Pavel Tomasta
Decompositions of graphs and hypergraphs into isomorphic factors with a given diameter

*Czechoslovak Mathematical Journal*, Vol. 27 (1977), No. 4, 598–608

Persistent URL: [http://dml.cz/dmlcz/101497](http://dml.cz/dmlcz/101497)

Terms of use:

© Institute of Mathematics AS CR, 1977

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://dml.cz](http://dml.cz)
DECOMPOSITIONS OF GRAPHS AND HYPERGRAPHS INTO ISOMORPHIC FACTORS WITH A GIVEN DIAMETER

PAVOL TOMASTA, Bratislava

(Received September 23, 1975)

INTRODUCTION

This paper deals with \( k \)-uniform hypergraphs for \( k \geq 2 \). D. Palumbiny in [2], [3] studies the problem of decomposing a complete graph into factors with equal diameters. He proved in [2] that \( F^2_m(d) = 2m \) for \( m \geq 2 \) and \( 3 \leq d \leq 2m - 1 \), where \( F^2_m(d) \) is the smallest natural number such that the complete graph with \( F^2_m(d) \) vertices can be decomposed into \( m \) factors with a diameter \( d \). Even though his aim was not to find a decomposition into isomorphic factors with the diameter equal to \( d \), the \( m \) factors of his decomposition of the complete graph with \( 2m \) vertices are isomorphic for \( d \) odd.

In this paper we shall systematically study the problem of decomposing a complete \( k \)-uniform hypergraph into isomorphic factors with a given diameter. The study of decompositions of complete graphs into isomorphic factors with a given diameter was initiated by [3], where the problem of decomposing a complete graph into three isomorphic factors with a given diameter \( d \geq 2 \) is considered.

* * *

First we give some definitions. A hypergraph is an ordered pair of sets \( G = (V, H) \), where \( H \subset P(V) \) (the potence of \( V \)). The set \( V \) is called the vertex set, \( H \) is the edge set of \( G \). A path of length \( q \) is a sequence \( x_1, h_1, \ldots, h_q, x_{q+1} \) such that \( x_1, \ldots, x_{q+1} \) are distinct vertices of \( V \), \( h_1, \ldots, h_q \) are distinct edges of \( H \) and \( x_k, x_{k+1} \in h_k \) for \( k = 1, 2, \ldots, q \). The distance \( d(x, y) \) of two vertices \( x \) and \( y \) is the length of the shortest path joining them. The diameter of a hypergraph is defined as

\[
    d = \sup_{x, y \in V} d(x, y)
\]

A hypergraph is said to be a \( k \)-uniform if for each \( h \in H \) we have \( |h| = k \). If the set \( H \) contains all \( k \)-element subsets of \( V \) we say that \( G \) is a complete \( k \)-uniform hypergraph and we denote \( G \) by \( \langle n \rangle_k \), where \( n = |V| \). A factor of \( G \) is a subhyper-
graph of \( G \) which contains all vertices of \( G \). We shall say that \( G_1 = (V_1, H_1) \) and \( G_2 = (V_2, H_2) \) are isomorphic and write \( G_1 \cong G_2 \) if there exists a bijection \( f : V_1 \to V_2 \) such that \( h \in H_1 \) if and only if \( f(h) \in H_2 \).

Denote by \( G_m^k(d) \) the smallest cardinal number such that \( \langle G_m^k(d) \rangle_k \) can be decomposed into \( m \) isomorphic factors with diameter \( d \).

A question arises whether \( G_m^k(d) \) has the same property as the number \( F_m^k(d) \), with the additional condition of isomorphism, i.e. whether \( \langle n \rangle_k \) can be decomposed into \( m \) isomorphic factors with diameter \( d \) if and only if \( n \geq G_m^k(d) \). However, if the factors of a decomposition of \( \langle n \rangle_k \) are mutually isomorphic they have the same number of edges so that \( m \) divides \( \binom{n}{k} \). This implies the negative answer to our question.

We shall call the numbers \( n \) for which \( m \) divides \( \binom{n}{k} \) the suitable numbers. This leads us to the following definition.

**Definition 1.** Let \( H_m^k(d) \) be the smallest cardinal number with the following property: A decomposition of the hypergraph \( \langle n \rangle_k \) into \( m \) isomorphic factors with diameter \( d \) exists if and only if \( n \geq H_m^k(d) \) and \( n \) is a suitable number.

Now we introduce a concept which makes it possible to bring a common point of view into the problems concerning decompositions.

