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MINIMAL COMPATIBLE TOLERANCES ON LATTICES

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1. In the paper [7] the concept of tolerance relation has been introduced. This concept was obtained by abstraction from the concept of tolerance which is well-known in the technical practice. In the paper [2] tolerance relations are studied on graphs and the concept of a compatible tolerance is introduced (although it is not called so). For universal algebras the concept of compatible tolerance is introduced in the paper [3]. There and in [4] fundamental characteristics of these relations are studied. In [4] it is proved that some theorems which are valid for congruences hold also for compatible tolerances. In [4] and [5] it is also proved that for some special algebras (groups, Boolean algebras) each compatible tolerance is a congruence. The existence conditions for compatible tolerances which are not congruences are studied in detail especially in [5], [8] and [9]. In [6] it is proved that the set of all compatible tolerances on a given algebra forms a complete lattice. Many properties of this lattice are the same as those of the lattice of all congruences on a given algebra. Therefore it is natural to study the analogies between these lattices on particular algebras. In the paper [10] the minimal congruence containing a given set is defined. This concept plays a considerable part in the study of congruences, because each congruence C on a given algebra $\mathfrak{A}$ is the join of minimal congruences containing two-element sets $\{a, b\}$, where a, b belong to $\mathfrak{A}$ and $a\ C\ b$ holds. Further, these minimal congruences are compact elements in the lattice of all congruences on a given algebra. But in [6] it is proved that the same assertions hold also for compatible tolerances. Thus it seems to be useful to study minimal compatible tolerances on various types of algebras. The aim of this paper is to give fundamental characteristics of minimal compatible tolerances on some types of lattices.

2. By the symbol $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ we denote an algebra $\mathfrak{A}$ with the support A and with the set $\mathscr{F}$ of fundamental operations.

Definition 1. Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let R be a binary relation on A. The relation R is called *compatible with* $\mathfrak{A}$, if for each n-ary operation $f \in \mathscr{F}$, where n is a positive integer, and for any $2n$ elements $a_1, \ldots, a_n, b_1, \ldots. b_n$ of A such that $a_i R b_i$ for $i = 1, \ldots, n$ we have $f(a_1, \ldots, a_n) R f(b_1, \ldots, b_n)$.

Definition 2. A reflexive and symmetric binary relation on a set is called a *tolerance* on this set. If a tolerance T on the support A of an algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ is compatible with A, we call it a *compatible tolerance on* $\mathfrak{A}$.

In [4], [5], [8] and [9] examples of compatible tolerances which are not congruences are given.

Definition 3. Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, $S \subseteq A$, $S \neq \emptyset$. Let $\mathscr{C}$ be a class of compatible relations on $\mathfrak{A}$. Then $R_S \in \mathscr{C}$ is called the minimal relation of the class $\mathscr{C}$ containing S, if for each $x \in S$, $y \in S$ we have $x R_S y$ and for each relation $R \in \mathscr{C}$ with this property $R_S \subseteq R$ holds. If $\mathscr{C}$ is the class of all compatible tolerances (or congruences) on $\mathfrak{A}$, then such an R_S is called the minimal compatible tolerance (or the minimal congruence respectively) on $\mathfrak{A}$ containing S.

Notation. Let $\mathfrak{A}$ be an algebra, let $\mathscr{C}$ be a given class of compatible relations on $\mathfrak{A}$. In the whole paper the symbol R_S (or T_S, or C_S) denotes the minimal relation of the class $\mathscr{C}$ (or the minimal compatible tolerance on $\mathfrak{A}$, or the minimal congruence on $\mathfrak{A}$ respectively) containing S. If $S = \{a, b\}$, we denote shortly $R_{ab} = R_{\{a,b\}}$, $T_{ab} = T_{\{a,b\}}$, $C_{ab} = C_{\{a,b\}}$.

