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PERIODIC SOLUTIONS TO CERTAIN EVOLUTION INEQUALITIES

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INTRODUCTION

Let X be a real reflexive Banach space with norm $\| \ \|$. We denote by X^* the dual of X and by (v^*, v) the dual pairing between $v^* \in X^*$ and $v \in X$.

Suppose we are given a real Hilbert space H with norm $| \ |$ such that X is continuously and densely imbedded into H. Identifying H with its dual we get the continuous and dense imbedding $H \subset X^*$, and if $w \in H$ and $v \in X$, the dual pairing (w, v) coincides with the scalar product of w and v in H.

Further, let $\varphi : X \to (-\infty, +\infty]$ be a convex, lower semi-continuous functional, $\varphi \not\equiv +\infty$. Let $D(\varphi)$ denote its effective domain, i.e.

$$D(\varphi) = \{v \in X \colon \varphi(v) < +\infty\} .$$

Let $A : X \to X^*$ be a (possibly nonlinear) mapping. We then ask for a function $u \in L^p(0, T; X)$ $(0 < T < \infty)$ such that

$$(1) \qquad u' + Au + \partial\varphi(u) \ni f \quad \textit{for a.a.} \quad t \in [0, T] , \quad u(0) = u(T)$$

where the derivative $u' = \mathrm{d}u/\mathrm{d}t$ is to be understood in the sense of vector-valued distributions, f being a given function. In particular, let X be a real Hilbert space, and let A and B be two linear bounded mappings from X into X^*. Under these assumptions we consider the problem of finding a function $u \in L^2(0, T; X)$ such that

$$(2) \qquad u'' + Au' + Bu + \partial\varphi(u') \ni f \quad \textit{for a.a.} \quad t \in [0, T] ,$$
$$u(0) = u(T) , \quad u'(0) = u'(T) .$$

Both in (1) and (2) $\partial\varphi$ denotes the subdifferential mapping of φ (see e.g. [1], [4]).

The existence of a solution to the problem

$$(1') \qquad u' + Mu + \omega u \ni f \quad \textit{for a.a.} \quad t \in [0, T] , \quad u(0) = u(T)$$

where M is an m-accretive operator in a Banach space with uniformly convex dual, and $\omega = \text{const} > 0$ has been proved in [1], [3]. The existence of a solution to (1′) for $\omega = 0$ has been established in [3] for M to be the subdifferential mapping of a convex, lower semi-continuous and coercive functional on a Hilbert space. For general maximal monotone and coercive mappings M in a Hilbert space the existence of a weak solution to (1′) for $\omega = 0$ has been proved in [4]. In [5], the authors have studied (1′) for t-dependent M. An existence theorem for weak solutions to the problem (1) for a wide range of nonlinear mappings A can be found in [3].

Some results on the existence of a solution to special cases of (2) have been presented in [6].

In Section 1 of the present paper we prove the existence of a solution to (1) for A to be the sum of a monotone gradient operator and a certain "lower order" operator. Our method of proof consists in starting with a weak solution to (1) and proving its regularity then.

The existence of a solution to (2) is proved in Section 2. Following [1], [3] we replace (2) by a first order problem to which the theory of [1]–[4] applies ($\omega > 0$). After establishing a-priori-estimates we are able to carry through the passage to limit $\omega \to 0$.

SECTION 1

For $v \in L^p(0, T; X)$ $(1 < p < \infty)$ we define

$$\Phi(u) = \begin{cases} \int_0^T \varphi(v(t))\, dt & \text{if} \quad \varphi(v(\cdot)) \in L^1(0, T), \\ +\infty & \text{otherwise}. \end{cases}$$

Φ is a convex, lower semi-continuous functional from $L^p(0, T; X)$ into $(-\infty, +\infty]$ (see [3], [4]). Let $D(\Phi)$ denote the effective domain of Φ.

