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COMPLETENESS IN SEMI-LOCAL IDEAL LATTICES

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1. INTRODUCTION

In this paper we investigate the nature of semi-local rings whose lattice of ideals is topologically complete in its semi-local topology. In particular (see §§ 2–3 for definitions), if $(R, M_1, M_2, \ldots, M_n)$ is a semi-local ring, $M = \bigcap_{i=1}^{n} M_i$, and R^* is the M-adic ring completion of R, we show (Theorem 4.6) that the following conditions are equivalent:

(1.1) The ideal lattice $L(R)$ of R is complete in the M-adic metric.
(1.2) If $\langle A_i \rangle$ is a decreasing sequence of elements in $L(R)$, then $A_i \to \bigcap_{i=1}^{\infty} A_i$ as $i \to \infty$ in the M-adic metric.
(1.3) R is quasi-M-complete.
(1.4) $L(R)$ and $L(R^*)$ are isomorphic is multiplicative lattices.

Consequently, from an ideal-theoretic viewpoint, if the lattice of ideals of a semi-local ring is topologically complete, one can assume the ring is topologically complete. We also give an example (Example 4.11) of a semi-local ring which is not topologically complete but whose ideal lattice is topologically complete.

2. NOTATION AND DEFINITIONS

By *a multiplicative lattice* we will mean a complete lattice on which there is defined a commutative, associative, totally join distributive multiplication for which the unit element of the lattice is an identity for multiplication (written juxtaposition). Let L be a multiplicative lattice. An element P of L is said to be *meet principal* if

$$AP \wedge B = A \wedge (B : P),$$

for all A and B in L; P is said to be *join principal* if

$$(A \vee BP) : P = B \vee (A : P),$$

for all A and B in L; and P is said to be *principal* if P is both meet and join principal. L will be called *principally generated* if each element of L is a join (finite or infinite) of principal elements of L. L is called *a Noether lattice* in case L is modular, principally generated, and satisfies the ascending chain condition on elements. In general we shall adopt the Noether lattice terminology given in [3] and [4].

All rings considered in this paper are assumed to be commutative rings with an identity element. If R is a ring, we shall denote the multiplicative lattice of ideals of R by $L(R)$ and in general we will adopt the ring notation of [7]. In particular, if A is an ideal of R such that $\bigcap_{i=1}^{\infty} A^i = 0$ (i.e. R is Hausdorff in the A-adic topology), we will denote the A-adic ring completion of R by R^*, and, for an ideal B of R, the ideal generated by B in R^* will be denoted by BR^*. If R is a Noetherian ring, then it is well known that $L(R)$ is a Noether lattice (cf. [3], p. 486) (it is also known that not every Noether lattice is the lattice of ideals of some Noetherian ring (cf. [1], p. 169, and [2], p. 131). By a semilocal ring we shall mean a Noetherian ring with finitely many maximal ideals. If R is a semi-local ring with maximal ideals $M_1, M_2, \ldots, M_n$, we will indicate this by saying $(R, M_1, M_2, \ldots, M_n)$ is a semi-local ring. Finally, we let $I(R)$ denote the set of ideals of R.

Let L be a Noether lattice and let A be an element of L. For every element B and C in L define

$$\delta_A(B, C) = \sup \{n \mid B \vee A^n = C \vee A^n\}$$

and

$$d_A(B, C) = 2^{-\delta_A(B,C)} .$$

The distance function d_A is a pseudometric and is called *the A-adic pseudometric* on L (see § 3, p. 350, of [4] for more details). If

$$\bigwedge_n (C \vee A^n) = C , \quad \text{for all } C \text{ in } L, \tag{2.0}$$

then it is known that d_A is a metric (cf. [4], p. 352, Theorem 3.10) and the A-adic completion of L can be developed (see § 6 of [4], p. 360, for details of this construction). If $(R, M_1, M_2, \ldots, M_n)$ is a semi-local ring and we set $M = \bigcap_{i=1}^{n} M_i$, then it follows from (2.0) and ([7], (16.7) Theorem, p. 52) that d_M is a metric on the Noether lattice $L(R)$. The M-adic completion of $L(R)$ will be denoted by $L(R)^*$.

3. PRELIMINARY RESULTS

The results in this section are preliminary in nature and will be required in the sequel. We begin with the following definition which we will refer to several times.

