(\cos\alpha \,.\, \Omega^1 - \sin\alpha \,.\, \Omega^2), \quad \omega^2 = \sin\alpha \,.\, \Omega^1 + \cos\alpha \,.\, \Omega^2, \tag{7}$$
$$\omega_1^3 = \varepsilon_1\varepsilon_2(\cos\alpha\cos\beta \,.\, \Omega_1^3 - \sin\alpha\cos\beta \,.\, \Omega_2^3 - \cos\alpha\sin\beta \,.\, \Omega_1^4 +$$
$$+ \sin\alpha\sin\beta \,.\, \Omega_2^4),$$
$$\omega_1^4 = \varepsilon_1(\cos\alpha\sin\beta \,.\, \Omega_1^3 - \sin\alpha\sin\beta \,.\, \Omega_2^3 + \cos\alpha\cos\beta \,.\, \Omega_1^4 -$$
$$- \sin\alpha\cos\beta \,.\, \Omega_2^4),$$
$$\omega_2^3 = \varepsilon_2(\sin\alpha\cos\beta \,.\, \Omega_1^3 + \cos\alpha\cos\beta \,.\, \Omega_2^3 - \sin\alpha\sin\beta \,.\, \Omega_1^4 -$$
$$- \cos\alpha\sin\beta \,.\, \Omega_2^4),$$
$$\omega_2^4 = \sin\alpha\sin\beta \,.\, \Omega_1^3 + \cos\alpha\sin\beta \,.\, \Omega_2^3 + \sin\alpha\cos\beta \,.\, \Omega_1^4 +$$
$$+ \cos\alpha\cos\beta \,.\, \Omega_2^4.$$

Consider the functions

$$K = a_1a_3 - a_2^2 + b_1b_3 - b_2^2, \quad k = a_1b_2 - a_2b_1 + a_2b_3 - a_3b_2 \tag{8}$$

defined by

$$\omega_1^3 \wedge \omega_2^3 + \omega_1^4 \wedge \omega_2^4 = K\omega^1 \wedge \omega^2, \quad \omega_1^3 \wedge \omega_1^4 + \omega_2^3 \wedge \omega_2^4 = k\omega^1 \wedge \omega^2 \tag{9}$$

resp. Similarly, K^* and k^* be defined by

$$\Omega_1^3 \wedge \Omega_2^3 + \Omega_1^4 \wedge \Omega_2^4 = K^*\Omega^1 \wedge \Omega^2, \quad \Omega_1^3 \wedge \Omega_1^4 + \Omega_2^3 \wedge \Omega_2^4 = k^*\Omega^1 \wedge \Omega^2 \tag{10}$$

resp. From (7), we get

$$K^* = K\,, \quad k^* = \varepsilon_1\varepsilon_2 k\,. \tag{11}$$

Thus K is an invariant; it is easy to see that it is the Gauss curvature of M. The expression k depends just on the orientation of the moving frames. Let us look for the existence of a tangent vector field $V = \xi v_1 + \eta v_2$ such that there is a non-trivial tangent vector field $W = \xi' v_1 + \eta' v_2$ satisfying $WV \subset T(M)$. Because of

$$\mathrm{d}V = \mathrm{d}x\,.\,v_1 + \mathrm{d}y\,.\,v_2 + x(\omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4) + \\ + y(-\omega_1^2 v_2 + \omega_2^3 v_3 + \omega_2^4 v_4)\,,$$

we get

$$WV \equiv (a_1\xi\xi' + a_2\xi\eta' + a_2\eta\xi' + a_3\eta\eta')\,v_3 + \\ + (b_1\xi\xi' + b_2\xi\eta' + b_2\eta\xi' + b_3\eta\eta')\,v_4 \pmod{T(M)}\,;$$

because of $WV \equiv 0$ and $\xi'^2 + \eta'^2 \neq 0$, we get

$$(a_1b_2 - a_2b_1)\,\xi^2 + (a_1b_3 - a_3b_1)\,\xi\eta + (a_2b_3 - a_3b_2)\,\eta^2 = 0\,. \tag{12}$$

The equation (12) is the equation of the so-called conjugate directions $\{\xi v_1 + \eta v_2\}$ of M.

