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COMMUTATIVE SEMI-PRIMARY x -SEMIGROUPS

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In this paper we intend to prove some results on semi-primary semigroups equipped with a system of x -ideals. Our effort was motivated by the results of the almost coincidentally entitled paper [3].

Let S be a commutative semigroup written multiplicatively. We say that a *system of x -ideals* (or an *x -system*) is defined in S if to every subset A of S there corresponds a subset A_x of S such that

- (1) $A \subseteq A_x$
- (2) $A \subseteq B_x$ implies $A_x \subseteq B_x$
- (3) $AB_x \subseteq B_x \cap (AB)_x$

where AB is the set of all products ab with a in A and b in B . If $A = A_x$ we shall say that A is an *x -ideal*. For the sake of brevity we shall call a semigroup equipped with an x -system *x -semigroup*. This concept of x -ideals was introduced by AUBERT in [1] following an idea of Lorenzen. For the details and the relationships to the known ideal theories [1] may be consulted.

Proposition 1 of [1] shows that the family of all x -ideals of S forms a complete lattice with respect to set-inclusion. The union within this lattice will be called the *x -union* and denoted by \bigcup_x , i.e.

$$\bigcup_x A_x^{(i)} = \left(\bigcup_{i \in I} A_x^{(i)} \right)_x.$$

The *x -product* of two subsets A and B of S is defined as the set $(AB)_x$ and denoted by $A \circ B$. It follows from Theorem 1 of [1] that $A \circ B = A \circ B_x = A_x \circ B_x$ for any subsets A and B of S . In the sequel A_x^n or A^n means the x -product of n factors A_x or the usual product of n factors A , respectively.

An x -ideal P_x is said to be prime if $ab \in P_x$ implies $a \in P_x$ or $b \in P_x$. An x -ideal Q_x is said to be primary if $ab \in Q_x$ and $a \notin Q_x$ imply $b^n \in Q_x$ for some positive integer n .

Lemma 1 (Aubert [1]). *The x -ideal P_x is prime if and only if $A_x \circ B_x \subseteq P_x$ and $A_x \not\subseteq P_x$ imply $B_x \subseteq P_x$.*

The (nilpotent) radical $\sqrt{A_x}$ of A_x is the set of all elements b in S such that $b^n \in A_x$ for some (positive) integer n . The operation of forming the radical has a number of expected properties, e.g. $\sqrt{(A_x \circ B_x)} = \sqrt{(A_x \cap B_x)} = \sqrt{A_x} \cap \sqrt{B_x}$ and the following lemma holds.

Lemma 2 (Aubert [1]). *The radical of an x -ideal A_x is the intersection of all prime x -ideals containing A_x .*

An x -system is said to be of *finite character* if for N finite, $A_x = \bigcup_{N \subseteq A} N_x$ for every subset A of S .

Lemma 3 (Aubert [1]). *An x -system is of finite character if and only if the set-theoretic union of any chain of x -ideals is an x -ideal.*

Let S be an x -semigroup. Put $\Omega_x = \bigcap_{A \subseteq S} A_x$. (It may happen that Ω_x is void.) An element t is *nilpotent* if $t^n \in \Omega_x$ for some n . An x -ideal A_x is *nil* if each element of A_x is nilpotent. The radical of Ω_x (the *x -nilradical* of S) is the set of all nilpotent elements in S and according to Lemma 2 it equals the intersection of all prime x -ideals in S . The x -nilradical of S is *half-prime*, that is it coincides with its radical. Proposition 12 of [1] immediately yields the following lemma.

Lemma 4. *For any two x -ideals A_x and B_x we have $A_x \circ B_x \subseteq \sqrt{\Omega_x}$ if and only if $A_x \cap B_x \subseteq \sqrt{\Omega_x}$.*

An x -ideal A_x is called *semi-primary* if its radical $\sqrt{A_x}$ is a prime x -ideal. Commutative x -semigroup is *semi-primary* if its each x -ideal is semi-primary. For instance, valuation rings are semi-primary.

Theorem 1. *Let S be an x -semigroup with non-void x -radical $\sqrt{\Omega_x}$ and let $\sqrt{\Omega_x}$ be a proper ($\neq S$) prime x -ideal. Then an x -ideal A_x is nil if and only if there is a non-nil B_x with $A_x \cap B_x \subseteq \sqrt{\Omega_x}$.*

Proof. We know from Lemma 4 that $A_x \cap B_x \subseteq \sqrt{\Omega_x}$ is equivalent to $A_x \circ B_x \subseteq \sqrt{\Omega_x}$. Since $\sqrt{\Omega_x}$ is a proper x -ideal, there is a non-nilpotent element t in S . Now, if A_x is nil we have, say, $(t)_x \circ A_x \subseteq A_x \subseteq \sqrt{\Omega_x}$. The reverse statement of the theorem follows from the fact that $\sqrt{\Omega_x}$ is prime.