**Definition 2.** Let \( G \) be an arbitrary group of automorphisms of the hypergraph \( \langle n \rangle_k \) and let there exist a surjection \( h : G \to R \), where \( R \) is a decomposition of the hypergraph \( \langle n \rangle_k \) into isomorphic factors, with the following property:

\[
x(h(y)) = h(xy) \quad \text{for every} \quad x, y \in G.
\]

Then we shall say that \( R \) is a decomposition of \( \langle n \rangle_k \) by the group \( G \). If the mapping \( h \) is a bijection then we shall say that \( R \) is a simple decomposition of \( \langle n \rangle_k \) by \( G \). The factor \( h(x) \) will be denoted by \( G_x \).

The following lemma makes it possible to prove a necessary and sufficient condition for the existence of a simple decomposition of \( \langle n \rangle_k \) by an Abelian group of a finite order.

**Lemma 1.** Let \( R \) be a simple decomposition of the hypergraph \( \langle n \rangle_k \) by an Abelian group \( H \) and let a group \( H_1 \) be a subgroup of \( H \). Then there exists a simple decomposition \( R_1 \) of \( \langle n \rangle_k \) by the group \( H/H_1 \).

**Proof.** Denote \( H_0^1 = \bigcup_{a \in H_1} H_a \).

Let \( x, y \in zH_1 \) for some \( z \in H \). Then

\[
x(H_0^1) = \bigcup_{a \in H_1} H_{ax} = \bigcup_{b \in H_1} H_b, \quad y(H_0^1) = \bigcup_{b \in H_1} H_{ay} = \bigcup_{b \in H_1} H_b.
\]
But $x, y \in zH_1$ if and only if $xH_1 = yH_1$ and so we have $x(H_1^{1}) = y(H_1^{1})$ if and only if $x, y \in zH_1$ for some $z \in H$. The desired decomposition $R_1$ is formed by factors $x(H_1^{1})$, where $x$ are representants of the classes of $H/H_1$. The lemma is proved.

**Theorem 1.** Let $H$ be an Abelian group of a finite order $m > 1$ and let $k \geq 3$ be a natural number such that $(m, k!) = 1$. Then the following two statements are equivalent:

1. There exists a group $H_1 \cong H$ such that a hypergraph $\langle n \rangle_k$ has a simple decomposition by the group $H_1$.

2. $m$ divides $\binom{n}{k}$ and divides precisely one of the numbers $n, n - 1, \ldots, n - k + 1$.

**Proof.** Let $R$ be a simple decomposition of the hypergraph $\langle n \rangle_k$ by an Abelian group $H_1$ of order $m$. It is evident that $m$ divides $\binom{n}{k}$ because the factors are isomorphic and so each factor contains the same number of edges.

The condition $(m, k!) = 1$ implies that the number $m$ can be written in the form $m = m_1 \cdot m_2 \cdots \cdot m_k$, where $m_i$ are mutually prime and $m_i$ divides $n - i + 1$. Then we can express the group $H_1$ as the direct product of cyclic groups $H_1 = F_1 \times F_2 \times \cdots \times F_k$, where the order of the group $F_i$ is equal to $m_i$.

Let $m_t > 1$ for some $1 \leq t \leq k$. We shall show that $m_t = 1$ for every $i \neq t$. Lemma 1 implies the existence of a simple decomposition $R_t$ of the hypergraph $\langle n \rangle_k$ by $F_t$. Let $v_1$ be a vertex which is not a fix-point in all elements of $F_t$. Thus there exists $\alpha \in F_t$ such that $\alpha(v_2) = v_2, \alpha(v_2) = v_3, \ldots, \alpha(v_{k-1}) = v_k$. Let now $\beta(v_1) = v_1$ for some $\beta \in F_t$. Then we have by induction $\beta(v_j) = \beta(\alpha(v_{j-1})) = \alpha(v_j) = v_j$ for every $j = 2, 3, \ldots, k$ and thus $\beta$ is conforming with the zero element of $F_t$ on the set $h = \{v_1, v_2, \ldots, v_k\}$. The edge $h$ is contained in a factor $G_\gamma$ of the decomposition $R_t$. However, $h \in \beta(G_\gamma) = G_\beta$. This implies $\beta(\gamma) = \gamma$ and thus $\beta = \varepsilon$.