Definition 3.1. If A is an ideal of a ring R, then we shall say that R is *quasi-A-complete*, if, whenever given a decreasing sequence $\langle A_i \rangle$, $i = 1, 2, \ldots$ of ideals of R and a natural number n, then there exists a natural number $s(n)$ such that

$$(3.0) \qquad A_i \subseteq \left(\bigcap_{i=1}^{\infty} A_i\right) + A^n,$$

for all integers $i \geqq s(n)$.

This concept will play a central role for our investigations in the next section of this paper. Our first theorem relates the quasi-M-completness of a semi-local ring $(R, M_1, \ldots, M_n)$ to the M-adic metric completness of the Noether lattice $L(R)$.

Theorem 3.2. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring and let $M = \bigcap_{i=1}^{\infty} M_i$. Then the following three statements are equivalent*

(3.1) *The Noether lattice $L(R)$ is complete in the M-adic metric.*
(3.2) *The ring R is quasi-M-complete.*
(3.3) *If $\langle A_i \rangle$ is a decreasing sequence of elements of $L(R)$, then $A_i \to \bigcap_{i=1}^{n} A_i$ as $i \to \infty$ in the M-adic metric.*

Proof. We shall give a cyclic proof and begin by showing that (3.1) implies (3.2). Thus, assume that the Noether lattice $L(R)$ is complete in the M-adic metric, let $\langle B_i \rangle$, $i = 1, 2, \ldots$ be a decreasing sequence of elements of $L(R)$, and let n be a positive integer. Since the sequence $\langle B_i \rangle$ is decreasing it is easily seen that $\langle B_i \rangle$ is a Cauchy sequence in $L(R)$ (with the M-adic metric). Consequently there exists an element B of $L(R)$ such that

$$(3.4) \qquad B_i \to B \quad \text{as} \quad i \to +\infty \quad (\text{in the } M\text{-adic metric}).$$

Hence, by Remark 3.6 of [4], there exists a positive integer N such that

$$(3.5) \qquad B_i + M^n = B + M^n,$$

for all integers $i \geqq N$. So, if it were the case that

$$(3.6) \qquad B \subseteq \bigcap_{j=1}^{\infty} B_j,$$

then it would follow from (3.5) that

$$B_i \subseteq B_i + M^n = B + M^n \subseteq \left(\bigcap_{j=1}^{\infty} B_j\right) + M^n,$$

for all integers $i \geqq N$, and consequently R would be quasi-M-complete by (3.0). Hence, in order to show that R is quasi-M-complete it is sufficient to establish (3.6) which we proceed to do as follows. Fix a positive integer j, and let m be an arbitrary nonnegative integer. As a consequence of (3.4), there exists a positive integer $s(m)$ such that

$$B \subseteq B + M^m = B_{s(m)} + M^m \subseteq B_j + M^m$$

and so we obtain

$$B \subseteq B_j + M^m, \tag{3.7}$$

for each integer $m \geqq 0$. From (3.7) and the Krull Intersection Theorem it follows that

$$B \subseteq B_j, \quad j = 1, 2, \ldots$$

which establishes (3.6). Thus (3.1) implies (3.2).

Assume now that the ring R is quasi-M-complete, let $\langle A_i \rangle$ be a decreasing sequence of elements of $L(R)$, and let ε be a positive real number. Choose n to be the least positive integer k such that $2^{-k} < \varepsilon$. Since R is quasi-M-complete, there exists a positive integer $s(n)$ such that

$$A_i \subseteq \left(\bigcap_{i=1}^{\infty} A_i\right) + M^n \tag{3.8}$$

for all integers $i \geqq s(n)$. (3.8) yields

$$A_i + M^n = \left(\bigcap_{i=1}^{\infty} A_i\right) + M^n,$$

for all integers $i \geqq s(n)$, and thus

$$A_i \to \bigcap_{i=1}^{\infty} A_i \quad \text{as} \quad i \to \infty \quad (\text{in the } M\text{-adic metric})$$

by Remark 3.6 of [4]. Hence (3.2) implies (3.3).