Remark. If $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ is an algebra and $\mathscr{C}$ is the class of all compatible reflexive relations on $\mathfrak{A}$, then for each $a \in A$, $b \in A$ we have $R_{ab} \subseteq T_{ab} \subseteq C_{ab}$, because every congruence is a compatible tolerance and every compatible tolerance is a compatible reflexive relation.

3. Let L be a lattice, let $a \in L$, $b \in L$. Denote $I(a, b) = \{x \in L \mid a \wedge b \leqq x \leqq a \vee b\}$. Evidently $I(a, b) = I(a \wedge b, a \vee b)$ and $I(a, b)$ for each $a \in L$, $b \in L$ is the interval on L bounded by the elements $a \wedge b$, $a \vee b$.

Lemma 1. *Let L be a distributive lattice, let $a \in L$, $b \in L$, $a \leqq b$. Let $\mathscr{C}$ be the class of all compatible reflexive relations on L with the property that if $R \in \mathscr{C}$, $r \in L$, $s \in L$, $r R s$, then $z R t$ for each $z \in I(r, s)$, $t \in I(r, s)$. Then the relation R defined so that $x R y$ if and only if*

(1) $$b \vee (x \wedge y) \geqq x \vee y\,,$$

(2) $$a \wedge (x \vee y) \leqq x \wedge y$$

is the minimal relation of the class $\mathscr{C}$ containing the pairs (a, b), (b, a) and, moreover, $R = T_{ab}$.

Proof. Evidently R is reflexive and symmetric, therefore it is a tolerance. We shall prove its compatibility. For each four elements x, y, z, t of L the following inequalities hold:

$$b \vee ((x \vee z) \wedge (y \vee t)) \geqq b \vee (x \wedge y),$$
$$b \vee ((x \vee z) \wedge (y \vee t)) \geqq b \vee (z \wedge t).$$

This implies

$$b \vee ((x \vee z) \wedge (y \vee t)) \geqq (b \vee (x \wedge y)) \vee (b \vee (z \wedge t)). \tag{3}$$

Let $x\ R\ y$, $z\ R\ t$, then according to (1) and (3)

$$b \vee ((x \vee z) \wedge (y \vee t)) \geqq (x \vee y) \vee (z \vee t) = (x \vee z) \vee (y \vee t).$$

From (2) and from the distributivity of L we obtain

$$a \wedge ((x \vee z) \vee (y \vee t)) = a \wedge ((x \vee y) \vee (z \vee t)) =$$
$$= ((a \wedge (x \vee y)) \vee (a \wedge (z \vee t)) \leqq$$
$$\leqq (x \wedge y) \vee (z \wedge t) \leqq (x \vee z) \wedge (y \vee t).$$

According to (1) and (2) it is $x \vee z\ R\ y \vee t$. Dually we can prove $x \wedge z\ R\ y \wedge t$, thus R is a compatible relation on L. Evidently $a\ R\ b$. According to Theorem 1 in [5], if R is a compatible tolerance on L, then $r\ R\ s \Rightarrow z\ R\ t$ for each $z \in I(r, s)$, $t \in I(r, s)$, thus $R \in \mathscr{C}$. This means $R_{ab} \subseteq T_{ab} \subseteq R$. We shall prove the converse inclusion. Let $x\ R\ y$. Then from (2) and from the distributivity of L we have

$$x \wedge y = (a \wedge (x \vee y)) \vee (x \wedge y) = (a \vee (x \wedge y)) \wedge ((x \vee y) \wedge (x \wedge y)) =$$
$$= (a \vee (x \wedge y)) \wedge (x \vee y).$$

From (1) we have

$$x \vee y = (b \vee (x \wedge y)) \wedge (x \vee y).$$

As R_{ab} is reflexive, $x\ R_{ab}\ x$ holds for each $x \in L$. Further $a\ R_{ab}\ b$, $R_{ab} \in \mathscr{C}$. Thus

$$x \wedge y = (a \vee (x \wedge y)) \wedge (x \vee y)\ R_{ab}\ (b \vee (x \wedge y)) \wedge (x \vee y) = x \vee y,$$

which means $x \wedge y\ R_{ab}\ x \vee y$. But $R_{ab} \in \mathscr{C}$ and $x \in I(x \wedge y, x \vee y)$, $y \in I(x \wedge y, x \vee y)$, therefore also $x\ R_{ab}\ y$. Thus we have $R \subseteq R_{ab}$, which means $R = T_{ab} = R_{ab}$.