Throughout this section we assume that $2 \leqq p < \infty$ and $\tilde{A}v \in L^{p'}(0, T; X^*)$ $(1/p + 1/p' = 1)$ for any $v \in L^p(0, T; X)$, where $(\tilde{A}v)(t) = A\, v(t)$ for a.a. $t \in [0, T]$.[1])

We impose the following additional conditions upon A:

(1.1) $A = A_1 + A_2$ *where*: $A_1 : X \to X^*$ *is monotone, there exists a functional* $F : X \to \mathbb{R}^1$ *such that* $A_1 = \operatorname{grad} F$, *and* $A_2 : X \to H$;

(1.2) $\tilde{A}$ *is pseudo-monotone*[2]) *and maps bounded sets into bounded sets*;

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[1]) Note that this condition can be verified when imposing certain continuity and boundedness conditions upon A.

[2]) Let X be a real Banach space with dual X^*, the dual pairing between X^* and X being denoted by $\langle\ ,\ \rangle$. A mapping $S : X \to X^*$ is called *pseudo-monotone* if for any sequence $\{u_j\} \subset X$ such that $u_j \to u$ weakly in X and $\limsup \langle Su_j, u_j - u\rangle \leqq 0$, it follows that $\langle Su, u - v\rangle \leqq \liminf \langle Su_j, u_j - v\rangle$ for all $v \in X$.

(1.3) *there exists* $v_0 \in D(\Phi)$ *with* $v_0' \in L^{p'}(0, T; X^*)$ *and* $v_0(0) = v_0(T)$ *such that*

$$\left[\int_0^T (Av, v - v_0)\,dt + \Phi(v)\right] \|v\|^{-1}_{L^p(0,T;X)} \to +\infty \quad \text{as} \quad v \in D(\Phi)\,, \quad \|v\|_{L^p(0,T;X)} \to \infty\,.$$

We then have

Theorem 1. *Let the conditions* (1.1)–(1.3) *be satisfied. Suppose that* $f = f_1 + f_2$ *where*

$$f_1 \in L^2(0, T; H)\,, \quad f_2, f_2' \in L^{p'}(0, T; X^*)\,.$$

Then there exists a solution $u \in D(\Phi)$ *to* (1) *such that*

$$u \in C([0, T]; H)\,, \quad u' \in L^2(0, T; H)\,.$$

Proof. Based on the conditions (1.2), (1.3) we obtain from [3] the existence of a function $u \in D(\Phi) \cap C([0, T]; H)$ such that

$$\text{(1.4)} \qquad \int_0^T (v' + Au, v - u)\,dt + \Phi(v) - \Phi(u) \geqq \int_0^T (f, v - u)\,dt$$

$$\forall v \in D(\Phi) \quad \text{with} \quad v' \in L^{p'}(0, T; X^*)\,, \quad v(0) = v(T)\,.$$

Moreover, it holds $u(0) = u(T)$.

Let $\varepsilon > 0$. We then consider the function

$$u_\varepsilon(t) = e^{-t/\varepsilon} z_\varepsilon + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\, u(s)\,ds\,, \quad t \in [0, T]$$

where

$$z_\varepsilon = \frac{1}{\varepsilon(1 - e^{-T/\varepsilon})}\int_0^T e^{(s-T)/\varepsilon}\, u(s)\,ds\,.$$

In other words, u_ε solves the problem

$$u_\varepsilon(t) + \varepsilon\, u_\varepsilon'(t) = u(t) \quad \text{for a.a.} \quad t \in [0, T]\,, \quad u_\varepsilon(0) = u_\varepsilon(T)\,.$$

The following properties of u_ε are readily verified (cf. [3]):

$$\text{(1.5)} \qquad \Phi(u_\varepsilon) \leqq \Phi(u) \quad \forall \varepsilon > 0\,;$$

(1.6) *there exists a sequence of reals* ε_j ($\varepsilon_j > 0$ *for* $j = 1, 2, \ldots$) *such that* $u_{\varepsilon_j} \to u$ *weakly in* $L^p(0, T; X)$ *as* $j \to \infty$.