It remains to be shown that (3.3) implies (3.1). So assume (3.3) and let $\langle B_i \rangle$, $i = 1, 2, \ldots$, be a Cauchy sequence of elements of $L(R)$ (relative to the M-adic metric). Let $\langle D_i \rangle$, $i = 1, 2, \ldots$, be a regular subsequence of $\langle B_i \rangle$ (see [4], Lemma 4.11, p. 355) and define A_i by

$$A_i = D_i + M^i, \quad \text{for} \quad i = 1, 2, \ldots .$$

Then $\langle A_i \rangle$ is a decreasing sequence of elements of $L(R)$ and thus we have

$$A_i \to \bigcap_{i=1}^{\infty} A_i \quad \text{as} \quad i \to \infty \tag{3.9}$$

in the M-adic metric. Since the Cauchy sequences $\langle B_i \rangle$, $\langle D_i \rangle$, and $\langle A_i \rangle$ will all have the same limit if any one of them exists (see Lemmas 4.11 and 4.12 of [4], p. 355) it follows from (3.9) that

$$B_i \to \bigcap_{i=1}^{\infty} A_i \quad \text{as} \quad i \to +\infty$$

in the M-adic metric. Thus the Noether lattice $L(R)$ is complete in the M-adic metric which establishes (3.1) and completes the proof of the theorem.

We will also need the following results in the next section.

Theorem 3.3. *Let* $(R, M_1, \ldots, M_n)$ *be a semi-local ring, let* $M = \bigcap_{i=1}^{n} M_i$, *and let* R^* *be the* M*-adic ring completion of* R. *If* X *is an element of* $I(R^*)$ *such that* $M^{*n} \subseteq X$, *for some natural number* n, *then there exists an element* B *of* $I(R)$ *such that*

$$M^n \subseteq B \,, \tag{3.10}$$

and

$$BR^* = X \,. \tag{3.11}$$

Proof. Let X be an element of $I(R^*)$ such that $M^{*n} \subseteq X$, for some n. Then there exist $a_1, \ldots, a_k$ in R^* such that

$$X = \sum_{i=1}^{k} a_i R^* \,. \tag{3.12}$$

Since R is dense in R^*, there exist elements $b_1, \ldots, b_k$ in R such that

$$a_i - b_i \in M^{*n+1} \subseteq M^{*n} \subseteq X \,, \tag{3.13}$$

for $i = 1, 2, \ldots, k$. Now set $B = \sum_{i=1}^{k} b_i R^*$ and observe that

$$BR^* \subseteq X \tag{3.14}$$

by construction. Suppose next that a is an element of X. From (3.12) there exist elements $r_1, \ldots, r_m$ in R^* such that $a = \sum_{i=1}^{m} r_i a_i$.

Hence

$$a = \sum_{i=1}^{m} r_i b_i + \sum_{i=1}^{m} r_i (a_i - b_i) \,,$$

from which it follows that

$$a \in BR^* + M^{*n+1} \subseteq BR^* + M^* X$$

by (3.13), and so

$$X \subseteq BR^* + M^{*n+1} \subseteq BR^* + M^* X \tag{3.15}$$

since a was an arbitrary element of X. From (3.14) and (3.15) we obtain

$$X = BR^* + M^* X$$

which gives us (3.11) by the Krull-Azumaya Lemma ([7], (4.1) Theorem, p. 12).

Since $M^{*n} \subseteq X$ by hypotheses, we have

$$M^n = M^{*n} \cap R \subseteq X \cap R = BR^* \cap R = B$$

by ([7], (17.9) Corollary, p. 57) which establishes (3.10) and completes the proof of the theorem.

For an ideal B of a ring R, we introduce the symbol $P(R, B)$ which we define as follows:

$$P(R, B) = \{A \in I(R) \mid B^n \subseteq A, \text{ for some natural number } n\}$$

The following two results will be needed in the next section.

Corollary 3.4. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring, let $M = \bigcap_{i=1}^{n} M_i$, and let R^* be the M-adic ring completion of R. Then the map $\varphi : P(R, M) \rightarrow$ $\rightarrow P(R^*, M^*)$ defined by $\varphi(A) = AR^*$ is bijective.*

Proof. It follows from (17.9) Corollary, p. 57, of [7] that φ is injective and from Theorem 3.3 we have that φ is surjective.

Corollary 3.5. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring, let $M = \bigcap_{i=1}^{n} M_i$, and let R^* be the M-adic ring completion of R. If A is an element of $I(R)$ such that $M^{*n} \subseteq A$, for some natural number n, then $(A \cap R) R^* = A$.*

Proof. Follows from Corollary 3.4.