Corollary. *If S is a semi-primary x -semigroup with at least one nilpotent element and $A_x \cap B_x$ is nil then at least one of the x -ideals A_x and B_x is nil.*

Theorem 2. *Let S be an x -semigroup and \mathcal{S} such a system of x -ideals in S that each x -ideal not in \mathcal{S} is semi-primary. Then for any two prime x -ideals A_x and B_x we have either $A_x \cap B_x \in \mathcal{S}$ or A_x and B_x are ordered under the set-inclusion.*

Proof. If $A_x \cap B_x$ is not in \mathcal{S} then $A_x \cap B_x$ is semi-primary. But $A_x \cap B_x = \sqrt{A_x \cap B_x} = \sqrt{A_x \cap B_x}$ which implies that $A_x \cap B_x$ is prime. On the hand, $A_x \cap B_x \supseteq A_x \circ B_x$ and therefore $A_x \cap B_x \supseteq A_x$ or $A_x \cap B_x \supseteq B_x$.

Corollary 1. *In a semi-primary x -semigroup, prime x -ideals form a chain.*

Corollary 2. *If in an x -semigroup the x -ideals different from Ω_x are semi-primary then for any two prime x -ideals A_x and B_x either $A_x \cap B_x = \Omega_x$ or $A_x \supseteq B_x$ or $B_x \supseteq A_x$.*

Corollary 3. *A semi-primary x -semigroup S with $S \circ S = S$ is quasi-local (i.e. with a unique maximal x -ideal).*

Theorem 3. *Let S be an x -semigroup. Then the following statements are equivalent:*

- (1) S is a semi-primary x -semigroup.
- (2) Every principal x -ideal of S is semi-primary.
- (3) Prime x -ideals of S form a chain.
- (4) The radicals of all the x -ideals in S form a chain.

The proof of the equivalence of the first three statements runs along the same lines as the proof of Theorem 1 in [3]. The equivalence between (3) and (4) is an easy exercise.

Theorem 4. *Let S be a semi-primary x -semigroup. Then for any two finitely generated x -ideals A_x and B_x we have either $A_x^n \subseteq B_x$ or $B_x^n \subseteq A_x$ for some n .*

Proof. Since S is semi-primary, $\sqrt{A_x} \subseteq \sqrt{B_x}$ may be assumed. Let $A_x = \bigcup_{i=1}^k (a_i)_x$. Then $a_i^m \in B_x$ for some m and every $i = 1, \dots, k$. According to Theorem 1 of [1] the x -multiplication is distributive with respect to the x -union and therefore $A_x^{km} \subseteq B_x$.

Corollary. *In a semi-primary x -semigroup the principal x -ideals generated by idempotents form a chain.*

To show that Theorem 4 cannot be extended to all x -ideals we borrow the following example from [4]. Let $R = F[x_1] \cup F[x_1, x_2] \cup F[x_1, x_2, x_3] \cup \dots$, where F is a field and $\{x_1, x_2, x_3, \dots\}$ is a countable set of indeterminates such that $x_i^2 = 0$, $x_i x_j = x_j x_i$ and $x_i a = a x_i$ for all $a \in F$ and i . Every ideal in R is primary and thus R is semi-primary. Let A_1 and A_2 be ideals generated by $\{x_1, x_3, x_5, \dots\}$ and $\{x_2, x_4, x_6, \dots\}$ respectively. If $A_1^n \subseteq A_2$ for some n , then $f^n = 0$ for every $f \in A_1$ which is not true. Similarly $A_2^n \not\subseteq A_1$ for all n .

Theorem 5. *A sufficient condition for an x -semigroup to be semi-primary is that for any two x -ideals A_x and B_x there is an integer n (depending on A_x and B_x) with $A_x^n \subseteq B_x$ or $B_x^n \subseteq A_x$.*

Proof. If A_x and B_x are any two prime x -ideals in S then we have, say, $A_x^n \subseteq B_x$ for some n . But then $A_x \subseteq B_x$ and Theorem 3 completes the proof.

Corollary. *Let S be an x -semigroup of finite character satisfying the ascending chain condition for x -ideals. Then S is semi-primary if and only if to any two x -ideals A_x and B_x there is an integer n with $A_x^n \subseteq B_x$ or $B_x^n \subseteq A_x$.*

To prove this corollary recall that in an x -semigroup of finite character with ACC for x -ideals each x -ideal is finitely generated.