2. Let Φ be an infinitesimal second order deformation of our surface M; Φ is obviously given by (1) where we have to replace $\omega_3^4, \omega_3^5, \omega_4^5$ by $\omega_3^4 + t\varphi_4^3 + \ldots$, $\omega_3^5 + t\varphi_3^5 + \ldots$, $\omega_4^5 + t\varphi_4^5 + \ldots$ resp. Comparing the terms at t in the integrability conditions, we get

$$\omega_1^4 \wedge \varphi_3^4 + \omega_1^5 \wedge \varphi_3^5 = 0\,, \quad \omega_1^3 \wedge \varphi_3^4 - \omega_1^5 \wedge \varphi_4^5 = 0\,, \\ \omega_2^4 \wedge \varphi_3^4 + \omega_2^5 \wedge \varphi_3^5 = 0\,, \\ \omega_2^3 \wedge \varphi_3^4 - \omega_2^5 \wedge \varphi_4^5 = 0\,, \quad \omega_1^3 \wedge \varphi_3^5 + \omega_1^4 \wedge \varphi_4^5 = 0\,, \\ \omega_2^3 \wedge \varphi_3^5 + \omega_2^4 \wedge \varphi_4^5 = 0\,; \tag{13}$$

$$\mathrm{d}\varphi_3^4 = -\omega_3^5 \wedge \varphi_4^5 - \varphi_3^5 \wedge \omega_4^5\,, \\ \mathrm{d}\varphi_3^5 = \omega_3^4 \wedge \varphi_4^5 + \varphi_3^4 \wedge \omega_4^5\,, \quad \mathrm{d}\varphi_4^5 = -\omega_3^4 \wedge \varphi_3^5 - \varphi_3^4 \wedge \omega_3^5\,. \tag{14}$$

Because of (4) and (13), we get the existence of functions $A_1, \ldots, C_4$ such that

$$b_1\varphi_3^4 + c_1\varphi_3^5 = A_1\omega^1 + A_2\omega^2\,, \quad a_1\varphi_3^4 - c_1\varphi_4^5 = B_1\omega^1 + B_2\omega^2\,, \\ b_2\varphi_3^4 + c_2\varphi_3^5 = A_2\omega^1 + A_3\omega^2\,, \quad a_2\varphi_3^4 - c_2\varphi_4^5 = B_2\omega^1 + B_3\omega^3\,, \\ b_3\varphi_3^4 + c_3\varphi_3^5 = A_3\omega^1 + A_4\omega^2\,, \quad a_3\varphi_3^4 - c_3\varphi_4^5 = B_3\omega^1 + B_4\omega^2\,, \\ a_1\varphi_3^5 + b_1\varphi_4^5 = C_1\omega^1 + C_2\omega^2\,, \\ a_2\varphi_3^5 + b_2\varphi_4^5 = C_2\omega^1 + C_3\omega^2\,, \\ a_3\varphi_3^5 + b_3\varphi_4^5 = C_3\omega^1 + C_4\omega^2\,. \tag{15}$$

Write

$$\varphi_3^4 = x_1\omega^1 + x_2\omega^2\,, \quad \varphi_3^5 = y_1\omega^1 + y_2\omega^2\,, \quad \varphi_4^5 = z_1\omega^1 + z_2\omega^2\,; \tag{16}$$

from (15), we obtain

$$\begin{aligned} b_2x_1 - b_1x_2 + c_2y_1 - c_1y_2 &= 0\,, \\ b_3x_1 - b_2x_2 + c_3y_1 - c_2y_2 &= 0\,, \\ a_2x_1 - a_1x_2 - c_2z_1 + c_1z_2 &= 0\,, \\ a_3x_1 - a_2x_2 - c_3z_1 + c_2z_2 &= 0\,, \\ a_2y_1 - a_1y_2 + b_2z_1 - b_1z_2 &= 0\,, \\ a_3y_1 - a_2y_2 + b_3z_1 - b_2z_2 &= 0\,. \end{aligned} \tag{17}$$