From this we have that for every vertex which is not a fixpoint with regard to the group $F_t$ there exists a set of vertices which are images of this vertex by mappings $\alpha \in F_t$ and which has a cardinality equal to $m_t$. These sets are either disjoint or identical. Denote by $S$ the system of these disjoint sets. Let $u \in A \in S$ and $\zeta \in F_r$, $r \neq t$. Let $\zeta(u) \in B \in S$. Now let us have $v \in A$. Then there exists $\alpha \in F_r$ such that $\zeta(u) = v$ on the other hand $\zeta(v) = \zeta \alpha(u) = \alpha \zeta(u) \in B$ and we have $\zeta(A) \subseteq B$.

The converse inclusion can be proved analogously and so we have $\zeta(A) = B$. Now let $A = B$. Then $\zeta(u) = \beta(u)$ for some $\beta \in F_r$. Let us have $x \in A$. Then $x = \gamma(u)$ for some $\gamma \in F_r$ and $\zeta(x) = \zeta \gamma(u) = \gamma \zeta(u) = \gamma \beta(u) = \beta \gamma(u) = \beta(x)$. This implies that $\zeta$ and $\beta$ are identical on the set $A$. If we take now some $k$-tuple $g$ from the set $A$ we get $\beta(g) = \zeta(g)$ and thus $\beta = \zeta$, because otherwise the edge $g$ would be included in two different factors which is a contradiction.
However, the equality $\beta = \zeta$ holds if and only if $\beta = \zeta = \varepsilon$. This implies: If 
$\zeta \neq \eta$ then $\zeta(A) \neq \eta(A)$, $\zeta, \eta \in F_r$, and thus for every set $A \in S$ there exists a system of cardinality $m_r$ of disjoint sets of cardinality $m_r$. These systems are for different $A$ either disjoint or identical. This implies that the system $S$ splits into $(n - t + 1)/m_r$ subsystems. This is possible if and only if $m_r = 1$ which we want to prove.

Proof of the sufficient condition: Let $H$ be a group of order $m$ and let $m$ satisfy the
condition (2) in Theorem 1. Thus there exists such $i$ that $m$ divides $n - i$, $0 \leq i \leq k - 1$.

We shall define a group $H_1 \cong H$ by which we shall be able to construct the simple
decomposition just found.

Choose $i$ vertices from $\langle n \rangle_k$ and divide the remaining $n - i$ vertices into $m$-tuples
which will be denote by $A_1, A_2, \ldots, A_m$. To every natural $x \leq z$ and to every $\alpha \in H$
assign a vertex $u_x(\alpha) \in A_x$ such that

$$
\beta u_x(\alpha) = u_x(\beta \alpha) \quad \text{for every} \quad \beta \in H.
$$

In this way we define a group of automorphisms $H_1 \cong H$ on the set consisting
of $n - i$ vertices. On the remaining $i$ vertices define $H_1$ to be point stationary.

A simple decomposition of the hypergraph $\langle n \rangle_k$ by $H_1$ is constructed in the fol-
lowing way: We choose an arbitrary edge $h_1$ and insert it into the factor $G_\alpha$ corre-
sponding to the unit element of $H_1$. We insert the edges $\alpha(h_1)$ into the factors
$G_\alpha = \alpha(G_\alpha)$ for every $\alpha \in H_1$. If we do not use all edges in this way, we insert an
arbitrary one of them — for example $h_2$ — into the factor $G_\alpha$. Then we insert $\alpha(h_2)$
into $\alpha(G_\alpha) = G_\alpha$. We continue in this way, while we exhaust all edges. The decomposi-
tion obtained in this way is obviously a simple decomposition of $\langle n \rangle_k$ by the group
$H_1 \cong H$. The proof is complete.

Remark 1. In the construction described above a weaker condition is sufficient
for the existence of a simple decomposition, namely $(m, k) = 1$. We shall often use
this fact in the sequel.

**Theorem 2.** Let $m, k \geq 3$, $m > k$, $(m, k) = 1$, $d \geq 2$ be integers. Then $G_m^k(d)$
exists and

$$
G_m^k(d) \leq m[(d - 2)(k - 1) + 1] \quad \text{if} \quad d \geq 3,
$$

$$
G_m^k(2) \leq 2m.
$$

Proof. 1. Let $d \geq 3$. Denote $n = mt$, where $t = (d - 2)(k - 1) + 1$. Denote
the vertices of $\langle n \rangle_k$ by $i_j$, $1 \leq i \leq m; 1 \leq j \leq t$. Obviously $m$ divides $\binom{n}{k}$. Moreover,
$m$ divides $n$. Because $(m, k) = 1$, the sufficient condition for the existence of a simple
decomposition of $\langle n \rangle_k$ by a cyclic group $H$ of order $m$, generated by the element
$\beta = (1, \ldots, m_1) \ldots (1, \ldots, m_t)$, is satisfied. Now we shall show that there exists
a special simple decomposition of $\langle n \rangle_k$ by the group $H$, whose factors have the diame-
ters equal to $d$. We construct the factors $G_\alpha, \alpha \in H$ as follows:

601
1. Let the factor $G_e$ corresponding to zero of $H$ contain a path of length $d - 2$ which is formed by the edges
\[ h_i = \{1, \ldots, 1, t_i, \ldots, 1, t_{i+1}\}, \quad \text{where} \quad t_i = 1 + (i - 1)(k - 1); \quad i = 1, \ldots, d - 2; \quad f = \{1, 2, \ldots, k_1\}. \]

2. Let $A_i = \{2, \ldots, m_i\}, i = 1, \ldots, t$; let $\{B \cup \{2_j\}\} \in G_e$ for an arbitrary $(k - 1)$-tuple $B \subset A_i$ and for every $j \neq i$.

3. Let $G_{\alpha} = \alpha(G_e)$ for every $\alpha \in H$.

The diameter of the factors defined in this way is obviously equal to $d$. This is true for $G_e$: the shortest path of length $d$ is formed by the edges $\{2_3, 3_3, \ldots, k_3, 2_1\}$, $f, h_1, h_2, \ldots, h_{d-2}$.

The other factors are isomorphic to $G_e$, thus their diameters are also equal to $d$.

The factors $G_{\alpha}$ need not form a decomposition of the hypergraph $\langle n \rangle_k$. Let $S$ be a system of all edges, which are not included in any factor. The group $H$ decomposes $S$ into disjoint sets of cardinality $m$. Since $m > k$, there exists in each of these sets an edge $h$ that it does not contain the vertices $1_i$, where $1 \leq i \leq t$. Let then $h \in G_e$ and $\alpha(h) \in G_{\alpha}$ for every $\alpha \in H$.

In this way we do not change the diameter of the factors and so we obtain a simple decomposition of $\langle n \rangle_k$ by the cyclic group of order $m$, which implies the existence of the number $G_m^k(d)$ and at the same time its upper bound.

II. Let $d = 2$. We decompose the hypergraph $\langle 2m \rangle_k$ into $m$ factors with diameter two.

Obviously, the sufficient condition for the existence of a simple decomposition of $\langle 2m \rangle_k$ by a group $H$ generated by the element $\beta = (1, \ldots, m_1)(1, \ldots, m_2)$ is satisfied.

Let $A_i = \{1, \ldots, m_i\}, i = 1, 2$ and let $B$ be an arbitrary $(k - 1)$-tuple, $B \subset A_i$. Then let $h = \{B \cup \{j_1\}\} \in G_e$ for $i \neq j$; $i, j = 1, 2$ and $\alpha(h) \in G_{\alpha}$ for every $\alpha \in H$.

The diameter of the factors constructed in this way is obviously two, because $d_G(m_1, m_2) = 2$.

The factors $G_{\alpha}$ need not form a decomposition of the hypergraph $\langle 2m \rangle_k$. Let $S$ be a system of all edges, which are not included in any factor. The group $H$ decomposes $S$ into disjoint sets of cardinality $m$. Since $m > k$, there exists in each of these sets an edge $h$ that it does not contain the vertices $m_1, m_2$. Then let $h \in G_e$ and $\alpha(h) \in G_{\alpha}$ for every $\alpha \in H$.

The edges added to $G_{\alpha}$ in this way do not change the diameter of $G_e$. Thus the factors $G_{\alpha}$ form a simple decomposition of $\langle 2m \rangle_k$ by the cyclic group $H$ of order $m$ into factors with diameter two, which implies the existence of the number $G_m^k(2)$ and its upper bound. The theorem is proved.

In the following considerations the concept of a "simple decomposition by a group is not sufficient. Thus we shall use decompositions by a group of greater order than the number of factors of the decomposition.
Theorem 3. Let $m > k$, $(m, k) > 1$, $k \geq 3$, $d \geq 2$ be integers. Then $G^k_m(d)$ exists and

$$G^k_m(d) \leq km[(d - 2)(k - 1) + 1] \quad \text{if} \quad d \geq 3,$$

$$G^k_m(2) \leq 2mk.$$

Proof. I. Let $d \geq 3$. Denote $n = kmt$, where $t = (d - 2)(k - 1) + 1$. We shall show the existence of a decomposition of $\langle n \rangle_k$ by a cyclic group $H$ of order $mk$ into $m$ factors with diameter $d$.

