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RIESZ GROUPS WITH A FINITE NUMBER
OF DISJOINT ELEMENTS

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Let $G = (G, +, \leq)$ be an ordered group (henceforth po-group). Two elements $a_1, a_2 \in G$ are disjoint if $a_1 > 0, a_2 > 0, a_1 \wedge a_2 = 0$, where $a_1 \wedge a_2$ denotes $\inf_G(a_1, a_2)$. $A = \{a_1, \dots, a_n\}$ is called a disjoint subset of G if $A \subseteq G^+ \setminus \{0\}$ and any two elements $a_i, a_j \in A, i \neq j$ are disjoint.

P. CONRAD in [1] has studied the structure of a lattice-ordered group G satisfying the following condition:

(c_n) G contains an n -element disjoint subset but does not contain an $(n + 1)$ -element disjoint subset.

l -groups with the property (c_2) had been studied by P. CONRAD and A. CLIFFORD in [2] and by F. ŠIK in [8].

Similarly J. JAKUBÍK in [4] has studied a po-group G having the property:

(q_2) There exist two m -disjoint elements $x, y \in G$ such that if $A \subseteq G$ is an m -disjoint subset and $\text{card } A > 1$, then $A = \{x, y\}$.

($x, y \in G$ will be called m -disjoint if $0 \in x \wedge y$, where $x \wedge y$ is a multilattice operation in G .)

In this paper, Riesz groups with the property (c_n) are investigated.

0. Let $G = (G, +, \leq)$ be a po-group. G will be called an *interpolation group* if to any $a_1, a_2, b_1, b_2 \in G$ satisfying $a_i \leq b_j$ ($i = 1, 2; j = 1, 2$), there exists $c \in G$ such that $a_i \leq c \leq b_j$ ($i = 1, 2; j = 1, 2$) (i.e. the ordered set (po-set) (G, \leq) satisfies the interpolation property). A directed interpolation group is said to be a *Riesz group*. A po-set S satisfying the interpolation property is said to be an *antilattice-ordered set* if it holds: If $a \wedge b[a \vee b]$ exists in S , then $a \wedge b = a$ or $a \wedge b = b[a \vee b = a$ or $a \vee b = b]$. A Riesz group $G = (G, +, \leq)$ is said to be an *antilattice* if the po-set (G, \leq) is an antilattice-ordered set. A Riesz group G is an antilattice if and only if it holds: If $a \wedge b = 0$ ($a, b \in G$), then $a = 0$ or $b = 0$

(See [3, Lemma 7.1].) A po-group G is said to be *regular* if the existence of $\inf_{G^+}(x, y)$ implies the existence of $\inf_G(x, y)$ for $x, y \in G^+$. (G^+ denotes the positive cone of G .) If G is regular, then $c = \inf_{G^+}(x, y)$ implies $c = \inf_G(x, y)$.

If $\emptyset \neq A$ is a subset of a group G , then $\langle A \rangle$ will always denote the subgroup of G that is generated by A .

1. Any interpolation group is regular. (See [6].)

2. Let G be a Riesz group satisfying the property (c_n) ($n \geq 2$) and let $\{a_1, \dots, a_n\}$ be an n -element disjoint subset of G . Then

$$H_i = \{x \in G : x \wedge a_j = 0 \text{ for all } j \neq i\}$$

is an antilattice-ordered convex subsemigroup with 0 of G^+ and $G_i = \langle H_i \rangle$ is an antilattice-ordered directed convex subgroup of G .

Proof. a) Let $x, y \in H_i$, i.e. $x \wedge a_j = y \wedge a_j = 0$ for all $j \neq i$. Then, by [7, Hilfssatz 2], $(x + y) \wedge a_j = 0$ for all $j \neq i$, and hence H_i is a subsemigroup with 0 of G^+ . It is evident that H_i contains with each element x the whole interval $[0, x]$, therefore H_i is convex.

b) By a) and by [5, Theorem 2.1], $G_i = \langle H_i \rangle$ is a directed convex subgroup of G and $G_i^+ = H_i$. Since G_i is convex and G is an interpolation group, it follows that also G_i is an interpolation group. Let us show that G_i is antilattice-ordered. Let $0 \leq x, y \in G_i$ (hence $x, y \in H_i$) and let $x \wedge y = 0$. Then $x = 0$ or $y = 0$, for otherwise $\{x, y, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$ would be an $(n + 1)$ -element disjoint subset of G .

c) From b) and from the regularity of G it follows that H_i is antilattice-ordered.

3. Let G be a group, H_1, \dots, H_n subsemigroups of G , and let A be the subsemigroup of G that is generated by H_1, \dots, H_n . Then $A = H_1 \oplus \dots \oplus H_n$ (see also [1, p. 173]) will mean that

- (1) $A = H_1 + \dots + H_n$,
- (2) $H_i \cap (H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n) = \{0\}$ for all $i = 1, \dots, n$,
- (3) $x_i + x_j = x_j + x_i$ for all $x_i \in H_i, x_j \in H_j, i \neq j$.

