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ON A CONJECTURE OF THE SEMIGROUP OF FULLY INDECOMPOSABLE RELATIONS

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The purpose of this note is to show that the conjecture in [4] does not hold in general. In order to avoid the multiplications of large matrices, we shall use some of the properties of directed graphs to show that, for each integer \( n \geq 5 \), there exists a primitive binary relation \( \rho \) on a set of \( n \) points such that none of the \( \rho^i \)'s is fully indecomposable for \( i = 1, 2, \ldots, n \).

A binary relation on a finite set \( \Omega = \{a_1, a_2, \ldots, a_n\} \) of \( n \) elements, \( n > 1 \), is a subset of \( \Omega \times \Omega = \{(a_i, a_j); a_i, a_j \in \Omega\} \). Let \( B = B(\Omega) \) be the set of all binary relations on \( \Omega \). (When there is no confusion, an element in \( B \) is also called a relation on \( \Omega \), or just a relation.) Then \( B \) is a semigroup with the multiplication defined as follows: for \( \rho \) and \( \tau \) in \( B \), \( (a_i, a_j) \in \rho \tau \) if there is a \( a_k \in \Omega \) such that \( (a_i, a_k) \in \rho \) and \( (a_k, a_j) \in \tau \). Let \( \omega \) be the universal relation on \( \Omega \), i.e., \( \omega = \Omega \times \Omega \), and \( \Lambda = \{(a_i, a_i); a_i \in \Omega\} \). Also, let \( M_n \) denote the set of all \( n \times n \) matrices over the Boolean algebra of \( \{0, 1\} \), then \( M_n \) is a semigroup under the ordinary matrix multiplication, and the map

\[ \rho \rightarrow M(\rho) = (m_{i,j}) \]

where

\[ m_{i,j} = \begin{cases} 1 & \text{if } (a_i, a_j) \in \rho, \\ 0 & \text{otherwise,} \end{cases} \]

is an isomorphism of \( B \) onto \( M_n \). Let \( X_n \) be the set of all directed graphs on \( n \) vertices with allowable loops and simple directed edges. Each matrix in \( M_n \) can be considered as the adjacency matrix of a directed graph \( Y \) in \( X_n \), and it determines \( Y \) uniquely up to isomorphism. Also, each graph in \( X_n \) with labelled vertices determines a unique matrix in \( M_n \) as its adjacency matrix. Hence, there is an one-to-one correspondence among \( B, M_n \) and \( X_n \):

\[ \rho \rightarrow M(\rho) \rightarrow Y(\rho). \]
Let $B_0 = B_0(\Omega)$ consist of all binary relations on $\Omega$ with $\text{pr}_1(\varrho) = \text{pr}_2(\varrho) = \Omega$ where
\[
a_i \varrho = \{ x \in \Omega; (a_i, x) \in \varrho \}, \quad a_i \varrho = \{ y \in \Omega; (y, a_i) \in \varrho \},
\]
\[
\text{pr}_1(\varrho) = \bigcup_{j=1}^{n} a_j \varrho \quad \text{and} \quad \text{pr}_2(\varrho) = \bigcup_{j=1}^{n} a_j \varrho.
\]
Clearly, $B_0$ is a subsemigroup of $B$. This means that, if $\varrho \in B_0$, then none of the columns and none of the rows in $M(\varrho)$ consist of all zeros, and every vertex in the graph $Y(\varrho) \in X_n$ is incident with at least one incoming edge, and at least one outgoing edge. (A loop is considered both as an incoming edge and as an outgoing edge).