Lemma 2. *Let L be a lattice. Then $T_{ab} = T_{a \wedge b, a \vee b}$ for each $a \in L$, $b \in L$.*

Proof. Evidently $a \in I(a \wedge b, a \vee b)$, $b \in I(a \wedge b, a \vee b)$, therefore by Theorem 1 from [5] $a\ T_{a \wedge b, a \vee b}\ b$, which means $T_{ab} \subseteq T_{a \wedge b, a \vee b}$ according to Definition 3. Further $a \wedge b \in I(a \wedge b, a \vee b) = I(a, b)$, $a \vee b \in I(a \wedge b, a \vee b) = I(a, b)$, hence $a \wedge b\ T_{ab}\ a \vee b$ and by Definition 3 the inclusion $T_{a \wedge b, a \vee b} \subseteq T_{ab}$ holds. Therefore $T_{ab} = T_{a \wedge b, a \vee b}$.

Theorem 1. *Let L be a distributive lattice. Then $C_{ab} = T_{ab}$ for each $a \in L$, $b \in L$.*

Proof. By Lemma 2, $T_{ab} = T_{a \wedge b, a \vee b}$. As $a \wedge b \leqq a \vee b$, it is sufficient to study T_{ab} for $a \leqq b$. Suppose $a \leqq b$. According to Theorem 1 from [5] each compatible tolerance T on L fulfils the implication $r\, T\, s \Rightarrow x\, T\, y$ for each $r \in L$, $s \in L$, $x \in I(r, s)$, $y \in I(r, s)$, therefore it fulfils the assumptions of Lemma 1. Thus $x\, T_{ab}\, y$ if and only if the elements x, y fulfil (1) and (2). By Theorem 2 from [1] the relation defined by the rule from Lemma 1 is equal to C_{ab}, therefore $T_{ab} = C_{ab}$.

Corollary 1. *Each compatible tolerance on a distributive lattice L is a join of congruences on L in the lattice of all compatible tolerances on L, namely, $T = \bigvee_{aTb} C_{ab}$ for each $T \in LT(L)$.*

Proof. By Theorem 15 from [6], $T = \bigvee_{aTb} T_{ab}$ holds for each compatible tolerance T on L. As $T_{ab} = C_{ab}$ for each $a \in L$, $b \in L$, we have $T = \bigvee_{aTb} C_{ab}$.

We have still another corollary which shows an interrelation between the lattice of all congruences and the lattice of all compatible tolerances on a distributive lattice.

Corollary 2. *Let L be a distributive lattice, let $K(L)$ be the lattice of all congruences on L, let $LT(L)$ be the lattice of all compatible tolerances on L. Then the following three assertions are equivalent:*

(α) *$K(L)$ is a sublattice of the lattice $LT(L)$.*
(β) *$K(L) = LT(L)$.*
(γ) *Each compatible tolerance on L is a congruence on L.*

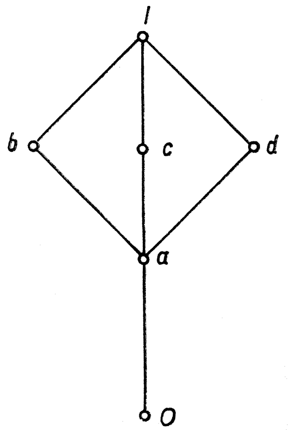

Fig. 1.

Proof. Evidently the implication $(\gamma) \Rightarrow (\beta) \Rightarrow (\alpha)$ holds. Let (α) be true. Then the join of congruences in the lattice $LT(L)$ is a congruence. But for each $T \in LT(L)$ we have $T = \bigvee_{aTb} C_{ab}$, where each C_{ab} is a congruence, thus (γ) holds. We have proved $(\alpha) \Leftrightarrow (\beta) \Leftrightarrow (\gamma)$.

