Czechoslovak Mathematical Journal

R. C. Solomon

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Czechoslovak Mathematical Journal, Vol. 27 (1977), No. 4, 556–559

Persistent URL: http://dml.cz/dmlcz/101493

Czechoslovak Mathematical Journal, 27 (102) 1977, Praha

FAMILIES OF SETS AND FUNCTIONS

R. C. SOLOMON, Lancaster

(Received August 11, 1975)

0. This somewhat picaresque article contains various results concerning cardinals of families of sets and functions. In section 1 we define two cardinals as the least cardinals of two such families, and prove inequalities between them and similar cardinals. A connection with the problem of uncountable sets with Hausdorff measure zero is discussed. This section uses several results of ROTHBERGER [7, 8, 9]. In section 2 we construct an ultrafilter with these cardinals. In section 3 it is shown that it is consistent with set-theory that strict inequality holds between two of the cardinals.

N denotes the set of natural numbers, and $P(N)$ the set of its subsets. ${}^N N$ is the set of functions from N to N. If $a, b \in P(N)$ we say $a \subset^* b$ if $a - b$ is finite. If $f, g \in {}^N N$ we say $f <^* g$ if $\{n : f(n) \geqq g(n)\}$ is finite.

1. We define $\varkappa$ to be the least cardinal of a family $F \subset P(N)$ so that the intersection of finitely many members of F is infinite, and for no infinite $a \in P(N)$ is $a \subset^* b$ for all $b \in F$.

We define λ to be the least cardinal of a family $F \subset {}^N N$ which is unbounded under $<^*$.

$\aleph_0 < \varkappa, \lambda \leqq 2^{\aleph_0}$. Rothberger proved [7] that $\lambda \geqq \varkappa$. Both cardinals have many equivalent definitions. In the terminology of [3], $\varkappa = \aleph_1$ is equivalent to the existence of Ω-limits, and $\lambda = \aleph_1$ is equivalent to the existence of (Ω, ω^*)-gaps. Martin's Axiom, in particular the Continuum Hypothesis, implies that $\varkappa = \lambda = 2^{\aleph_0}$.

Two infinite subsets a, b of N are called almost-disjoint if $a \cap b$ is finite. A maximal almost-disjoint family is an infinite subset F of $P(N)$ so that if $a, b \in F$, a and b are almost-disjoint, and if c is any infinite subset of N, $c \cap a$ is infinite for some $a \in F$.

Theorem 1. *Any maximal almost-disjoint family has cardinality at least λ.*

Proof. Suppose F is an almost-disjoint family of cardinality $\mu < \lambda$. Let $F = \{a_\alpha : \alpha < \mu\}$. We can remove at most finitely many members from each a_n to ensure that $\{a_n : n < \omega\}$ is disjoint. For each $\alpha \geqq \omega$, let $f_\alpha \in {}^N N$ be defined by $f_\alpha(n) = m$, where the greatest element of $a_\alpha \cap a_n$ is the m^{th} element of a_n. As $\mu < \lambda$, there is $f \in {}^N N$ so that $f^* > f_\alpha$ for all $\alpha \geqq \omega$. Define $b = \{\text{the } f(n)^{\text{th}} \text{ element o } a_n : n \in \omega\}$. Then b is infinite, and $b \cap a_n$ is finite for all n. If $\alpha \geqq \omega$ $f(n) > f_\alpha(n)$ for all but finitely many n's, and so $b \cap a_\alpha$ is finite. Hence F is not a maximal almost disjoint family.

Theorem 2. *No non-principal ultrafilter q over N is generated by less than λ sets.*

Proof. Suppose $F \subseteq P(N)$ generates a non principal filter, and $|F| = \mu < \lambda$. For each $a \in P(N)$ define $f_a \in {}^N N$ by $f_a(n) =$ the n^{th} member of a. As $\mu < \lambda$, there is $f \in {}^N N$ so that $f^* > f_a$ for all $a \in F$. For each $n \in \omega$ we define finite sets b_n, c_n as follows: $b_1 = \{i : i \leqq f(1)\}$. If we have defined b_j for $j \leqq n$, let $|b_1 \cup \ldots \cup b_n| = m$ and let $r = \max\{b_1 \cup \ldots \cup b_n\}$. Then let $c_n = \{i : r < i \leqq f(m+1)\}$ if this is non-empty, and $c_n = \{r+1\}$ otherwise. If we have defined c_j for $j \leqq n$, let $m = |c_1 \cup \ldots \cup c_n|$ and let $r = \max\{c_1 \cup \ldots \cup c_n\}$. Then let $b_{n+1} = \{i : r < i \leqq f(m+1)\}$ if this is non-empty, and $b_{n+1} = \{r+1\}$ otherwise.