Theorem 6. *Let S be a semi-primary x -semigroup. If A_x is finitely generated then A_x^n is contained in a principal x -ideal for some n .*

Proof. Let $A_x = \bigcup_{i=1}^k (a_i)_x$. We know that the radicals $\sqrt{(a_i)_x}$ for $i = 1, \dots, k$ form a chain. Let $\sqrt{(a_k)_x}$ contain all the remaining ones. Then $a_i^m \in (a_k)_x$ for some m and all $i = 1, \dots, k$ which yields $A_x^{km} \subseteq (a_k)_x$ and the proof is complete.

The (unique) maximal ideal of the ring R from the example above shows that the theorem does not hold for arbitrary x -ideals.

Corollary 1. *If P_x is a finitely generated prime x -ideal in a semi-primary x -semigroup then P_x is the radical of a principal ideal.*

We know from the previous proof that $P_x^n \subseteq (a)_x$ for some $a \in P_x$ and thus $\sqrt{(a)_x} = P_x$.

Corollary 2. *In a semi-primary x -semigroup no finitely generated prime x -ideal P_x is the set-theoretic union of prime x -ideals properly contained in P_x .*

P_x is the radical of a principal x -ideal which is generated by one of its elements and therefore this element cannot belong to a prime x -ideal properly contained in P_x .

The next theorem originates in [2]. It is only another version of Lemma 3.4 of [2] in terms of x -ideals and its proof can also be rewritten from [2] without difficulties.

Theorem 7. *Let S be a quasi-local x -semigroup of finite character with $S \circ S = S$. Then S is a semi-primary x -semigroup satisfying the ACC for prime x -ideals if and only if for any prime x -ideal P_x different from Ω_x there exists a prime x -ideal $N(P_x)$ properly contained in P_x such that for each prime x -ideal P'_x properly in P_x we have $P'_x \subseteq N(P_x)$.*

Corollary. *Let S be a local (i.e. quasi-local with ACC for x -ideals) x -semigroup of finite character with $S \circ S = S$. Then S is semi-primary if and only if for every prime x -ideal $P_x \neq \Omega_x$ the x -union of all prime x -ideals properly contained in P_x is a prime x -ideal properly contained in P_x .*

An x -semigroup S will be called (von Neumann) quasi-regular if $a \in (a)_x^2$ for each $a \in S$. Plainly, an x -semigroup is quasi-regular if and only if $A_x^2 = A_x$ for every x -ideal A_x . An x -semigroup S is said to be regular if it is quasi-regular and its x -system enjoys the property that $(a)_x = \{a\} \cup aS$ for every $a \in S$ (this is the so-called Lorenzen's x -system, cf. [1]). A semigroup S equipped with a Lorenzen's x -system is regular if and only if every principal x -ideal in S is generated by an idempotent. Quasi-regular x -semigroup S need not be regular. Perhaps the simplest example of this kind is obtained in the case when S is equipped with an x -system consisting from only one x -ideal S .

Theorem 8. *Let S be a quasi-regular x -semigroup. Then the following statements are equivalent:*

- (1) *Every x -ideal in S is prime.*
- (2) *S is a semi-primary x -semigroup.*
- (3) *Principal x -ideals of S form a chain.*
- (4) *All x -ideals of S form a chain.*

If in addition S is regular then the statements above are equivalent with

- (5) *Principal x -ideals generated by idempotents form a chain.*

Proof. (2) implies (3). For any two principal x -ideals $(a)_x$ and $(b)_x$ we have, say, $(a)_x^n \subseteq (b)_x$ according to Theorem 4. But S is quasi-regular and thus $(a)_x^n = (a)_x$.

(3) implies (4). Let $A_x \not\subseteq B_x$. Then there exists $a \in A_x - B_x$. Now for any b in B_x we have $b \in (b)_x \subseteq (a)_x$, that is $B_x \subseteq (a)_x$ and also $B_x \subseteq A_x$.

The rest of the proof can follow the lines of proof of Theorem 2 in [3].

Corollary 1. *Let S be a semi-primary x -semigroup. Then every x -ideal in S is prime if and only if S is quasi-regular.*

The proof of this and the next corollary is based on the same ideas as the proofs of Corollary 1 and 2 of [3].

Corollary 2. *Let S be a semigroup equipped with a Lorenzen's x -system. Then every x -ideal in S is prime if and only if S is regular and principal x -ideals generated by idempotents form a chain.*

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