We insert $v = u_\varepsilon$ in (1.4) and obtain

$$\text{(1.7)} \quad -\varepsilon\int_0^T |u_\varepsilon'|^2\,dt - \varepsilon\int_0^T (Au, u_\varepsilon')\,dt + \Phi(u_\varepsilon) - \Phi(u) \geqq -\varepsilon\int_0^T (f, u_\varepsilon')\,dt \quad \forall \varepsilon > 0\,.$$

By (1.1),

$$-\int_0^T (Au, u_\varepsilon')\,dt \leqq \left\{\int_0^T |A_2 u|^2\,dt\right\}^{1/2} \left\{\int_0^T |u_\varepsilon'|^2\,dt\right\}^{1/2}$$

for all $\varepsilon > 0$. Observing (1.5) one concludes from (1.7) that

$$\int_0^T |u_\varepsilon'|^2\,dt \leqq c_1 \left[1 + \int_0^T (f, u_\varepsilon')\,dt\right] \quad \forall \varepsilon > 0$$

where $c_1 = \text{const} > 0$. Taking into account that

$$|u_\varepsilon(t)| \leqq \|u\|_{C([0,T];H)} \quad \forall \varepsilon > 0\,, \quad \forall t \in [0, T]\,,$$

and that

$$\|u_\varepsilon\|_{L^p(0,T;X)} \leqq c_2(1 + \|u\|_{L^p(0,T;X)}) \quad \forall 0 < \varepsilon \leqq 1$$

where $c_2 = \text{const} > 0$, we find

$$\int_0^T (f_2, u_\varepsilon')\,dt \leqq \text{const} \quad \forall 0 < \varepsilon \leqq 1\,.$$

Thus

$$\int_0^T |u_\varepsilon'|^2\,dt \leqq \text{const} \quad \forall 0 < \varepsilon \leqq 1\,.$$

But the latter estimate together with (1.6) implies that u' exists and belongs to $L^2(0, T; H)$.

Let $\bar{v} \in D(\Phi)$ with $\bar{v}' \in L^{p'}(0, T; X^*)$, $\bar{v}(0) = \bar{v}(T)$ be given. Let $0 < \lambda < 1$. Replacing v in (1.4) by $(1 - \lambda)\,u + \lambda\bar{v}$, dividing by λ and letting $\lambda \to 0$ one obtains

$$\int_0^T (u' + Au,\ \bar{v} - u)\,dt + \Phi(\bar{v}) - \Phi(u) \geqq \int_0^T (f, \bar{v} - u)\,dt\,. \tag{1.8}$$

Let $v \in D(\Phi)$. Set, for any $\varepsilon > 0$,

$$v_\varepsilon(t) = e^{-t/\varepsilon} w_\varepsilon + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\, v(s)\,ds\,, \quad t \in [0, T]$$

where

$$w_\varepsilon = \frac{1}{\varepsilon(1 - e^{-T/\varepsilon})}\int_0^T e^{(s-T)/\varepsilon}\, v(s)\,ds\,.$$

Let $\{\varepsilon_j\}$ ($\varepsilon_j > 0$ for $j = 1, 2, \ldots$) be a sequence of reals such that $v_{\varepsilon_j} \to v$ weakly in $L^p(0, T; X)$ as $j \to \infty$. Inserting $v = v_{\varepsilon_j}$ in (1.8) and using that $\lim \Phi(v_{\varepsilon_j}) \geqq \Phi(v)$ we conclude from (1.8) after passing to limit that

$$\int_0^T (u' + Au,\ v - u)\,dt + \Phi(v) - \Phi(u) \geqq \int_0^T (f, v - u)\,dt\,.$$

Since this inequality is true for any $v \in D(\Phi)$ we get the first relation in (1).

Let $X = V$ be a real Hilbert space.