We are now in the position to prove the following local

Theorem 1. *Let $M \equiv M(G) \subset E^5$ be a surface such that* $\dim T_m^2(M) = 5$ *for each $m \in M$. Then each infinitesimal second order deformation Φ of M is trivial.*

Proof. From (1), we obtain

$$v_1m = v_1\,, \quad v_2m = v_2\,, \tag{18}$$

$$\begin{aligned} v_1v_1m &= (\cdot)\,v_2 + a_1v_3 + b_1v_4 + c_1v_5\,, \\ v_2v_1m &= (\cdot)\,v_2 + a_2v_3 + b_2v_4 + c_2v_5\,, \\ v_2v_2m &= (\cdot)\,v_1 + a_3v_3 + b_3v_4 + c_3v_5\,. \end{aligned} \tag{19}$$

The vectors (19) are linearly independent because of the condition $\dim T_m^2(M) = 5$, and we may choose the frames in such a way that

$$b_1 = c_1 = c_2 = 0\,, \quad a_1b_2c_3 \neq 0\,. \tag{20}$$

The system (17) is then reduced to

$$\begin{gathered} b_2x_1 = 0\,, \quad b_3x_1 - b_2x_2 + c_3y_1 = 0\,, \quad a_2x_1 - a_1x_2 = 0\,, \\ a_3x_1 - a_2x_2 - c_3z_1 = 0\,, \\ a_2y_1 - a_1y_2 + b_2z_1 = 0\,, \quad a_3y_1 - a_2y_2 + b_3z_1 - b_2z_2 = 0\,, \end{gathered} \tag{21}$$

and we get $x_1 = x_2 = y_1 = y_2 = z_1 = z_2 = 0$, i.e.,

$$\varphi_3^4 = \varphi_3^5 = \varphi_4^5 = 0\,. \tag{22}$$

Thus Φ is trivial. QED.

3. Now, let us prove a global result.

Theorem 2. *Let* $G \subset \mathscr{R}^2$ *be a bounded domain and let* $M : G \cup \partial G \to E^5$ *be a surface such that:* (i) $\dim T^2_m(M) = 4$ *for each* $m \in M$; (ii) *there is* $K \neq k$ *or* $K \neq -k$ *on* M; (iii) *there are no real conjugate directions on* M. *Let* Φ *be an infinitesimal second order deformation of* M *which is trivial on the boundary of* M. *Then* Φ *is trivial on* M.

Proof. Choose moving frames $\{v_i\}$ of M and suppose $K \neq k$ on M; of course, we have (5). From (17) and (i), $x_1 = x_2 = 0$, i.e.,

$$\varphi_3^4 = 0 . \tag{23}$$

From (13), (14) and (17),

$$\omega_1^3 \wedge \varphi_3^5 + \omega_1^4 \wedge \varphi_4^5 = 0 , \quad \omega_2^3 \wedge \varphi_3^5 + \omega_2^4 \wedge \varphi_4^5 = 0 , \tag{24}$$

$$d\varphi_3^5 = \omega_3^4 \wedge \varphi_4^5 , \quad d\varphi_4^5 = -\omega_3^4 \wedge \varphi_3^5 , \tag{25}$$

$$a_2 y_1 - a_1 y_2 + b_2 z_1 - b_1 z_2 = 0 , \quad a_3 y_1 - a_2 y_2 + b_3 z_1 - b_2 z_2 = 0 . \tag{26}$$

From $(16_{2,3})$ and (25), we get the existence of functions $S_1, \ldots, S_6$ such that

$$\begin{aligned} dy_1 - y_2\omega_1^2 - z_1\omega_3^4 &= S_1\omega^1 + S_2\omega^2 , \\ dy_2 + y_1\omega_1^2 - z_2\omega_3^4 &= S_2\omega^1 + S_3\omega^2 , \\ dz_1 - z_2\omega_1^2 + y_1\omega_3^4 &= S_4\omega^1 + S_5\omega^2 , \\ dz_2 + z_1\omega_1^2 + y_2\omega_3^4 &= S_5\omega^1 + S_6\omega^2 . \end{aligned} \tag{27}$$