4. Let G be a Riesz group, H_1, \dots, H_n ($n \geq 2$) convex subsemigroups with 0 of G^+ such that $H_i \cap H_j = \{0\}$ for all $i \neq j$, and let A be the subsemigroup of G that is generated by H_1, \dots, H_n . Then

- a) $A = H_1 \oplus \dots \oplus H_n$;
- b) if $x = x_1 + \dots + x_n$ where $x_i \in H_i$ ($i = 1, \dots, n$), then $x = x_1 \vee \dots \vee x_n$;
- c) A is convex.

Proof. a) Let $x \in H_i, y \in H_j, i \neq j$. If $0 \leq z \leq x, y$, then the convexity of the subsemigroups H_i, H_j implies $z \in H_i \cap H_j$, hence $z = 0$. Since (by 1) any Riesz group is regular, it is $x \wedge y = 0$. Hence by [7, Hilfssatz 2] it holds $x + y = x \vee y = y + x$, therefore $A = H_1 + \dots + H_n$.

Let $x_i \in H_i \cap (H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n)$. Then $x_i = x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_n$, where $x_k \in H_k, k \in \{1, \dots, n\} \setminus \{i\}$. Thus the preceding part implies $x_i = x_1 \vee \dots \vee x_{i-1} \vee x_{i+1} \vee \dots \vee x_n$.

Let further $x_i \in H_i, x_j \in H_j, i \neq j$. Then $x_j \in \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ implies $0 = x_i \wedge x_j = (x_1 \vee \dots \vee x_{i-1} \vee x_{i+1} \vee \dots \vee x_n) \wedge x_j = x_j$. Hence $0 = x_j$ for all $j \neq i$ and thus also $x_i = 0$. Therefore $A = H_1 \oplus \dots \oplus H_n$.

b) The assertion b) is now evident.

c) Let $0 \leq y \leq x, x \in A$. Then $0 \leq y \leq x_1 + \dots + x_n$, where $x_i \in H_i, i = 1, \dots, n$. G is a Riesz group, hence there exist $0 \leq x'_i \leq x_i (i = 1, \dots, n)$ such that $y = x'_1 + \dots + x'_n$. The subsemigroups H_1, \dots, H_n are convex, therefore $x'_i \in H_i (i = 1, \dots, n)$, i.e. $y \in A$.

If G is a po-group, then $G = G_1 \boxplus \dots \boxplus G_n$ means that G is an o -direct sum of its o -ideals (i.e. normal directed convex subgroups) G_i .

5. Let G be a Riesz group satisfying the property $(c_n) (n \geq 2), \{a_1, \dots, a_n\}$ an n -element disjoint subset of $G, H_i = \{x \in G; x \wedge a_j = 0 \text{ for all } j \neq i\} (i = 1, \dots, n), A$ the subsemigroup of G generated by H_1, \dots, H_n . Then $\langle A \rangle = \langle H_1 \rangle \boxplus \dots \boxplus \langle H_n \rangle$.

Proof. First let us show that $\langle A \rangle$ is the direct sum $\langle H_1 \rangle \oplus \dots \oplus \langle H_n \rangle$ of the subgroups $\langle H_1 \rangle, \dots, \langle H_n \rangle$. Let us prove that for $i \neq j$ it is $H_i \cap H_j = \{0\}$. Let $x \in H_i \cap H_j$. But then $x \wedge a_k = 0$ for all $k = 1, \dots, n$ and since G has the property $(c_n), x = 0$. Hence (by 4) it holds $A = H_1 \oplus \dots \oplus H_n$ and A is convex with 0 . Therefore (by [5, Theorem 2.1]) $\langle A \rangle$ is a directed convex subgroup of G and $\langle A \rangle^+ = A$.

Now let us show that $H_i (i = 1, \dots, n)$ is invariant in A . Let $y \in A, y = h_1 + \dots + h_n, h_i \in H_i (i = 1, \dots, n), x \in H_i$. Then

$$-y + x + y = -h_n - \dots - h_1 + x + h_1 + \dots + h_n,$$

hence by 4

$$\begin{aligned} -y + x + y &= -h_i - h_n - \dots - h_{i+1} - h_{i-1} - \dots - h_1 + h_1 + \dots \\ &\dots + h_{i-1} + h_{i+1} + \dots + h_n + x + h_i = -h_i + x + h_i. \end{aligned}$$

Let $j \neq i$. Then $0 = x \wedge a_j = -h_i + (x \wedge a_j) + h_i$, therefore by [7, Hilfssatz 2]

$$\begin{aligned} 0 &= (-h_i + x + h_i) \wedge (-h_i + a_j + h_i) = \\ &= (-h_i + x + h_i) \wedge (-h_i + h_i + a_j) = (-h_i + x + h_i) \wedge a_j. \end{aligned}$$

Hence $-h_i + x + h_i \in H_i$. This implies by [5, Theorem 3.1] that $\langle H_i \rangle$ ($i = 1, \dots, n$) is a normal subgroup of $\langle A \rangle$.