An element $\varrho \in B_0$ is said to be decomposable if there is a $\pi$ belonging to the group $\Pi$ of all permutation relations on $\Omega$ such that $M(\pi \varrho \pi^{-1})$ is of the form
\[
\begin{pmatrix}
B & 0 \\
C & D
\end{pmatrix}
\]
where $B$ and $D$ are square matrices of sizes $s \times s$ and $(n - s) \times (n - s)$ respectively, and $1 \leq s \leq n - 1$. Otherwise it is called indecomposable. An element $\varrho \in B_0$ is said to be partly decomposable if there are two elements $\pi_1$ and $\pi_2$ in $\Pi$ such that $M(\pi_1 \varrho \pi_2)$ is of form (1). Otherwise it is called fully indecomposable. Let $I = I(\Omega)$, $F = F(\Omega)$ and $H = H(\Omega)$ be, respectively, the set of all indecomposable relations in $B_0$, the set of all fully indecomposable relations in $B_0$ and the set of all relations in $B_0$ each of which contains a permutation relation. $H$ is called the Hall relations on $\Omega$. $F$ and $H$ are semigroups, in fact, $F$ is a two sided ideal of $H$ (see Theorems 1.2 and 2.3 in [4]). We note that if a matrix contains an $s \times (n - s)$ zero submatrix for some $s$, $1 \leq s \leq n - 1$, then the matrix does not belong to $F$. A relation $\varrho \in B_0$ is said to be primitive if there is an integer $k = k(\varrho)$ such that $\varrho^k = \omega$. Clearly, a primitive relation is indecomposable. The set of all primitive relations in $B_0$ is denoted by $P = P(\Omega)$. As stated in [4], we have
\[
B_0 \triangleright I \triangleright P \triangleright F.
\]
A graph $Y$ in $X_n$ is said to be strongly connected if, for any two vertices in $Y$, there is a directed path in $Y$ from one vertex to the other. If $\varrho$ is decomposable, then the corresponding graph $Y(\varrho)$ is not strongly connected. If $\varrho \in P$, then the corresponding graph $Y(\varrho)$ is strongly connected. However, the converse does not hold, e.g., a directed $n$-cycle is strongly connected, but its corresponding binary relation does not belong to $P$. WIELANDT, in [6], was the first to state that for any $\varrho \in P$, there is a least integer $k = k(\varrho)$, called the index of primitivity of $\varrho$, such that $\varrho^k = \omega$ and $k \leq (n - 1)^2 + 1$. It was proved by many others, e.g., HOLLADAY and VARGA [3]. (Wielandt and others dealt with the $n \times n$ matrices with non-negative real entries, but as far as the primitivity and the index of primitivity concern, they are the same as the $n \times n$ matrices over the Boolean algebra of $\{0, 1\}$). As stated on pp. 162–163

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in [4]: "To any \( q \in P \) there is a least integer \( l_1 = l_1(q) \geq 1 \) such that \( q^{l_1} \in H \), and a least integer \( l_2 = l_2(q) \geq l_1 \) such that \( q^{l_2} \in F \). The problem to find exact upper bounds for \( l_1 \) and \( l_2 \) (in terms of \( n \)) seems to be (at this writing) rather difficult. There are some reasons for the following

**Conjecture.** For any \( q \in P \), we have \( l_2 = l_2(q) \leq n^* \).

Here, we show that the conjecture does not hold in general. To find the exact upper bounds for \( l_1 \) and \( l_2 \) remains to be very difficult and unanswered. Let \( \Omega = \{ a_1, a_2, \ldots, a_5 \} \) and \( q \in B_0 = B_0(\Omega) \) such that

\[
M = M(q) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Then \( M^2, M^3, M^4, M^5, M^6 \) and \( M^{10} \) are respectively

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

Hence, \( q^{11} = \omega \) and \( q \) is primitive. With some suitable permutations of rows and columns, we see none of \( q^i \) for \( i = 1, 2, \ldots, 5 \) belonging to \( F \), e.g., for \( M^4 \), we have

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}.
\]
which is of form (1), and \( \varphi^k \) is partly decomposable, i.e., \( \varphi^k \notin F \). The corresponding graph \( Y(\varphi) \) is

![Graph](image)

In fact, we can prove the following

**Theorem.** For each integer \( n \geq 5 \), there exists a primitive binary relation \( \varphi \) on \( \Omega = \{a_1, a_2, \ldots, a_n\} \) such that none of the \( \varphi^i \) is fully indecomposable for \( i = 1, 2, \ldots, n \).

In order to avoid the multiplication of large matrices in our proof, we shall use some of the elementary properties of directed graphs, namely, the following

**Lemma 1.** Let \( M = M(\varphi) \) be the adjacency matrix of the graph \( Y = Y(\varphi) \). Then, in \( M^t = (m_{i,j}^t) \), \( m_{k,1}^t \) is 1 (is 0) if and only if there is at least one directed path (no directed path) of length \( t \) from the vertex \( k \) in \( Y \) to the vertex 1 in \( Y \).

**Proof.** It follows from the definition of adjacency matrix and the definition of matrix multiplication over the Boolean algebra of \( \{0, 1\} \).

**Lemma 2.** Let \( \varphi \in B_0 \) and \( Y = Y(\varphi) \) be the corresponding graph. If every vertex of \( Y \) is on a \( k \)-cycle (not necessarily a simple \( k \)-cycle), then \( \Delta \leq \varphi^k \).