Denote by $i_j$, $1 \leq i \leq km; 1 \leq j \leq t$ the vertices of $\langle n \rangle_k$. Let the group $H$ be generated by $\beta = (1_1, \ldots, (km)_1) \ldots (1_t, \ldots, (km)_t)$. The construction of the factors with diameter $d$ proceeds as follows: Let the edges $h_j = \{m_j, 2m_j, \ldots, (km)_j\} \in G_\epsilon$ for $1 \leq j \leq t$ where $G_\epsilon$ is the factor corresponding to the zero element of $H$.

Let $g_1 \in A_1^0 = \{1_1, \ldots, (km)_1\}$ be an edge containing the vertices $m_1, (m + 1)_1$. The remaining vertices belonging to $g_1$ let be different from the vertices of $h_1$. The element $\beta^r$ is obviously of order $k$ and so let $\beta^m(g_1) \in G_\epsilon$ for $1 \leq r \leq k$.

Now we insert into the factor $G_\epsilon$ a path of length $d - 2$: Denote

$$f_s = \{m_{s1}, 2m_{s2}, \ldots, (km)_{s(t+1)}\},$$

where $t_s = 1 + s(k - 1), 0 \leq s < d - 2$. Then let $\beta^m(f_s) \in G_\epsilon$ for every $1 \leq r \leq k$.

Denote $A_j = \{i_j \mid 1 \leq i \leq km\} - h_j$ and take an arbitrary $(k - 1)$-tuple $B \subset A_j$. Then insert $\beta^m(B \cup (m + 1)_j)$ into $G_\epsilon$ for every $i \neq j; i, j = 1, 2, \ldots, t, 1 \leq r \leq k$.

The factor $G_\epsilon$ constructed in this way has a diameter equal to $d$, since

$$d_{G_\epsilon}((m + 1)_2, (m(k - q))_2) = d, \quad q \equiv d - 3 \pmod{k}.$$

The shortest path of length $d$ is formed by the edges

$$\{(m + 1)_2, (m + 1)_1, (m + 2)_1, \ldots, (m + k - 1)_1\},$$

$$g_1, f_0, \beta^{m(k-1)}(f_1), \beta^{m(k-2)}(f_2), \ldots, f_{k-1}, \beta^{m(k-1)}(f_{k-1}), \ldots, \beta^m(f_{d-3}).$$

Now define $G_{\beta^i} = \beta^i(G_\epsilon), 0 \leq i < m$. The factors $G_{\beta^i}$ need not form a decomposition of $\langle n \rangle_k$. Let $S$ be a system of all edges which are not in any factor. The group $H$ decomposes $S$ into disjoint sets. Since $m > k$, in each of these sets there exists an edge $g$ that does not contain the vertices of $h_j, 1 \leq j \leq t$. Let us insert the edges $\beta^r(g), 1 \leq r \leq k$ into $G_\epsilon$ and their images in the mappings $\beta^i$ into the factors $G_{\beta^i}$, $0 \leq i < m$. In this way we obviously do not change the diameter of $G_\epsilon$ and the factors $G_{\beta^i}$ form a decomposition of $\langle n \rangle_k$ into $m$ factors with a diameter $d$ which implies the existence of the number $G^k_m(d)$ and its upper bound.

II. If $d = 2$, the proof is analogous to the above one. It is not difficult to prove the existence of a decomposition of $\langle 2mk \rangle_k$ into $m$ isomorphic factors with diameter two, because now we need not construct a path of length $d$. The theorem is proved.
In the end of this part we can say that in Theorems 2 and 3 the problem of
the existence of a decomposition of a complete \(k\)-uniform hypergraph into isomorphic
factors with a diameter \(d\) is affirmatively solved for \(d\) greater or equal to two and for
the number of factors greater the than uniformity of the hypergraph.

DECOMPOSITIONS OF GRAPHS INTO ISOMORPHIC
FACTORS WITH A GIVEN DIAMETER

In this part we shall prove the existence of the number \(G_m^2(d)\) for \(d \geq 3\) and \(m \geq 4\).
We also prove the existence of the number \(H_m^2(d)\) for \(d \geq 3\) and for \(m\) which is a power
of a prime different from two. Moreover, we shall show that the existence of the
number \(H_m^2(d)\) for \(m\) which is not a power of a prime, cannot be proved by the method
of a simple decomposition by an Abelian group.