From $(4_{1-4}) + (5)$, we get the existence of functions $\alpha_1, \ldots, \beta_4$ such that

$$\begin{aligned} da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 &= \alpha_1\omega^1 + \alpha_2\omega^2 , \\ da_2 + (a_1 - a_3)\,\omega_1^2 - b_2\omega_3^4 &= \alpha_2\omega^1 + \alpha_3\omega^2 , \\ da_3 + 2a_2\omega_1^2 - b_3\omega_3^4 &= \alpha_3\omega^1 + \alpha_4\omega^2 , \\ db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 &= \beta_1\omega^1 + \beta_2\omega^2 , \\ db_2 + (b_1 - b_3)\,\omega_1^2 + a_2\omega_3^4 &= \beta_2\omega^1 + \beta_3\omega^2 , \\ db_3 + 2b_2\omega_1^2 + a_3\omega_3^4 &= \beta_3\omega^1 = \beta_4\omega^2 . \end{aligned} \tag{28}$$

The differential consequences of (26) are then

$$\begin{aligned} a_2S_1 - a_1S_2 + b_2S_4 - b_1S_5 &= -\alpha_2y_1 + \alpha_1y_2 - \beta_2z_1 + \beta_1z_2 , \\ a_2S_2 - a_1S_3 + b_2S_5 - b_1S_6 &= -\alpha_3y_1 + \alpha_2y_2 - \beta_3z_1 + \beta_2z_2 , \\ a_3S_1 - a_2S_2 + b_3S_4 - b_2S_5 &= -\alpha_3y_1 + \alpha_2y_2 - \beta_3z_1 + \beta_2z_2 , \\ a_3S_2 - a_2S_3 + b_3S_5 - b_2S_6 &= -\alpha_4y_1 + \alpha_3y_2 - \beta_4z_1 + \beta_3z_2 . \end{aligned} \tag{29}$$

Consider the system

$$(30)\qquad a_1X + b_1Y - a_2Z - b_2T = Q_1\,,\quad a_2X + b_2Y - a_3Z - b_3T = Q_2$$

for X, Y, Z, T. It is easy to show that the convenable combinations of the equations (30) lead to

$$(31)\qquad (K - k)X = (a_3b_2 - a_2b_3 + b_1b_3 - b_2^2)(X + T) + \\ + (a_2b_2 - a_3b_1)(Y - Z) + (a_3 - b_2)\,Q_1 + (b_1 - a_2)\,Q_2\,,$$

$$(K - k)\,Y = (a_2b_2 - a_1b_3)(X + T) + (a_1a_3 - a_2^2 + a_3b_2 - a_2b_3)(Y - Z) + \\ + (a_2 + b_3)\,Q_1 - (a_1 + b_2)\,Q_2\,,$$

$$(K - k)\,Z = (a_2b_2 - a_1b_3)(X + T) + (a_1b_2 - a_2b_1 - b_1b_3 + b_2^2)(Y - Z) + \\ + (a_2 + b_3)\,Q_1 - (a_1 + b_2)\,Q_2\,,$$

$$(K - k)\,T = (a_1a_3 - a_2^2 - a_1b_2 + a_2b_1)(X + T) + (a_3b_1 - a_2b_2)(Y - Z) + \\ + (b_2 - a_3)\,Q_1 + (a_2 - b_1)\,Q_2\,.$$