Let $b = \bigcup_{n \geqq 1} b_n$ and $c = \bigcup_{n \geqq 1} c_n$. Then $b \cup c = \omega$. But if b is in the filter generated by F, there is $a \in F$ so that $a \subseteq b$. Certainly $f_a(n) \geqq f_b(n)$ for all n. By the construction of b, for infinitely many m's the $m+1^{\text{th}}$ member of b occurs after $f(m+1)$. Hence $f_a(m+1) \geqq f_b(m+1) > f(m+1)$ for such an m. This contradicts $f_a <^* f$. So b is not in the filter, and by a similar argument c is not either. Hence F cannot generate an ultrafilter.

Now we turn to properties of sets of reals. If $A \subseteq R$, we say A has property C if whenever $\{a_n\}$ is a sequence of positive reals there are intervals I_n, each of length a_n, so that $A \subseteq \bigcup I_n$. A set A is concentrated if there is a countable set D so that whenever G is an open set containing D, $A - G$ is countable.

It is easy to show that a concentrated set has property C. A has property C iff $\mu^h(A) = 0$ for every Hausdorff h-measure, [6]. An uncountable set with property C was first constructed, using the Continuum Hypothesis, by BESICOVITCH [1]. Rothberger showed, [8], that there is a concentrated set iff $\lambda = \aleph_1$. Also he proved, [9], that every set of cardinality less than $\varkappa$ has property C. So in particular if Martin's Axiom and $2^{\aleph_0} > \aleph_1$ are true there are no concentrated sets but there are uncountable sets with property C. The only situation in which we might be unable to construct an uncountable set with property C is if $\varkappa = \aleph_1$, $\lambda > \aleph_1$. But we shall show in section 3 that this is consistent with set-theory.

2. The structure of the space βN is connected with these cardinals.

An ultrafilter $q \in \beta N - N$ is called a μ-p-point if whenever $F \subseteq q$, $|F| < \mu$, there is $a \in q$ so that $a \subset^* b$ for all $b \in F$. $\aleph_1$-p-points are just called p-points, and their existence was proved in [10] assuming the Continuum Hypothesis. In [2], BOOTH proved the existence of $2^{\aleph_0}$-p-points, assuming Martin's Axiom. In fact, $\varkappa = 2^{\aleph_0}$ is sufficient for this. And to construct p-points we need only assume $\lambda = 2^{\aleph_0}$. We show a bit more than this.

Theorem 3. *Assume $\lambda = 2^{\aleph_0} > \aleph_1$. Then there is a p-point $q \in \beta N - N$ which is not an $\aleph_2$-p-point.*

Proof. Let $\{a_\alpha : \alpha < \omega_1\}$ be a sequence of sets so that $\alpha > \beta$ implies $a_\alpha \subset^* a_\beta$ but $a_\beta \not\subset^* a_\alpha$. We will construct a p-point q so that $a_\alpha \in q$ for all $\alpha < \omega$, but for no $a \in q$ is $a \subset^* a_\alpha$ for all α.

Enumerate ${}^N N$ as $\{f_\beta : \omega_1 \leqq \beta < 2^{\aleph_0}\}$. For q to be a p-point it is obviously sufficient that for every $f \in {}^N N$ there is a set $a \in q$ so that f restricted to a is either constant or finite-to-one.

Suppose we have added d_γ for every $\gamma < \beta$, and $d_\gamma = a_\gamma$ for $\gamma < \omega_1$, and f_γ is either constant or finite-to-one on d_γ for $\gamma \geqq \omega_1$. Let $|\beta| = \mu$, and let $\{e_\gamma : \gamma < \mu\}$ consist of all the finite intersections of the d_γ. Our induction assumption is that for every γ there is $\alpha < \omega_1$, with $e_\gamma - a_\alpha$ infinite.

First we try to make f_β constant on d_β:

Case 1. For some $n \in \omega$, for all γ there is α, so that $(e_\gamma \cap f_\beta^{-1}[n]) - a_\alpha$ is infinite. Then we let $d_\beta = f_\beta^{-1}[n]$.

Case 2. Not case 1. Now we try to make f_β finite-to-one on d_β.