4. MAIN THEOREM

Let $(R, M_1, \ldots, M_n)$ be a semi-local ring, let $M \doteq \bigcap_{i=1}^{n} M_i$, and let R^* be the M-adic ring completion of R. In this section we establish several equivalent conditions in order that $L(R)$ and $L(R^*)$ be isomorphic as multiplicative lattices (Theorem 4.6) and thus, when any one of these conditions is satisfied, it follows that the ideal structure of R and R^* are the same. Consequently, in this case (speaking from an ideal theoretical viewpoint) nothing is gained by passing to the completion of R.

Theorem 4.1. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring and let $M = \bigcap_{i=1}^{n} M_i$. If R is complete in the M-adic topology, then R is quasi-M-complete.*

Proof. $L(R) \approx L(R^*) \approx L(R)^*$ ([6], Theorem 5) and $L(R)^*$ is a complete semi-local Noether lattice with its semi-local metric ([5], Theorem 2.9, p. 95, and Corollary 3.6, p. 98). It follows that R is quasi-M-complete by Theorem 3.2.

Theorem 4.2. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring, let $M = \bigcap_{i=1}^{n} M_i$, and let R^* be the M-adic ring completion of R. Then*

(4.1) *R^* is a semi-local ring with maximal elements $M_1R^*, M_2R^*, \ldots, M_nR^*$,*

(4.2) $\bigcap_{i=1}^{n} (M_iR^*) = (\bigcap_{i=1}^{n} M_i) R^* = MR^*$,

(4.3) *R^* is quasi-MR^*-complete.*

Proof. (4.1) and (4.2) can be obtained from (18.1) Theorem, p. 58, of [7] and (4.3) follows from (17.6) Corollary, p. 55, of [7] together with Theorem 4.1 above.

Lemma 4.3. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring, let $M = \bigcap_{i=1}^{n} M_i$, and let $\langle A_i \rangle$, $i = 1, 2, \ldots$, be a decreasing sequence of elements of $L(R)$. If R is quasi-M-complete, then*

$$(4.4) \qquad \Big(\bigcap_{i=1}^{\infty} A_i\Big) R^* = \bigcap_{i=1}^{\infty} (A_i R^*) .$$

Proof. Assume R is quasi-M-complete. From Theorem 3.2 we obtain

$$A_i \rightarrow \bigcap_{i=1}^{\infty} A_i \quad \text{as } i \rightarrow \infty$$

in the M-adic metric on $L(R)$. Thus, given a natural number n, there exists a natural number $s(n)$ such that

$$A_i + M^n = \Big(\bigcap_{i=1}^{\infty} A_i\Big) + M^n$$

for all integers $i \geqq s(n)$, and thus

$$A_i R^* + (MR^*)^n = \Big(\bigcap_{i=1}^{\infty} A_i\Big) R^* + (MR^*)^n$$

for all integers $i \geqq s(n)$, which implies that

$$A_i R^* \rightarrow \Big(\bigcap_{i=1}^{\infty} A_i\Big) R^* \quad \text{as} \quad i \rightarrow \infty$$

in the (MR^*)-adic metric on $L(R)$. But

$$A_i R^* \rightarrow \bigcap_{i=1}^{\infty} (A_i R^*) \quad \text{as} \quad i \rightarrow \infty$$

in the MR^*-adic metric by Theorem 4.2 and Theorem 3.2, and thus

$$\Big(\bigcap_{i=1} A_i\Big) R^* = \bigcap_{i=1} (A_i R^*)$$

by the uniqueness of limits which is the desired result.

Lemma 4.4. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring and let $M = \bigcap_{i=1}^{n} M_i$. If R is quasi-M-complete, then the map $\varphi : I(R) \rightarrow I(R^*)$, defined by $\varphi(A) = AR^*$, for each A in $L(R)$, is a bijection.*

Proof. Let A be an element of $I(R^*)$. For each natural number n, set

$$A_n = (A + M^{*n}) \cap R$$

Then, for each n,

$$A_n R^* = ((A + M^{*n}) \cap R) R^* = A + M^{*n}$$

by Corollary 3.5, and thus

$$(\bigcap_{i=1}^{\infty} A_i) R^* = \bigcap_{i=1}^{n} (A_i R^*) = \bigcap_{i=1}^{\infty} (A + M^{*i}) = A$$

by Lemma 4.3 and (16.7) Theorem of [7] which shows φ is surjective. As indicated previously, φ is injective (cf. [7], (17.9) Corollary, p. 57) which completes the proof.