We shall show that for non-distributive lattices the assertion of Theorem 1 is not true. In Fig. 1 we see the Hasse diagram of a non-distributive modular lattice L. It is easy to verify that T_{Ob} on L is the tolerance defined so that $x\ T_{Ob}\ y$ if and only if either both x and y are in the interval $\langle O, b \rangle$, or both x and y are in the interval $\langle a, I \rangle$. We have $O\ T_{Ob}\ b$, $b\ T_{Ob}\ I$, but not $O\ T_{Ob}\ I$. Thus T_{Ob} is not transitive and is not a congruence.

4. In the paper [10] it is shown how it is possible to construct C_S on a distributive lattice L, if S is an ideal of L. From the equality $T_{ab} = C_{ab}$ an analogue for a compatible tolerance T_S follows.

Lemma 3. *Let L be a lattice with the least element O and let R be a reflexive compatible relation on L. Then $J = \{a \in L \mid a\ R\ O\}$ is an ideal of the lattice L.*

Proof. Let $x \in J$, $y \in J$, then $x\ R\ O$, $y\ R\ O$ implies $x \vee y\ R\ O \vee O = O$, thus $x \vee y \in J$. Let $x \in J$, $a \in L$. As R is reflexive, $a\ R\ a$ holds and from the compatibility $a \wedge x\ R\ a \wedge O = O$, therefore $a \wedge x \in J$ and J is an ideal of the lattice L.

Lemma 4. *Let J be an ideal of the lattice L with the least element O. Then $T_J = \bigvee_{x \in J} T_{Ox}$ in the lattice of all compatible tolerances on L.*

Proof. For each element $x \in J$ evidently $x\ T_J\ O$ (because $O \in J$), therefore $T_J \supseteq T_{Ox}$. This implies $T_J \supseteq \bigvee_{x \in J} T_{Ox}$. Let $a \in J$, $b \in J$, then $a\ T_{Oa}\ O$, $b\ T_{Ob}\ O$, from the symmetry $O\ T_{Ob}\ b$, thus $a = a \vee O(\bigvee_{x \in J} T_{Ox})\ O \vee b = b$, because according to Theorem 1 from [6] the relation $\bigvee_{x \in J} T_{Ox}$ is again a compatible tolerance and according to Theorem 2 from [6] (where $p(x_1, x_2) = x_1 \vee x_2$ is the corresponding lattice polynomial without constants) this equality holds. Thus $a(\bigvee_{x \in J} T_{Ox})\ b$. As a, b are two arbitrary elements of J, we have $T_J \subseteq \bigvee_{x \in J} T_{Ox}$ and thus $T_J = \bigvee_{x \in J} T_{Ox}$.

Theorem 2. *Let L be a distributive lattice with the least element O and let J be an ideal of L. Let $a \in L$, $b \in L$. If $a\ T_J\ b$, then there exists an element $v \in J$ such that $a \vee v = b \vee v$.*

Proof. Let $a\ T_J\ b$. According to Lemma 4 we have $a\ (\bigvee_{x \in J} T_{Ox})\ b$, therefore according to Theorem 2 from [6] there exist elements $a_1, \ldots, a_r$, $b_1, \ldots, b_r$ of L and elements $u_1, \ldots, u_r$ of J and a lattice polynomial without constants $p(x_1, \ldots, x_r)$