Now let us prove that $\langle A \rangle = \langle H_1 \rangle + \dots + \langle H_n \rangle$. Let $z \in \langle A \rangle$. Then $z = x - y$, where $x, y \in A$, i.e. $x = h_1^{(x)} + \dots + h_n^{(x)}$, $y = h_1^{(y)} + \dots + h_n^{(y)}$, $h_i^{(x)}, h_i^{(y)} \in H_i$, $i = 1, \dots, n$. Thus $z = h_1^{(x)} + \dots + h_n^{(x)} - h_1^{(y)} - \dots - h_n^{(y)} \in \langle \langle H_1 \rangle, \dots, \langle H_n \rangle \rangle$. Since $\langle H_i \rangle = H_i - H_i$ ($i = 1, \dots, n$) and since all elements of distinct subsemigroups H_i, H_j commute, it holds also that all elements of $\langle H_i \rangle, \langle H_j \rangle$ commute. Hence $\langle \langle H_1 \rangle, \dots, \langle H_n \rangle \rangle = \langle H_1 \rangle + \dots + \langle H_n \rangle$, and so $\langle A \rangle \subseteq \langle H_1 \rangle + \dots + \langle H_n \rangle$. The converse inclusion is evident.

Let now $x \in \langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle$. Then

$$x = h_1 - h'_1 + \dots + h_{i-1} - h'_{i-1} + h_{i+1} - h'_{i+1} + \dots + h_n - h'_n,$$

where $h_j, h'_j \in H_j$ ($j = 1, \dots, i-1, i+1, \dots, n$), and thus

$$\begin{aligned} x &= h_1 + \dots + h_{i-1} + h_{i+1} + \dots + h_n - h'_1 - \dots - h'_{i-1} - h'_{i+1} - \dots - h'_n = \\ &= (h_1 + \dots + h_{i-1} + h_{i+1} + \dots + h_n) - \\ &\quad - (h'_1 + \dots + h'_{i-1} + h'_{i+1} + \dots + h'_n). \end{aligned}$$

Hence $x \in \langle H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n \rangle$. Therefore $\langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle = \langle H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n \rangle$.

It is clear that $B^{(i)} = H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n$ is a subsemigroup with 0 of G^+ . Indeed, all elements from any distinct summands commute. Let us show that $B^{(i)}$ is convex. Let $0 \leq y \leq h_1 + \dots + h_{i-1} + h_{i+1} + \dots + h_n$, $h_j \in H_j$, $j = 1, \dots, i-1, i+1, \dots, n$. Since G is a Riesz group, $y = \bar{h}_1 + \dots + \bar{h}_{i-1} + \bar{h}_{i+1} + \dots + \bar{h}_n$ where $0 \leq \bar{h}_j \leq h_j$, $j = 1, \dots, i-1, i+1, \dots, n$. H_j being convex implies $\bar{h}_j \in H_j$, and hence $y \in B^{(i)}$.

Now, since G is a Riesz group it follows by [5, Theorems 2.1, 2.4, 3.1]

$$\begin{aligned} (\langle H_i \rangle \cap (\langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle))^+ &= \\ &= (\langle H_i \rangle \cap \langle H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n \rangle)^+ = \\ &= \langle H_i \rangle^+ \cap \langle H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n \rangle^+ = \\ &= H_i \cap (H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n) = \{0\}. \end{aligned}$$

The subgroup $\langle H_i \rangle \cap (\langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle)$ is directed, thus also $\langle H_i \rangle \cap (\langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle) = \{0\}$. Therefore $\langle A \rangle = \langle H_1 \rangle \oplus \dots \oplus \langle H_n \rangle$.

Let now $0 \leq x \in \langle A \rangle$, $x = x_1 + \dots + x_n$, $x_i \in \langle H_i \rangle$, $i = 1, \dots, n$. Since the subgroups $\langle H_i \rangle$ are directed, it holds

$$0 \leq x_1 + \dots + x_n \leq \bar{x}_1 + \dots + \bar{x}_n,$$

where $\bar{x}_i \in U(x_i, 0) \cap \langle H_i \rangle$, $i = 1, \dots, n$. ($U(x, y)$ means the set of all upper bounds

of a subset $\{x, y\}$ in G .) And since G is a Riesz group, there exist $0 \leq u_i \leq \bar{x}_i$ ($i = 1, \dots, n$) such that

$$x_1 + \dots + x_n = u_1 + \dots + u_n.$$

$\langle H_i \rangle$ being convex, it is $u_i \in \langle H_i \rangle$, $i = 1, \dots, n$. And since $\langle A \rangle$ is the direct sum of its subgroups $\langle H_i \rangle$, $0 \leq x_i = u_i$, $i = 1, \dots, n$. Therefore $\langle A \rangle = \langle H_1 \rangle \boxplus \dots \boxplus \langle H_n \rangle$.