**Proof.** It follows from Lemma 1.

**Lemma 3.** Let \( Z \) be a directed graph on \( n \) vertices such that each vertex of \( Z \) has a loop and \( Z \) contains a simple \( n \)-cycle, and let \( \mu \) be its corresponding binary relation in \( B_0 = B_0(\Omega) \). Then \( \mu^{n-1} = \omega \) and \( \mu \) is primitive.

**Proof.** It is sufficient to assume \( \mu = \Delta \cup \sigma \) where \( \sigma \) corresponds to the simple \( n \)-cycle in \( Z \). Let \( M = M(\Delta \cup \sigma) \), then \( M = I + X \) where \( I \) is the identity matrix corresponding to \( \Delta \) and \( X = (x_{i,j}) \) is the matrix corresponding to \( \sigma \). Since \( X \) is the adjacency matrix of the simple \( n \)-cycle, we have, for any \( i_1 \) such that \( 1 \leq i_1 \leq n \),

\[
x_{i_1,i_2} = x_{i_2,i_3} = x_{i_3,i_4} = \ldots = x_{i_{n-1},i_n} = x_{i_n,i_1} = 1
\]

where \( i_1, i_2, \ldots, i_n \) are pairwise distinct. By Lemma 1, in \( X^k = (x_{i,j}^k) \), \( x_{i_1,i_{k+1}}^k = 1 \)

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for \( k = 1, 2, \ldots, n - 1 \). Consequently,
\[
M^{n-1} = (I + X)^{n-1} = I + X + X^2 + \ldots + X^{n-1}
\]
consists of all one’s. Hence, \( \mu^{n-1} = \omega \) and \( \mu \) is primitive.

Now the proof of our Theorem goes as follows:

**Case 1.** \( n \) is odd \( \geq 5 \). Let \( m = (n + 3)/2 \). Construct a directed graph \( Y \) on \( n \) vertices with two directed cycles with length \( m \) and length \( m - 1 \) having one edge in common.

Let \( \varrho \) be the binary relation in \( B_0 = B_0(\Omega) \) corresponding to \( Y \). We claim that \( \varrho \) is primitive:

We show \( \Delta \subseteq \varrho^{n+2} \). Every vertex of \( Y \) is on the cycle of length \( n + 2 \).

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots \rightarrow m \rightarrow 1 \rightarrow 2 \rightarrow m + 1 \rightarrow m + 2 \rightarrow \ldots \rightarrow n \rightarrow 1
\]

Hence, by Lemma 2, \( \Delta \subseteq \varrho^{n+2} \).

Let \( Z \) be the directed graph on \( n \) vertices corresponding to \( \varrho^{n+2} \). We show that \( Z \) contains a simple \( (n + 2) \)-cycle. Here the notation \( 1 \rightarrow 1 \) means that, in \( Y \), the vertex 1 reaches the vertex \( m \) by \( (n + 2) \)-length. Since \( m = (n + 3)/2 \) and since the two cycles in \( Y \) differ by 1 length, we have

\[
1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1
\]

Since \( \Delta \subseteq \varrho^{n+2} \), every vertex in \( Z \) has a loop. By Lemma 3, \( \varrho^{n+2} \) is primitive, and so is \( \varrho \).

We claim that none of the \( \varrho^i \)'s belongs to \( F \) for \( i = 1, 2, \ldots, n \):

In the graph \( Y \), we know that both \( m - i \) and \( n - i \) vertices reach 1 by \( (i + 1) \)-length for \( i = 0, 1, 2, \ldots, m - 4 \), i.e., in the matrix \( M^{i+1} = (m_{i+1}^{i+1}) \), we have \( m_{i+1}^{i+1} = 1 \), and the rest of \( (m - i) \)th row and \( (n - i) \)th row are zeros. Since \( M^{i+1} \) contains an \( 2 \times (n - 2) \) zero submatrix, we have \( \varrho^{i+1} \notin F \) for \( i = 0, 1, 2, \ldots, m - 4 \).
Similarly, in $M^{m-2}$, $m^{m-2}_{4,2} = m^{m-2}_{(m+1),2} = 1$ and the rest of 4th row and $(m + 1)$th row are zeros. Since $M^{m-2}$ contains an $2 \times (n - 2)$ zero submatrix, we have $q^{m-2} \notin F$.

Similarly, for $l = 3, 4, \ldots, m + 1$, in $M^{m+l-4}$, $m^{m+l-4}_{4,1} = m^{m+l-4}_{(m+1),1} = m^{m+l-4}_{4,(m+l-2)} = m^{m+l-4}_{(m+1),(m+l-2)} = 1$, and the rest of 4th row and $(m + 1)$th row are zeros. Since $M^{m+l-4}$ contains an $2 \times (n - 2)$ zero submatrix, we have $q^{m+l-4} \notin F$ for $l = 3, 4, \ldots, m + 1$.