Let A and B be linear bounded mappings from V into V^* which satisfy the following conditions:

$$(2.1) \qquad (Av, v) \geqq \alpha_0 \|v\|^2 \quad \forall v \in V, \quad \alpha_0 = \text{const} > 0\,;$$

$$(2.2) \qquad (Bv, v) \geqq \beta_0 \|v\|^2 \quad \forall v \in V, \quad \beta_0 = \text{const} > 0\,,$$
$$(Bu, v) = (Bv, u) \quad \forall u, v \in V.$$

We further assume that

$$(2.3) \qquad \partial\Phi \textit{ maps bounded sets into bounded sets}\,.$$

The aim of this section is to prove

Theorem 2. *Let $f \in L^2(0, T; H)$ with $f' \in L^2(0, T; H)$ and $f(0) = f(T)$ be given. Then there exists a function $u \in C([0, T]; V)$ such that*

$$(2.4) \qquad u' \in D(\Phi)\,, \quad u'' \in L^2(0, T; V^*)\,,$$

$$(2.5) \qquad \int_0^T (u'' + Au' + Bu, v - u')\,\mathrm{d}t + \Phi(v) - \Phi(u') \geqq$$
$$\geqq \int_0^T (f, v - u')\,\mathrm{d}t \quad \forall v \in D(\Phi)\,,$$

$$(2.6) \qquad u(0) = u(T)\,, \quad u'(0) = u'(T)\,.$$

Proof. 1° Approximate solutions. Set $\boldsymbol{X} = V \times H$. $\boldsymbol{X}$ is a Hilbert space with respect to the scalar product $\langle \boldsymbol{U}_1, \boldsymbol{U}_2 \rangle = (Bu_1, u_2) + (v_1, v_2)$ where $\boldsymbol{U}_i = \{u_i, v_i\}$, $u_i \in V$, $v_i \in H$ $(i = 1, 2)$.

We define

$$D(\boldsymbol{M}) = \{\{u, v\} \in V \times H : v \in D(\varphi), (Bu + Av + \partial\varphi(v)) \cap H \neq \emptyset\}\,,$$

and

$$\boldsymbol{M}(\boldsymbol{U}) = \{-v, (Bu + Av + \partial\varphi(v)) \cap H\}$$

for any $\boldsymbol{U} \in D(\boldsymbol{M})$ $(\boldsymbol{U} = \{u, v\})$.

It is readily seen that $\boldsymbol{M}$ is monotone in $\boldsymbol{X}$. Moreover, $\boldsymbol{M}$ is maximal monotone in $\boldsymbol{X}$ (i.e., equivalently, $R(\boldsymbol{I} + \boldsymbol{M}) = \boldsymbol{X}$). Indeed, let $\{g, h\} \in \boldsymbol{X}$ be given. Note first of all that the mapping $I + A + B$ is monotone, hemi-continuous, bounded and coercive from V into V^*. Since $\partial\varphi$ is maximal monotone from V into 2^{V^*} we get $R(I + A + B + \partial\varphi) = V^*$ (cf. e.g. [1]). Hence there exists an element $v \in D(\partial\varphi)$ such that

$$v + Av + Bv + \partial\varphi(v) \ni h - Bg .$$

Setting $v + g = u$ it follows

$$u - v = g ,$$

$$v + Bu + Av + \partial\varphi(v) \ni h .$$

Taking into account that $h - v \in H$ we reach the desired assertion.

Thus, setting $\mathbf{F} = \{0, f\}$ for a.a. $t \in [0, T]$, we obtain from [1]–[4] for any $\omega > 0$ the existence and uniqueness of a function $\mathbf{U} \in C([0, T]; \mathbf{X})$[3]) which satisfies

$$\text{(2.7)} \qquad \mathbf{U}(t) \in D(\mathbf{M}) \ \forall t \in [0, T] , \quad \mathbf{U}' \in L^\infty(0, T; \mathbf{X}) ,$$

$$\text{(2.8)} \qquad \mathbf{U}' + \mathbf{M}(\mathbf{U}) + \omega\mathbf{U} \ni \mathbf{F} \quad \textit{for a.a.} \quad t \in [0, T] ,$$