Claim. *For all $\gamma < \beta$ there is $\alpha_\gamma < \omega_1$ so that f_β takes infinitely many values on $e_\gamma - a_{\alpha_\gamma}$.*

Proof. Suppose the claim fails at γ. So f_β takes only finitely many values on each $e_\gamma - a_\alpha$. Let $A_\alpha = \{n : f_\beta^{-1}[n] \cap (e_\gamma - a_\alpha) \text{ is infinite}\}$. Then A_α is finite for all α, and as $\alpha > \beta$ implies $a_\alpha \subset^* a_\beta$, A_α is increasing with α.

So for some α_0, A_α must remain fixed for $\alpha \geqq \alpha_0$. Case 1 did not hold. So for all $n \in \omega$, there is γ_n so that for all α, $(e_{\gamma_n} \cap f_\beta^{-1}[n]) - a_\alpha$ is finite. Let $e = \bigcap_{n \in A_{\alpha_0}} e_{\gamma_n}$. Then $(e \cap f_\beta^{-1}[n]) - a_\alpha$ is finite for all α and all $n \in A_{\alpha_0}$. Hence $(e \cap e_\gamma) - a_\alpha$ is finite for all α, contradicting the induction assumption for $e \cap e_\gamma$. This proves the claim.

For every $\gamma < \beta$ we define g_γ as follows:

$$g_\gamma(n) = m \text{ if the } m^{\text{th}} \text{ member of } f_\beta^{-1}[n] \text{ is in } e_\gamma - a_{\alpha_\gamma},\ g_\gamma(n) = 0 \text{ if } (e_\gamma - a_{\alpha_\gamma}) \cap f_\beta^{-1}[n] = \emptyset .$$

Then $g_\gamma(n) > 0$ for infinitely many n's, by the claim. $\mu < \lambda = 2^{\aleph_0}$, so let $g \in {}^N N$ be such that $g^* > g_\gamma$ for all γ. Define d_β to contain the first $g(n)$ members of $f_\beta^{-1}[n]$ for every n. Then obviously f_β restricted to d_β is finite-to-one.

Fix γ. Then there are infinitely many n's such that $0 < g_\gamma(n) < g(n)$, and then the $g_\gamma(n)^{\text{th}}$ member of $f_\beta^{-1}[n]$ will be in $(d_\beta \cap e_\gamma) - a_{\alpha_\gamma}$. So $(e_\gamma \cap d_\beta) - a_{\alpha_\gamma}$ is infinite for every γ. So the induction assumption remains true at β.

After completing this induction up to $2^{\aleph_0}$ we have a p-point that is not an $\aleph_2$-p-point.

Remark. This construction is essentially the same as that in [11], where a Ramsey ultrafilter which is not an $\aleph_2$-p-point was constructed, using Martin's Axiom and $2^{\aleph_0} > \aleph_1$.

3. Though Martin's Axiom implies $\varkappa = \lambda = 2^{\aleph_0}$, it is consistent that λ be any regular cardinal between $\aleph_1$ and $2^{\aleph_0}$, [4]. Here we give a sketch proof that it is consistent that $\lambda = \aleph_2 = 2^{\aleph_0}$ and $\varkappa = \aleph_1$. We need another result of Rothberger, [9], that if $\mu < \varkappa$ then $2^\mu = 2^{\aleph_0}$. The construction is similar to other independence proofs, so instead of giving it in detail we shall just refer the reader to [5], especially section 22, where the consistency of Martin's Axiom and $2^{\aleph_0} > \aleph_1$ is proved.

We start with a ground model $\mathfrak{M}$ in which $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_3$. For each $\alpha \leqq \aleph_2$ we construct a complete Boolean algebra B_α and let $\mathfrak{M}_\alpha = \mathfrak{M}[B_\alpha]$. If α is a limit ordinal B_α is the direct limit of B_β, $\beta < \alpha$. If $\alpha = \beta + 1$ then B_α is constructed so that $\mathfrak{M}_\alpha$ contains a function f_α which is $*>$ all functions in $\mathfrak{M}_\beta$. Let $\mathfrak{N} = \mathfrak{M}_{\aleph_2}$. All the Boolean algebras concerned obey the countable chain condition, and so cardinals are preserved. Hence in $\mathfrak{N}$, $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_3$. So $\varkappa = \aleph_1$. But if $A \subset {}^N N$ and $|A| < \aleph_2$, $A \subset \mathfrak{M}_\alpha$ for some $\alpha < \aleph_2$. Hence $f_{\alpha+1} *> f$ for all $f \in A$. This proves that $\lambda = \aleph_2$.

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Author's address: University of Lancaster, Department of Mathematics, Cartmel College, Bailrigg, Lancaster LA1 4YL, England. *Present address:* Kenyatta University College, P. O. Box 43844, Nairobi, Kenya.