6. Let A be a Riesz group such that $A = A_1 \boxplus \dots \boxplus A_n$, where A_1, \dots, A_n are antilattices, $A_i \neq \{0\}$ ($i = 1, \dots, n$). Then A satisfies the condition (c_n) .

Proof. Let $x_i \in A_i^+ \setminus \{0\}$, $i = 1, \dots, n$. Then, by the proof 4a), $x_i \wedge x_j = 0$ for $i \neq j$. Thus A contains an n -element disjoint subset. Let $Y = \{y_1, \dots, y_n, y_{n+1}\}$ be an $(n+1)$ -element disjoint subset in A , $y_j = y_{j1} + \dots + y_{jn}$, $y_{ji} \in A_i$, $j = 1, \dots, n$, $n+1$, $i = 1, \dots, n$. But then for each $j \neq k$ and for each $i = 1, \dots, n$ it is $y_{ji} \wedge y_{ki} = 0$. Since every A_i is an antilattice, $y_{ji} = 0$ or $y_{ki} = 0$. Therefore it must hold that at most one of the $y_{1i}, \dots, y_{ni}, y_{n+1,i}$ is strictly positive. But this means that some of the elements y_1, \dots, y_n, y_{n+1} is equal to 0, thus Y is not a disjoint subset in A . Therefore A has the property (c_n) .

Throughout the following G will denote a Riesz group with the property (c_n) ($n \geq 2$), $\{a_1, \dots, a_n\}$ an n -element disjoint subset in G , $H_i = \{x \in G; x \wedge a_j = 0 \text{ for all } j \neq i\}$ ($i = 1, \dots, n$), A a subsemigroup of G that is generated by the subsemigroups H_1, \dots, H_n .

7. Let $0 < b_i \in H_i$, $i = 1, \dots, n$, and let $K_i = \{x \in G; x \wedge b_j = 0 \text{ for all } j \neq i\}$. Then $H_i = K_i$, $i = 1, \dots, n$.

Proof. Let $x \in H_i$, $i \neq j$ and let $0 \leq y \in G$ such that $y \leq b_j$, x . Then the convexity of H_i, H_j yields $y \in H_j \cap H_i$, hence $y = 0$. Therefore $x \wedge b_j = 0$ for all $j \neq i$, and so $x \in K_i$. This implies $H_i \subseteq K_i$.

Similarly $K_i \subseteq H_i$.

8. If $\{b_1, \dots, b_n\}$ is an n -element disjoint subset of G , then $\{b_1, \dots, b_n\} \subseteq A$. Moreover, there exists a permutation φ on $\{1, \dots, n\}$ such that $b_i \in H_{i\varphi}$ for all $i = 1, \dots, n$.

Proof. Let $i \neq j$ and let $\neg(b_k \wedge a_i = 0)$, $\neg(b_k \wedge a_j = 0)$. Since G is a Riesz group, there exist c_{ki}, c_{kj} such that $0 < c_{ki} \leq b_k$, a_i ; $0 < c_{kj} \leq b_k$, a_j . But then $\{b_1, \dots, b_{k-1}, c_{ki}, c_{kj}, b_{k+1}, \dots, b_n\}$ is an $(n+1)$ -element disjoint subset of G . This means that it holds $\neg(b_k \wedge a_i = 0)$ for at most one $i \in \{1, \dots, n\}$, therefore $b_k \in H_i$ for some i . But since H_i is antilattice-ordered, no two of the b_k 's can belong to the same H_i .

9. $\langle A \rangle$ is a normal subgroup of G . *

Proof. Let $i \neq j$, $x, y \in G$, $y \leq -x + a_i + x$, $y \leq -x + a_j + x$. Then $x + y - x \leq a_i, a_j$, hence $x + y - x \leq 0$. This means $y \leq -x + x = 0$. Therefore it holds $(-x + a_i + x) \wedge (-x + a_j + x) = 0$. Hence by 8, $0 < -x + a_i + x \in H_{i\varphi}$ for all i , where φ is a permutation on $\{1, \dots, n\}$. Thus by 7,

$$-x + A + x = -x + (H_1 \oplus \dots \oplus H_n) + x \subseteq H_{1\varphi} \oplus \dots \oplus H_{n\varphi} = A.$$

Then, by [5, Theorem 3.1], $\langle A \rangle$ is normal in G .

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