$$\text{(2.9)} \qquad \mathbf{U}(0) = \mathbf{U}(T) .$$

Equivalently, when writing $\mathbf{U} = \{u, v\}$ we have $u \in C([0, T]; V)$, $v \in C([0, T]; H)$ and

$$v(t) \in D(\varphi) \ \forall t \in [0, T] ,$$

$$[B\,u(t) + A\,v(t) + \partial\varphi(v(t))] \cap H \neq \emptyset \ \forall t \in [0, T] ,$$

$$\text{(2.7')} \qquad u' \in L^\infty(0, T; V) , \quad v' \in L^\infty(0, T; H) ,$$

$$\text{(2.8')} \qquad u' - v + \omega u = 0 \quad \textit{for a.a.} \quad t \in [0, T] ,$$

$$\text{(2.8'')} \qquad v' + Av + Bu + \partial\varphi(v) + \omega v \ni f \quad \textit{for a.a.} \quad t \in [0, T] ,$$

$$\text{(2.9')} \qquad u(0) = u(T) , \quad v(0) = v(T) .$$

By (2.7′) and (2.8′), $v \in L^\infty(0, T; V)$. Setting $w = f - v' - Av - Bu - \omega v$ for a.a. $t \in [0, T]$, we have $w \in \partial\varphi(v)$ for a.a. $t \in [0, T]$ and $w \in L^2(0, T; V^*)$. Further, observing that v is weakly continuous from $[0, T]$ into V[4]) one easily verifies that the function $t \longmapsto \varphi(v(t))$ is integrable on $[0, T]$, i.e. $v \in D(\Phi)$. We now infer from (2.8″) that $w \in \partial\Phi(v)$.

2° A-priori-estimates. From (2.8′) it follows

$$\text{(2.10)} \qquad Bu' = Bv - \omega Bu \quad \textit{for a.a.} \quad t \in [0, T] .$$

[3]) More precisely, U_ω should be written to indicate the dependence of the solution on ω. However, for notational convenience, we drop the suffix ω.

[4]) Cf. *Lions, J.-L.* et *Magenes, E.:* Problèmes aux limites non homogènes et applications, vol. *1* (chap. 3, 8.4). Dunod, Paris 1968.

Since $u(0) = u(T)$ we find

$$\int_0^T (Bv, u)\, dt \geqq 0 \quad \forall \omega > 0 . \tag{2.11}$$

Recall that

$$v' + Av + Bu + w + \omega v = f \quad \textit{for a.a.} \quad t \in [0, T] \tag{2.12}$$

where $w \in \partial\varphi(v)$ for a.a. $t \in [0, T]$ (cf. (2.8″)). Observing (2.11) we conclude from the latter equality after multiplying by v that

$$\|v\|_{L^2(0,T;V)} \leqq \text{const} \quad \forall \omega > 0 . \tag{2.13}$$

Since $w \in \partial\Phi(v)$ the hypothesis (2.3) implies

$$\|w\|_{L^2(0,T;V^*)} \leqq \text{const} \quad \forall \omega > 0 . \tag{2.14}$$

Next, by the aid of (2.13) one easily derives from (2.10) the estimate

$$\|u'\|_{L^2(0,T;V)} \leqq \text{const} \quad \forall \omega > 0 . \tag{2.15}$$

Let $\omega_0 = \text{const} > 0$ be arbitrary, but fixed. We multiply (2.12) by u. Using that

$$\int_0^T (v', u)\, dt = - \int_0^T (v, u')\, dt ,$$

one obtains, for any $0 < \omega \leqq \omega_0$,

$$\int_0^T (Bu, u)\, dt \leqq \|v\|_{L^2(0,T;H)} \|u'\|_{L^2(0,T;H)} +$$
$$+ c\left(\|f\|_{L^2(0,T;H)} + \|v\|_{L^2(0,T;V)} + \|w\|_{L^2(0,T;V^*)}\right) \|u\|_{L^2(0,T;V)}$$

where $c = \text{const} > 0$. By (2.13)–(2.15),

$$\|u\|_{L^2(0,T;V)} \leqq \text{const} \quad \forall 0 < \omega \leqq \omega_0 . \tag{2.16}$$