Hence, $q$ is primitive and none of $q^i$ belongs to $F$ for $i = 1, 2, \ldots, n$.

Case 2. $n$ is even $> 5$. Let $m = (n + 4)/2$. Construct a directed graph $U$ on $n$ vertices with two directed cycles of length $m$ and length $m - 1$ having two edges in common.

![Diagram of a directed graph](image)

Let $\tau$ be the binary relation in $B_0 = B_0(\Omega)$ corresponding to $U$. We claim that $\tau$ is primitive:

We show $\Delta \subseteq \tau^{n+3}$. Every vertex of $U$ is on the cycle of length $n + 3$.

\[1 \to 2 \to 3 \to \ldots \to m \to 1 \to 2 \to 3 \to m + 1 \to m + 2 \to \ldots \to n \to 1.\]

Hence, by Lemma 2, $\Delta \subseteq \tau^{n+3}$.

Let $V$ be the directed graph on $n$ vertices corresponding to $\tau^{n+3}$. We show that $V$ contains a simple $(n + 3)$-cycle. Since $m = (n + 4)/2$ and since the two cycles in $U$ differ by one length, we have

\[1 \to \star m \to \star n \to \star m - 1 \to \star n - 1 \to \star m - 2 \to \ldots \to \star m + 1 \to \star 4 \to \star 3 \to \star 2 \to \star 1\]

where $1 \to \star m$ means, in $U$, the vertex 1 reaches the vertex $m$ by $(n + 3)$-length. Since $\Delta \subseteq \tau^{n+3}$, every vertex in $V$ has a loop. By Lemma 3, $\tau^{n+3}$ is primitive, and so is $\tau$.

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In the graph $U$, we know that both $m - i$ and $n - i$ vertices reach 1 by $(i + 1)$-length for $i = 0, 1, 2, \ldots, m - 5$, i.e., in the matrix $M^{i+1} = (m^{i+1}_{i,1})$, we have $m^{i+1}_{(m-i),1} = m^{i+1}_{(n-i),1} = 1$, and the rest of $(m - i)$th row and $(n - i)$th row are zero. Since $M^{i+1}$ contains an $2 \times (n - 2)$ zero submatrix, we have $\tau^{i+1} \notin F$ for $i = 0, 1, 2, \ldots, m - 5$.

Similarly, in $M^{m-3}$ and $M^{m-2}$, $m^{m-3}_{k,2} = m^{m-3}_{(m+1),2} = 1$, and $m^{m-2}_{5,3} = m^{m-2}_{(m+1),3} = 1$, and the rest of 5th row and $(m + 1)$th row are zeros. Since each of $M^{m-3}$ and $M^{m-2}$ contains an $2 \times (n - 2)$ zero submatrix, we have $\tau^{m-3} \notin F$ and $\tau^{m-2} \notin F$.

Similarly, for $l = 4, 5, \ldots, m + 1$, in $M^{m+1-5}$, $m^{m+1-5}_{5,1} = m^{m+1-5}_{(m+1),1} = m^{m+1-5}_{5,(m+1-3)} = m^{m+1-5}_{(m+1),(m+1-3)} = 1$, and the rest of 5th row and $(m + 1)$th row are zeros. Since $M^{m+1-5}$ contains an $2 \times (n - 2)$ zero submatrix, we have $\tau^{m+1-5} \notin F$ for $l = 4, 5, \ldots, m + 1$.

Hence, $\tau$ is primitive and none of $\tau^i$ belongs to $F$ for $i = 1, 2, \ldots, n$.

**Remark.** It is well known [6, 5, 3, 2, 1] that if $q \in P$ and $A \subseteq q$, then the index of primitivity is $\leq n - 1$. Also, Proposition 3.2 in [4] states that if $q \in P$ and $A \subseteq q$ then $q \in F$. Consider $q \in P$ and $A \subseteq q$. Since $q \in P$, $Y(q)$ is strongly connected and every vertex in $Y(q)$ is on a cycle, not necessarily a simple cycle. Say, the smallest length of such a cycle in $Y(q)$ is $t$, then $q^t \in P$, $A \subseteq q^t$ and $q^t \in F$. However, unfortunately, in general this $t$ is not the least integer such that $q^t \in F$. In the above example of the directed graph on 5 vertices, we have $t = 7$, but $q^6 \in F$.

**References**


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