such that $a_i\, T_{Ou_i}\, b_i$ for $i = 1, \ldots, r$, $p(a_1, \ldots, a_r) = a$, $p(b_1, \ldots, b_r) = b$. (The definition of a lattice polynomial without constants can be found in [6].) Let $v = \bigvee_{i=1}^{r} u_i$. Then $v \in J$, thus $v\, T_{Ov}\, O$. According to Theorem 1 in [5] then $u_i\, T_{Ov}\, O$ for $i = 1, \ldots, r$. This implies $T_{Ov} \supseteqq T_{Ou_i}$ and also $a_i\, T_{Ov}\, b_i$ for $i = 1, \ldots, r$. Now $a = p(a_1, \ldots, a_r)\, T_{Ov}\, p(b_1, \ldots, b_r) = b$, because T_{Ov} is compatible. According to Theorem 1 we have $T_{Ov} = C_{Ov}$ and according to Theorem 2 in [1] the inequality (1) from Lemma 1 holds, i.e. $v \vee (a \wedge b) \geqq a \vee b$. (We obtain this by substituting v, a, b for b, x, y.) The distributivity of L then yields

$$(v \vee a) \wedge (v \vee b) = v \vee (a \wedge b) = v \vee (a \wedge b) \vee (a \vee b) =$$
$$= v \vee (a \vee b) = (v \vee a) \vee (v \vee b),$$

which means $v \vee a = v \vee b$.

Corollary 3. *Let L be a distributive lattice with the least element O and let J be an ideal of L. Then $T_J = C_J$.*

Proof. By (P) we denote the following equivalence: $u\, T_J\, O \Leftrightarrow u \in J$. We prove (P). Let $u \in J$, then evidently $u\, T_J\, O$. On the other hand, if $u\, T_J\, O$, then according to Theorem 2 there exists $v \in J$ such that $u \vee v = O \vee v = v$, therefore $u \vee v = v$, which means $u \leqq v$. But $v \in J$, therefore also $u \in J$ and (P) is true. According to Corollary to Theorem 92 in [10], p. 185, the relation defined as the least relation T_J compatible with L and satisfying (P) is a minimal congruence on L containing J, therefore $T_J = C_J$.

For non-distributive lattices this assertion is not true. This can be shown again by the conterexample in Fig. 1. In the lattice L whose Hasse diagram is in Fig. 1 the set $I(O, b)$ is an ideal, thus $T_{Ob} = T_J$, where $J = I(O, b)$. It was proved above that T_{Ob} is not a congruence.

5. We shall add still one result concerning semilattices. An operation in a semilattice will be denoted by $\circ$ and we put $x \geqq y$ if and only if $x \circ y = x$. The relations T_{ab}, C_{ab} are defined analogously as in a lattice.

Theorem 3. *Let S be a semilattice, let $a \in S$, $b \in S$. A necessary condition for the equality $T_{ab} = C_{ab}$ is that either $a = b$, or a covers b, or b covers a, or a and b are incomparable and $a \circ b$ covers both a and b.*

Proof. Let $a \in S$, $b \in S$. It is easy to see that $x\, T_{ab}\, y$ if and only if either $x = y$, or $x = a$, $y = b$, or $x = b$, $y = a$, or $x = a \circ z$, $y = b \circ z$, or $x = b \circ z$, $y = a \circ z$, where z is an element of S. Now suppose that $a < b$ and b does not cover a. This means that there exists $c \in S$ such that $a < c < b$. We have $c\, T_{ab}\, b$, because $c = a \circ c$, $b = b \circ c$. Obviously $b\, T_{ab}\, a$. If T_{ab} were transitive, we should have $c\, T_{ab}\, a$, but this does not hold, because $c \neq a$, none of the elements a, c is equal to b and none of them can be expressed as $b \circ z$ for $z \in S$, because they are both less than b. Analogously if $b < a$ and a does not cover b. Now suppose that a, b are incomparable and $a \circ b$ does not cover a. There exists $d \in S$ such that $a < d < a \circ b$. We have $a \circ b \; T_{ab} \; a$ because $a \circ b = b \circ a$, $a = a \circ a$. Further, $d \; T_{ab} \; a \circ b$ because $d = a \circ d$, $a \circ b = b \circ d$. But $d \; T_{ab} \; a$ does not hold because $a \neq d$, none of the elements a, d is equal to b and none of them can be expressed as $b \circ z$ for $z \in S$, because they are both incomparable with b. Thus T_{ab} is not transitive. Analogously if $a \circ b$ does not cover b.