Finally, we infer from (2.12) by virtue of (2.13)–(2.16) that

$$\|v'\|_{L^2(0,T;V^*)} \leqq \text{const} \quad \forall 0 < \omega \leqq \omega_0 . \tag{2.17}$$

3° Passage to limit. Let $\{\omega_n\}$ be a sequence of reals such that $0 < \omega_n \leqq \omega_0$ $(n = 1, 2, \ldots)$ and $\omega_n \to 0$ as $n \to \infty$.

From the preceding two sections we obtain for each n the existence of functions $u_n \in C([0, T]; V)$, $v_n \in C([0, T]; H)$ and $w_n \in L^2(0, T; V^*)$ with $u_n' \in L^\infty(0, T; V)$, $v_n' \in L^\infty(0, T; H)$ and $w_n \in \partial\varphi(v_n)$ for a.a. $t \in [0, T]$ such that

$$u_n' - v_n + \omega_n u_n = 0 \quad \textit{for a.a.} \quad t \in [0, T] , \tag{2.18}$$

$$v_n' + Av_n + Bu_n + w_n + \omega_n v_n = f \quad \textit{for a.a.} \quad t \in [0, T] , \tag{2.19}$$

$$u_n(0) = u_n(T) , \quad v_n(0) = v_n(T) \tag{2.20}$$

(cf. (2.7′)−(2.9′)) and, without any loss of generality,

$$(2.21)\qquad u_n \to u \quad \textit{weakly in} \quad L^2(0, T; V),$$

$$u_n' \to u' \quad \textit{weakly in} \quad L^2(0, T; V),$$

$$(2.22)\qquad v_n \to v \quad \textit{weakly in} \quad L^2(0, T; V),$$

$$v_n' \to v' \quad \textit{weakly in} \quad L^2(0, T; V^*),$$

$$(2.23)\qquad w_n \to w \quad \textit{weakly in} \quad L^2(0, T; V^*)$$

as $n \to \infty$ (cf. (2.13)−(2.17)).

The passage to limit in (2.18) yields $u' = v$ for a.a. $t \in [0, T]$. Using this we conclude from (2.19) after passing to limit that

$$(2.24)\qquad u'' + Au' + Bu + w = f \quad \textit{for a.a.} \ t \in [0, T].$$

Further, by (2.21) and (2.22), the conditions (2.20) are preserved when letting $n \to \infty$. Thus

$$u(0) = u(T), \quad u'(0) = u'(T).$$

It remains to show that $w \in \partial\Phi(v)$. To this end, we note that, for each n,

$$B(u_n' - u') = B(v_n - v) - \omega_n B u_n,$$

which implies

$$(2.25)\qquad \int_0^T (Bu_n, v_n - v)\, dt = \int_0^T (B(v_n - v), u)\, dt + \omega_n \int_0^T (Bu_n, u_n - u)\, dt.$$

On the other hand, we obtain from (2.19)

$$\int_0^T (w_n, v_n - v)\, dt = \int_0^T (f, v_n - v)\, dt + \int_0^T (v_n', v)\, dt - \int_0^T (Av_n, v_n - v)\, dt - \\ - \int_0^T (Bu_n, v_n - v)\, dt - \omega_n \int_0^T (v_n, v_n - v)\, dt.$$

Observing (2.25) one finds

$$\limsup \int_0^T (w_n, v_n - v)\, dt \leqq \int_0^T (v', v)\, dt = 0.$$

Since $\partial\Phi$ is maximal monotone the first convergence property in (2.22), (2.23) and the latter inequality imply $w \in \partial\Phi(v)$.

Thus, the function u obtained in (2.21) satisfies (2.4)−(2.6).

Let us finally mention a unilateral boundary value problem in linear viscoelasticity to which Theorem 2 applies.