**Theorem 4.** Let \(t, d\) and \(m\) be integers, \(t > 2, d \in \{3, ..., t+2\}\), \(m > 3\) and \(m\)
odd. Then the graphs \(\langle mt \rangle_2\) and \(\langle mt+1 \rangle_2\) can be decomposed into \(m\) isomorphic
factors with diameter \(d\).

**Proof.** I. First we prove the existence of a decomposition of \(\langle mt \rangle_2\). Denote its
vertices by \(0_1, 1_1, ..., (m-1)_1, ..., 0_t, 1_t, ..., (m-1)_t\). Let \(q \in \{0, 1, ..., t-1\}\).

We construct the factor \(G^q_t\) as follows:

a) \(q > 1\):

Let

\[
X_1 = \{[0_1, 2_1], [1_1, 2_1], [2_2, 3_2]\},
\]

\[
X_2 = \{[0_i, 1_{i+1}], [0_{i+1}, 1_i] \mid 1 \leq i < q + 1\},
\]

\[
X_3 = \{[2_i, (2k + 1)_i] \mid i = 1, ..., t; 2 \leq k \leq \frac{1}{2}(m - 1)\},
\]

\[
X_4 = \{[0_i, 1_i] \mid 3 \leq i \leq q + 1\},
\]

\[
X_5 = \{[2_i, 4_i] \mid 2 \leq i \leq q + 1\},
\]

\[
X_6 = \{[1_i, 2_i], [0_i, 2_i] \mid q + 1 < i \leq t\},
\]

\[
X_7 = \{[2_i, (2k)_j] \mid i \neq j; i, j = 1, ..., t; k \geq 1\},
\]

\[
X_8 = \{[2_i, 3_j] \mid |i - j| > 1; i, j = 1, ..., t\}.
\]

Then let \(G^q_t = \bigcup_{a=1}^{8} X_a\) and the decomposition has the form

\[
R = \{p(G^q_t), ..., p^n(G^q_t)\},
\]

where \(p\) is a cyclic permutation of order \(m\) on the set of vertices of the graph \(\langle mt \rangle_2\)
with the orbits \(\{0_i, 1_i, ..., (m-1)_i\}, i = 1, 2, ..., t\).

\(R\) is relly a decomposition which we can verify by simply summing the edges in
the factor \(G^q_t\) and making sure that no edge repeats. This follows immediately

604
from the construction. The diameter of $G^q_1$ is equal to $q + 3$ because, for example, $d(3_2, 0_{q+1}) = q + 3$. The shortest path of length $q + 3$ is formed by the edges

\[ [3_2, 2_3], [2_2, 2_1], [2_1, 0_1], [0_1, 1_2], [1_2, 0_3], \ldots, [1_q, 0_{q+1}] \] for $q$ even,

\[ [3_2, 2_2], [2_2, 2_1], [2_1, 1_1], [1_1, 0_2], [0_2, 1_3], \ldots, [1_q, 0_{q+1}] \] for $q$ odd.

b) $q = 0$:

Let

\[ Y_1 = \{[0_i, 2_i], [1_i, 2_i] \mid i = 1, 2, \ldots, t\} \],

\[ Y_2 = \{[2_i, (2k)_j], [2_i, 3_j] \mid i \neq j; i, j = 1, 2, \ldots, t; k \geq 1\} \],

\[ Y_3 = \{[2_i, (2k + 1)_j] \mid i = 1, 2, \ldots, t; k \geq 2\} . \]

Then let the factor $G^0_t = Y_1 \cup Y_2 \cup Y_3$. Now we easily obtain the required decomposition by the permutation $p$. The factors of the decomposition have diameter $d = q + 3 = 3$ since, for example,

\[ d(0_1, 0_i) = 3. \]

c) $q = 1$:

Delete the edge $[2_1, 1_1]$ from $G^0_t$ and insert there the edge $[0_1, 1_1]$. Then obviously $d_{G^q_t}(1_1, 0_1) = q + 3 = 4$ and we obtain the required decomposition by the permutation $p$.

II. We construct a decomposition of $\langle mt + 1 \rangle_2$ as follows: We add a vertex $v$ and the edges $[v, 2_i], i = 1, 2, \ldots, t$ into the factor $G^q_t$. It is evident that the diameter is preserved. The other factors of the decomposition are obtained by the permutation $p$, which coincides with $p$ on its definition area, and $v$ is a fix-point. The theorem is proved.