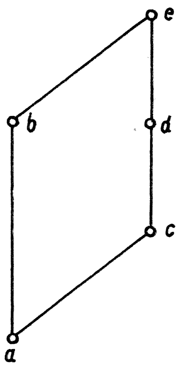

Fig. 2.

The condition is not sufficient. In Fig. 2 we see the Hasse diagram of a semilattice S in which $c \; T_{ab} \; e$ because $c = a \circ c$, $e = b \circ c$, and $e \; T_{ab} \; d$ because $e = b \circ d$, $d = a \circ d$, but not $c \; T_{ab} \; d$ because $c \neq d$, none of the elements c, d is equal to e and none of them can be expressed as $b \circ z$ for $z \in S$, because they are both incomparable with b.

Corollary 4. *Let S be a semilattice. Then $T_{ab} = C_{ab}$ for each $a \in S$, $b \in S$, if and only if there exists an element $o \in S$ such that $x \circ y = o$ for any $x \in S$, $y \in S$, $x \neq y$.*

Proof. Suppose that there exist two pairs a, b and c, d of elements of S such that $a \neq b$, $c \neq d$ and $a \circ b \neq c \circ d$. Then either $a \circ b < a \circ b \circ c \circ d$ or $c \circ d < a \circ b \circ c \circ d$. In the first case $a < a \circ b < a \circ b \circ c \circ d$ and $T_{a, a \circ b \circ c \circ d}$ is not a congruence. Analogously in the second case. Thus we must have $a \circ b = c \circ d$. As these elements were chosen arbitrarily, the products of all pairs of distinct elements must be equal to an element o. On the other hand, if such an element o exists, the tolerance T_{ab} has the property that $x \; T_{ab} \; y$ if and only if either $x = y$, or both x, y belong to the set $\{a, b, a \circ b\}$, therefore it is a congruence and $T_{ab} = C_{ab}$.

6. In the end we shall prove a theorem on the existence of a maximal compatible tolerance which does not contain a given pair of elements.

Theorem 4. *Let* $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ *be an algebra. Let* $a \in A$, $b \in A$, $a \neq b$. *Then there exists a compatible tolerance* T *such that*

(i) a non $T b$;

(ii) T *is a maximal (with respect to the set inclusion) compatible tolerance on* $\mathfrak{A}$ *with the property* (i).

Proof. Let $\mathscr{S} = \{T \in LT(\mathfrak{A}) \mid a \text{ non } T b\}$. Then $\mathscr{S} \neq \emptyset$ because the identical relation on A is in $\mathscr{S}$. The set $\mathscr{S}$ is a partially ordered set with respect to the set inclusion. Let $\mathscr{C} = \{T_\alpha, \alpha \in \Gamma\}$ be a chain in $\mathscr{S}$ (where Γ is a subscript set). Then evidently $\bigvee_{\alpha \in \Gamma} T_\alpha = \bigcup_{\alpha \in \Gamma} T_\alpha$. Denote $T' = \bigvee_{\alpha \in \Gamma} T_\alpha = \bigcup_{\alpha \in \Gamma} T_\alpha$. Then T' is a compatible tolerance on A and $x\, T'\, y$ if and only if $x\, T_\alpha\, y$ for some $\alpha \in \Gamma$. Thus a non $T'\, b$, which implies $T' \in \mathscr{S}$. The conditions of Zorn's Lemma are fulfilled and thus $\mathscr{S}$ has a maximal element T.

This maximal tolerance need not be unique. Let L be a lattice with the elements O, a, b, I such that $O < a < I$, $O < b < I$, $a \parallel b$. Let T_1 (or T_2) be a congruence on L with the classes $\{O, a\}$, $\{b, I\}$ (or $\{O, b\}$, $\{a, I\}$ respectively). Then both T_1 and T_2 are maximal compatible tolerances on L which do not contain the pair (O, I).

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