Theorem 4.5. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring, let $M = \bigcap_{i=1}^{n} M_i$, and let R^* be the M-adic ring completion of R. In order for $L(R)$ and $L(R^*)$ to be isomorphic as multiplicative lattices it is necessary and sufficient that R be quasi-M-complete.*

Proof. Assume R is quasi-M-complete and define $\varphi : L(R) \to L(R^*)$ by $\varphi(A) = AR^*$, for each A in $L(R)$. By Lemma 4.4, φ is bijective. It follows from (18.1) Theorem, p. 58, of [7] and other well known properties of ideals (cf. [8], p. 219) that φ is a multiplicative lattice morphism. Thus $L(R)$ and $L(R^*)$ are isomorphic as multiplicative lattices which proves that quasi-M-completeness is a sufficient condition.

Conversely, assume $\varphi : L(R^*) \to L(R)$ is a multiplicative lattice isomorphism. Since $L(R^*)$ is quasi-MR^*-complete (Theorem 4.2) it follows from Theorem 3.2 that the Noether lattice $L(R^*)$ is complete in the MR^*-adic metric. Since φ is a lattice isomorphism it must be the case that

$$\varphi(MR^*) = \varphi((\bigcap_{i=1}^{n} M_i) R^*) = \bigcap_{i=1}^{n} ((M_i R^*)) = M .$$

So, for each A and B in $L(R^*)$, and for each nonnegative integer n, we obtain

$$A + (MR^*)^n = B + (MR^*)^n$$

if and only if

$$\varphi(A) + M^n = \varphi(B) + M^n ,$$

which shows that φ is an isometry. Thus $L(R)$ is complete in the M-adic metric and consequently R is a quasi-M-complete which shows the necessity of quasi-M-completeness.

By combining Theorems 3.2 and 4.5 we obtain our main theorem.

Theorem 4.6. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring, let $M = \bigcap_{i=1}^{n} M_i$, and let R^* be the M-adic ring completion of R. Then the following four statements are equivalent.*

(i) *The Noether lattice $L(R)$ is complete in the M-adic metric.*
(ii) *If $\langle A_i \rangle$ is a decreasing sequence of elements of $L(R)$, then $A_i \to \bigcap_{i=1}^{\infty} A_i$ as $i \to \infty$ in the M-adic metric.*
(iii) *The ring R is quasi-M-complete.*
(iv) *$L(R)$ and $L(R^*)$ are isomorphic as multiplicative lattices.*

Corollary 4.7. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local ring, let $M = \bigcap_{i=1}^{n} M_i$, and let $L(R)$ be complete in the M-adic metric. Then R is analytically irreducible if and only if the zero element of $L(R)$ is prime.*

Proof. Follows from part (iv) of Theorem 4.6.

Corollary 4.8. *Let $(R, M_1, M_2, \ldots, M_n)$ be a semi-local domain and let $M = \bigcap_{i=1}^{n} M_i$. If R is quasi-M-complete, then R is analytically irreducible.*

Proof. Follows from Corollary 4.7.

Corollary 4.9. *Let $(R, M_1, \ldots, M_n)$ be a semi-local ring, let $M = \bigcap_{i=1}^{n} M_i$, and let $L(R)$ be complete in the M-adic metric. If zero is the only nilpotent meet-principal element of $L(R)$, then R is analytically unramified.*

Proof. Follows from (iv) of Theorem 4.6 and the fact that principal ideals of R^* are meet-principal elements in $L(R^*)$ ([3], p. 481).

Corollary 4.10. *Let $(R, M_1, \ldots, M_n)$ be a semi-local domain, let $M = \bigcap_{i=1}^{n} M_i$, and let $L(R)$ be complete in the M-adic metric. If zero is the only nilpotent meet-principal element of $L(R)$, then the derived normal ring of R is a finitely generated R-module.*

Proof. Follows from Corollary 4.9 and ([7], (32.2) Theorem, p. 114).

We now give an example of a semi-local ring (even local) which is quasi-M-complete but which is not complete in its ring topology.

Example 4.11. Let the rational integers be denoted by Z and let p be a prime element of Z. It is well-known that the ring $Z_{(p)}$ is not complete in its $pZ_{(p)}$ – adic ring topology and it is easily verified that $Z_{(p)}$ is quasi-$pZ_{(p)}$ – complete. We omit the details.

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