**Corollary 1.** Let $m$ be an arbitrary natural power of a prime different from two. Then $H^2_m(d)$ exists for $d \geq 3$.

**Proof.** Let $m = p^n$, where $p$ is a prime, $p \neq 2$. Let $p$ divide $\binom{N}{2}$. Then

1) either $p^n$ divides $N$,
2) or $p^n$ divides $N - 1$

and so either $p^n t = N$ or $p^n t + 1 = N$ for some $t$. This implies that all suitable numbers are of the form $mt$ or $mt + 1$. By Theorem 4 for an arbitrary $d \geq 3$ there exists $t_0$ such that for every $t \geq t_0$ there exists a decomposition of $\langle mt \rangle_2$ and $\langle mt + 1 \rangle_2$ into $m$ isomorphic factors with a diameter $d$ so that $H^2_m(d) \leq mt_0$.

**Remark 2.** We can take $t_0 = d$. For $d \geq 5$ we can take $t_0 = d - 2$ and thus $H^2_m(d) \leq md - 2m$. In the paper [1] an upper bound of the number $F^2_m(d)$ is found in the form

\[ F^2_m(d) \leq md - m \text{ for } d \geq 3. \]

Since $F^2_m(d) \leq H^2_m(d)$ we have $F^2_m(d) \leq md - 2m$ for $d \geq 5$.
Theorem 5. Let \( m \) be an odd natural number, which is not a power of a prime. Then there exists an arbitrarily large number \( N \) such that \( m \) divides \( \binom{N}{2} \) and there exists no Abelian group which simply decomposes the graph \( \langle N \rangle_2 \) into \( m \) isomorphic factors.

Proof. The assumptions imply that \( m \) can be written in the form \( m = m_1 \cdot m_2 \), where \( m_1, m_2 \neq 1; m_1, m_2 \) are coprime.

The diophantic equation

\[
m_1 x - m_2 y = 1
\]

has obviously an infinite number of solutions. Choose from them a solution \( x_0, y_0 \) which is sufficiently large and denote \( N = m_1 x_0 \). Then put \( m_2 y_0 = N - 1 \). It is evident that \( m \) divides neither \( N \) nor \( N - 1 \). Nonetheless, \( m \) divides \( \binom{N}{2} \).

Now we can use the theorem proved by B. ZELINKA in [4]: The graph \( \langle n \rangle_2 \) can be decomposed by an Abelian group of order \( m \) into \( m \) factors if and only if \( m \) is odd and

1) \( m \) divides \( \frac{1}{2}(n-1) \) or \( m \) divides \( n \), if \( n \) is odd

or

2) \( m \) divides \( \frac{1}{2}n \) or \( m \) divides \( n - 1 \), if \( n \) is even.

Obviously \( m \) does not satisfy the necessary condition of the existence of a simple decomposition of the graph \( \langle N \rangle_2 \) into \( m \) factors by an Abelian group of order \( m \). The theorem is proved.

This implies the following statement:

Corollary 2. The method used in the proof of Theorem 4 — i.e. a simple decomposition by an Abelian group — cannot be used for proving the existence of the number \( H^2_m(d) \) in the case that \( m \) is not a power of prime.

It remains to explore the existence of the number \( G^2_m(d) \) for \( m \) even.

Theorem 6. Let \( m, d \) be natural numbers, \( m \geq 4 \) and even, \( d \geq 3 \). Then the number \( G^2_m(d) \) exists and

\[
G^2_m(d) \leq 2m(d - 1) \quad \text{if} \quad d > 3,
\]

\[
G^2_m(3) \leq 2m.
\]

Proof. Let \( d \geq 4 \) and \( m = 2k, m \geq 4 \). Denote by \( i, j \) the vertices of the graph \( \langle 2m(d - 1) \rangle_2 \), where \( 1 \leq i \leq 2m; 1 \leq j \leq d - 1 \). Put

\[
X_1 = \left\{ [(2i, a_1), [(m + 2)_i, (m + a)_j]] \mid 2 \leq i \leq d - 1; 3 \leq a \leq m \right\},
\]

\[
X_2 = \left\{ [(1, 1, i+1), [(m + 1)_i, (m + 1)_j]] \mid 1 \leq i \leq d - 2 \right\},
\]

606
$X_3 = \{[1_1, 1_1] \mid 2 \leq i \leq d - 1\}$,
$X_4 = \{[1_1, a_1], [(m + 1)_1, (m + a)_1] \mid 2 \leq a \leq m\} \cup \{[1_1, (m + 1)_1]\}$,
$X_5 = \{[1_1, a_1], [(m + 1)_1, (m + a)_1] \mid 2 \leq a \leq m; 2 \leq i \leq d - 1\}$.