Applying this auxiliary result to the systems (26), $(29_{1,3})$ and $(29_{2,4})$ resp., we get

$$(32)\qquad (K - k)\,y_2 = (a_3b_2 - a_2b_3 + b_1b_3 - b_2^2)(y_2 + z_1) + \\ + (a_2b_2 - a_3b_1)(z_2 - y_1)\,,$$

$$(K - k)\,z_2 = (a_2b_2 - a_1b_3)(y_2 + z_1) + \\ + (a_1a_3 - a_2^2 + a_3b_2 - a_2b_3)(z_2 - y_1)\,,$$

$$(K - k)\,y_1 = (a_2b_2 - a_1b_3)(y_2 + z_1) + \\ + (a_1b_2 - a_2b_1 - b_1b_3 + b_2^2)(z_2 - y_1)\,,$$

$$(K - k)\,z_1 = (a_1a_3 - a_2^2 - a_1b_2 + a_2b_1)(y_2 + z_1) + \\ + (a_3b_1 - a_2b_2)(z_2 - y_1)\,;$$

$$(33)\qquad (K - k)\,S_2 = (a_3b_2 - a_2b_3 + b_1b_3 - b_2^2)(S_2 + S_4) + (a_2b_2 - a_3b_1)\,. \\ .\,(S_5 - S_1) + (a_3 - b_2)\,q_1 + (b_1 - a_2)\,q_2\,,$$

$$(K - k)\,S_5 = (a_2b_2 - a_1b_3)(S_2 + S_4) + (a_1a_3 - a_2^2 + a_3b_2 - a_2b_3)\,. \\ .\,(S_5 - S_1) + (a_2 + b_3)\,q_1 - (a_1 + b_2)\,q_2\,,$$

$$(K - k)\,S_1 = (a_2b_2 - a_1b_3)(S_2 + S_4) + (a_1b_2 - a_2b_1 - b_1b_3 + b_2^2)\,. \\ .\,(S_5 - S_1) + (a_2 + b_3)\,q_1 - (a_1 + b_2)\,q_2\,,$$

$$(K - k)\,S_4 = (a_1a_3 - a_2^2 - a_1b_2 + a_2b_1)(S_2 + S_4) + (a_3b_1 - a_2b_2)\,. \\ .\,(S_5 - S_1) + (b_2 - a_3)\,q_1 + (a_2 - b_1)\,q_2$$

with

$$q_1 = \alpha_2 y_1 - \alpha_1 y_2 + \beta_2 z_1 - \beta_1 z_2\,, \quad q_2 = \alpha_3 y_1 - \alpha_2 y_2 + \beta_3 z_1 - \beta_2 z_2\,;$$

(34) $$(K - k)\, S_3 = (a_3 b_2 - a_2 b_3 + b_1 b_3 - b_2^2)(S_3 + S_5) + (a_2 b_2 - a_3 b_1)\,. \\ .\,(S_6 - S_2) + (a_3 - b_2)\, q_1' + (b_1 - a_2)\, q_2'\,,$$

$$(K - k)\, S_6 = (a_2 b_2 - a_1 b_3)(S_3 + S_5) + (a_1 a_3 - a_2^2 + a_3 b_2 - a_2 b_3)\,. \\ .\,(S_6 - S_2) + (a_2 + b_3)\, q_1' - (a_1 + b_2)\, q_2'\,,$$

$$(K - k)\, S_2 = (a_2 b_2 - a_1 b_3)(S_3 + S_5) + (a_1 b_2 - a_2 b_1 - b_1 b_3 + b_2^2)\,. \\ .\,(S_6 - S_2) + (a_2 + b_3)\, q_1' - (a_1 + b_2)\, q_2'\,,$$

$$(K - k)\, S_5 = (a_1 a_3 - a_2^2 - a_1 b_2 + a_2 b_1)(S_3 + S_5) + (a_3 b_1 - a_2 b_2)\,. \\ .\,(S_6 - S_2) + (b_2 - a_3)\, q_1' + (a_2 - b_1)\, q_2'$$

with

$$q_1' = \alpha_3 y_1 - \alpha_2 y_2 + \beta_3 z_1 - \beta_2 z_2\,, \quad q_2' = \alpha_4 y_1 - \alpha_3 y_2 + \beta_4 z_1 - \beta_3 z_2\,.$$

It follows from (32) and $K \neq k$ that any linear combination of y_1, y_2, z_1, z_2 may be written as a combination of $y_2 + z_1$ and $y_1 - z_2$.