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary Γ. Denoting by $u = \{u_1, u_2, u_3\}$ the displacement vector in Ω, the vibrations of a viscoelastic body with short memory which occupies the region Ω are governed by the system of equations

$$(*) \qquad \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \sigma_{ij} = f_i \,^{5)} \quad in \quad \Omega \times [0, T], \quad i = 1, 2, 3$$

where

$$\sigma_{ij} = a^{(0)}_{ijkl} \varepsilon_{kl} + a^{(1)}_{ijkl} \frac{\partial}{\partial t} \varepsilon_{kl},$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and the coefficients $a^{(s)}_{ijkl}$ $(s = 0, 1)$ are assumed to satisfy the following conditions:

$$a^{(s)}_{ijkl} \textit{ is measurable and bounded in } \Omega,$$

$$a^{(s)}_{ijkl} = a^{(s)}_{jikl} = a^{(s)}_{klij},$$

$$a^{(s)}_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geqq \mu_0 \varepsilon^2_{ij} \textit{ for all symmetric tensors } \{\varepsilon_{ij}\}.$$

The vector $f = \{f_1, f_2, f_3\}$ represents the given body force.

In order to formulate boundary conditions for u, let $n = \{n_1, n_2, n_3\}$ denote the unit outer normal with respect to Ω and let

$$\sigma_N = \sigma_{ij} n_i n_j,$$

$$\sigma_T = \{\sigma_{T1}, \sigma_{T2}, \sigma_{T3}\} \quad where \quad \sigma_{Ti} = \sigma_{ij} n_j - \sigma_N n_i,$$

and

$$v_N = v_i n_i, \; v_T = v - v_N n.$$

Let $g \in L^2(\Gamma)$, $g > 0$ a.e. on Γ. We then consider the following boundary conditions:

$$(**) \qquad u_N = 0 \quad on \quad \Gamma \times [0, T],$$

$$\left.\begin{aligned} |\sigma_T| < g &\Rightarrow \frac{\partial u_T}{\partial t} = 0, \\ |\sigma_T| = g &\Rightarrow \exists \lambda \geqq 0 : \frac{\partial u_T}{\partial t} = -\lambda \sigma_T \end{aligned}\right\} \quad on \quad \Gamma \times [0, T].$$

[5]) We use the convention that a repeated suffix means summation over 1, 2, 3.

For introducing the weak formulation of boundary value problem (*), (**), let $W_2^1(\Omega)$ denote the usual Sobolev space[6]) and let

$$V = \{v \in [W_2^1(\Omega)]^3 : v_N = 0 \ a.e. \ on \ \Gamma\} .$$

We define, for any $u, v \in V$,

$$a^{(s)}(u, v) = \int_\Omega a_{ijkl}^{(s)} \varepsilon_{kl}(u) \varepsilon_{ij}(v) \, dx \quad (s = 0, 1) ,$$

$$\varphi(v) = \int_\Gamma g |v_T| \, d\Gamma .$$

Applying Theorem 2 we get: *Let* $f_i \in L^2(0, T; L^2(\Omega))$, $f_i' \in L^2(0, T; L^2(\Omega))$ *and* $f_i(0) = f_i(T)$ $(i = 1, 2, 3)$. *Then there exists a function* $u \in C([0, T]; V)$ *with* $u' \in L^2(0, T; V)$ *and* $u'' \in L^2(0, T; V^*)$ *such that*

$$\int_0^T (u'', v - u') \, dt + \int_0^T a^{(0)}(u, v - u') \, dt + \int_0^T a^{(1)}(u', v - u') \, dt +$$

$$+ \int_0^T \varphi(v) \, dt - \int_0^T \varphi(u') \, dt \geqq \int_0^T (f, v - u') \, dt$$

for all $v \in L^2(0, T; V)$, *and* $u(0) = u(T)$, $u'(0) = u'(T)$.

We dispense with further details and refer to the book: *Duvaut, G.* et *Lions, J.-L.*: Les inéquations en mécanique et en physique (chap. 3). Dunod, Paris 1972.

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