Then let $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \subseteq G_e$.

The group $H$ generated by $\alpha = (1_1, \ldots, (2m)_1), \ldots, (1_d, \ldots, (2m)_d)$ decomposes the set of all edges of $\langle 2m(d - 1) \rangle_2$ into disjoint sets of cardinality at least $m$.
Since $m > 2$, in each of these sets there exists an edge that does not contain the vertices of the form $1_i, m_i$, where $1 \leq i \leq d - 1$. Insert this edge into $G_e$. Then $\alpha^j(G_e) = G_{e}, j = 0, 1, \ldots, m$ evidently form the required decomposition, because for example $d_{G_e}(2_1, (m + 1)_d) = d$.

II. If $d = 3$ put $G_e = X_4$. It is evident that $d_{G_e}(2_1, (m + 2)_3) = 3$. The proof is complete.

Remark 3. In [2] it was proved that $G_{e}^2(3) = 2m$.

* *

In all above considerations we were not concerned with decompositions into smaller number of factors than the uniformity of the hypergraph. The following theorem gives a sufficient explanation.

The number $F^k_m(d)$ — if it exists — is the smallest number for which there exists a decomposition of $\langle f^k_m(d) \rangle_k$ into $m$ factors with diameter $d$.

Theorem 7. Let $m, k, d$ be natural numbers $m \leq k$, $k \geq 3$, $d \geq 4$. Then $F^k_m(d)$ does not exist.

Proof. Evidently it is sufficient to prove our statement for $m = k$. So let $m = k$ and let $F_k^m(d)$ exist. Then the hypergraph $\langle F_k^m(d) \rangle_m$ can be decomposed into $m$ factors with diameter $d$. Denote these factors by $F_1, F_2, \ldots, F_m$.

Let $x, y$ be vertices of $F_m$. Let $a, b$ be such vertices that their distance is $d$ in $F_1$. Then the distance between $x$ and either $a$ or $b$ is greater than one. Let this be the case for $x$ and $a$. The second case is analogous. In the factor $F_1$ there exists a vertex $v_i$ such that the distance between $x$ and $v_i$ is greater than one for every $i = 2, \ldots, m - 1$. Then the edge $h = \{x, a, v_2, \ldots, v_{m-1}\} \in F_m$. The distance either between $y$ and $a$ or between $y$ and $b$ in the factor $F_1$ is greater than one. Let $w_i$ be such vertex that the distance between $y$ and $w_i$ in $F_1$ is greater than one for every $i = 2, \ldots, m - 1$. Then it is evident that

$f = \{a, y, w_2, \ldots, w_{m-1}\} \in F_m$ or $g = \{b, y, w_2, \ldots, w_{m-1}\} \in F_m$.

If $f \in F_m$, then the distance between $x$ and $y$ in the factor $F_m$ is less than or equal to two.
Let now $f \in F_1$ and $g \in F_m$. Consider the edge $p = \{x, y, v_2, \ldots, v_{m-1}\}$. Two cases are possible:

1. $p \in F_m$. Then $d_{F_m}(x, y) = 1$.
2. $p \in F_1$. If $q = \{b, v_2, \ldots, v_{m-1}\} \in F_m$ then $d_{F_m}(x, y) \leq 2$. If $q \in F_1$ then $d_{F_1}(a, b) \leq 3$ since $q, g, f \in F_1$, which is a contradiction.

Since the vertices were chosen arbitrarily, we proved that the diameter of $F_m$ is smaller than or equal to two, which contradicts the assumption $d \geq 4$. The theorem is proved.

Theorem 1 involves the assumption $(m, k!) = 1$. A necessary and sufficient condition for the existence of a simple decomposition by an Abelian group was found also if $(m, k) = 1$ and the proof will appear in a future paper.

It remains an unsolved problem whether the number $H^1_m(d)$ exists for not a power of a prime. Another unsolved problem is whether $G^2_m(d) = H^2_m(d)$ for $m, k \geq 2$ and $d \geq 1$. In [5] it was conjectured that $G^2_m(d) = H^2_m(d)$. A further problem is to find an example that $F^m_m(d) \neq G^m_m(d)$.

References


Author’s address: 886 25 Bratislava, Obreancov mieru 49. ČSSR (Matematický ústav SAV).

608