From (27), we get

(35) $$\mathrm{d}(y_1 - z_2) - (y_2 + z_1)(\omega_1^2 + \omega_3^4) = (S_1 - S_5)\,\omega^1 + (S_2 - S_6)\,\omega^2\,, \\ \mathrm{d}(y_2 + z_1) + (y_1 - z_2)(\omega_1^2 + \omega_3^4) = (S_2 + S_4)\,\omega^1 + (S_3 + S_5)\,\omega^2\,.$$

Let G be covered by a system of isothermic coordinates (u, v); let

(36) $$I = r^2(\mathrm{d}u^2 + \mathrm{d}v^2)\,, \quad r(u, v) > 0\,, \quad \text{i.e.}\,, \quad \omega^1 = r\,\mathrm{d}u\,, \quad \omega^2 = r\,\mathrm{d}v\,.$$

Then

(37) $$\frac{\partial(y_1 - z_2)}{\partial u} = (S_1 - S_5)\, r + \varrho_1(y_2 + z_1)\,, \\ \frac{\partial(y_1 - z_2)}{\partial v} = (S_2 - S_6)\, r + \varrho_2(y_2 + z_1)\,, \\ \frac{\partial(y_2 + z_1)}{\partial u} = (S_2 + S_4)\, r + \varrho_3(y_1 - z_2)\,, \\ \frac{\partial(y_2 + z_1)}{\partial v} = (S_3 + S_5)\, r + \varrho_4(y_1 - z_2)\,,$$

$\varrho_1, \ldots, \varrho_4$ being easy to calculate. From (37) and $(33_1) + (34_3)$ or $(33_2) + (34_4)$ resp., we get a system of the form $(i = 1, 2)$

$$(38) \qquad a_{i1} \frac{\partial(y_1 - z_2)}{\partial u} + a_{i2} \frac{\partial(y_1 - z_2)}{\partial v} + b_{i1} \frac{\partial(y_2 + z_1)}{\partial u} + b_{i2} \frac{\partial(y_2 + z_1)}{\partial v} = c_{i1}(y_1 - z_2) + c_{i2}(y_2 + z_1)$$

with

$$(39) \qquad \begin{aligned} a_{11} &= a_2 b_2 - a_3 b_1 , & a_{12} &= b_1 b_3 - b_2^2 + a_2 b_1 - a_1 b_2 , \\ a_{21} &= a_1 a_3 - a_2^2 + a_3 b_2 - a_2 b_3 , & a_{22} &= a_2 b_2 - a_3 b_1 , \\ b_{11} &= b_2^2 - b_1 b_3 + a_2 b_3 - a_3 b_2 , & b_{12} &= a_2 b_2 - a_1 b_3 , \\ b_{21} &= a_1 b_3 - a_2 b_2 . & b_{22} &= a_1 a_3 - a_2^2 + a_2 b_1 - a_1 b_2 . \end{aligned}$$

Recall that the system (38) is called elliptic if the quadratic form

$$(40) \qquad \varphi = (a_{12} b_{22} - a_{22} b_{12}) \mu^2 - (a_{11} b_{22} - a_{21} b_{12} + a_{12} b_{21} - a_{22} b_{11}) \mu \nu + (a_{11} b_{21} - a_{21} b_{11}) \nu^2$$

is definite. In our case,

$$(41) \qquad \varphi = (k - K) \{(a_1 b_2 - a_2 b_1) \mu^2 + (a_1 b_3 - a_3 b_1) \mu \nu + (a_2 b_3 - a_3 b_2) \nu^2\} ,$$

and φ is definite because of the suppositions of our Theorem. Thus $y_1 - z_2 = y_2 + z_1 = 0$ in G and (32) implies $y_1 = y_2 = z_1 = z_2 = 0$ in G.

In the case $K \neq -k$, we have to use $y_1 + z_2$, $y_2 - z_1$ instead of $y_1 - z_2$, $y_2 + z_1$ resp. QED.

Let us remark that locally Φ does not need to be trivial.

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Author's address: 771 46 Olomouc, Leninova 26, ČSSR (Přírodovědecká fakulta University Palackého).