Czechoslovak Mathematical Journal

Tomáš Kepka

Structure of weakly abelian quasigroups

Czechoslovak Mathematical Journal, Vol. 28 (1978), No. 2, 181–188

Persistent URL: `http://dml.cz/dmlcz/101525`

Czechoslovak Mathematical Journal, 28 (103) 1978, Praha

STRUCTURE OF WEAKLY ABELIAN QUASIGROUPS

TOMÁŠ KEPKA, Praha

(Received May 26, 1975)

This paper is concerned with some properties of weakly abelian quasigroups and, it is a continuation of the last section from [4]. It will be shown (among other things) that the structure of weakly abelian D-quasigroups is very similar to that of distributive quasigroups.

First we recall some notions and definitions. A quasigroup Q is called

– abelian if it satisfies the identity $ab \,.\, cd = ac \,.\, bd$ (a)
– an LWA-quasigroup if it satisfies $aa \,.\, bc = ab \,.\, ac$ (b)
– an RWA-quasigroup if it satisfies $bc \,.\, aa = ba \,.\, ca$ (c)
– a WA-quasigroup if it satisfies both (b) and (c)
– a D-quasigroup if it satisfies $ab \,.\, ca = ac \,.\, ba$ (d)
– a WAD-quasigroup if it satisfies (b), (c) and (d)
– unipotent if $aa = bb$ for all $a, b \in Q$
– idempotent if $aa = a$ for every $a \in Q$
– distributive if it is an idempotent WA-quasigroup
– triabelian if every its subquasigroup which is generated by at most three elements is abelian.

If G is a groupoid and $x \in G$ then L_x and R_x will denote the left and right translation by x, respectively. If Q is a quasigroup and $x \in Q$ then $f(x)$ and $e(x)$ will be the left and right local unit of x, respectively. If Q is a commutative Moufang loop then $N(Q)$ denotes the nucleus of Q and a mapping g of Q into Q is said to be *nuclear* provided that $x^{-1} \,.\, g(x) \in N(Q)$ for each $x \in Q$. As is easy to see, the set of all nuclear permutations of Q is a subgroup in the symmetric group S_Q.

The following lemma is an easy consequence of [5, Theorem 2].

Lemma 1. *Let Q be a commutative loop and g a mapping of Q into Q. Then the following conditions are equivalent:*

(i) $(g(a) \,.\, a)(bc) = (g(a) \,.\, b)(ac)$ *for all* $a, b, c \in Q$.
(ii) *Q is a Moufang loop and g is nuclear.*

Theorem 1. *Let Q be a quasigroup. The following conditions are equivalent:*

(i) *Q is a WA-quasigroup and there is $a \in Q$ such that $ab \,.\, ca = ac \,.\, ba$ for all $b, c \in Q$.*

(ii) *Q is a WA-quasigroup and Q is isotopic to a commutative Moufang loop.*

(iii) *Q is a WA-quasigroup and Q is isotopic to a Moufang loop.*

(iv) *There are a commutative Moufang loop $Q(\circ)$, $\varphi, \psi \in \mathrm{Aut}\, Q(\circ)$ and $g \in Q$ such that $\varphi\psi = \psi\varphi$, $\varphi\psi^{-1}$ is a nuclear automorphism of $Q(\circ)$ and $ab = (\varphi(a) \circ \psi(b)) \circ g$ for all $a, b \in Q$.*

(v) *Q is a WAD-quasigroup.*

Proof. (i) implies (ii). If $b, c \in Q$ then $(aa \,.\, ab)(ac \,.\, aa) = (aa \,.\, ab)(aa \,.\, ca) = (aa \,.\, aa)(ab \,.\, ca) = (aa \,.\, aa)(ac \,.\, ba) = (aa \,.\, ac)(aa \,.\, ba) = (aa \,.\, ac)(ab \,.\, aa)$. Hence $(aa \,.\, x)(y \,.\, aa) = (aa \,.\, y)(x \,.\, aa)$ for all $x, y \in Q$ and we can use [4, Proposition 4.8] and Lemma 1.

The implication (ii) implies (iii) is trivial.

(iii) implies (iv). Let $x \in Q$ and $a \circ b = R_{xx}^{-1}(a) \,.\, L_{xx}^{-1}(b)$ for all $a, b \in Q$. As is proved in [4], $Q(\circ)$ is a CI-loop. However, $Q(\circ)$ is a Moufang loop, hence it is an IP-loop, and consequently $Q(\circ)$ is commutative. The rest follows from [4, Proposition 4.8, Theorem 4.9].

(iv) implies (v). Since $\varphi\psi^{-1}$ is a nuclear mapping and $\varphi\psi = \psi\varphi$, $\varphi^2\psi^{-2} = \varphi\psi^{-1}\varphi\psi^{-1}$ is nuclear. According to Lemma 1, we can write $ab \,.\, ca = (((\varphi^2(a) \circ \varphi\psi(b)) \circ \varphi(g)) \circ ((\psi\varphi(c) \circ \psi^2(a)) \circ \psi(g))) \circ g = (((\varphi^2(a) \circ \varphi\psi(b)) \circ (\varphi\psi(c) \circ \psi^2(a))) \circ (\varphi(g) \circ \psi(g))) \circ g = (((\varphi^2(a) \circ \varphi\psi(c)) \circ \varphi(g)) \circ ((\psi\varphi(b) \circ \psi^2(a)) \circ \psi(g))) \circ g = ac \,.\, ba$ for all $a, b, c \in Q$. Now the proof of the theorem is complete, the last implication being trivial.

Let Q be a WAD-quasigroup. A tetrad $(Q(\circ), \varphi, \psi, g)$ is called *an arithmetical form of Q* if it satisfies the condition (iv) from Theorem 1.

Lemma 2. *Let Q be a WAD-quasigroup and $x \in Q$. Then there exists an arithmetical form $(Q(\circ), \varphi, \psi, g)$ of Q such that the element $xx \,.\, xx$ is equal to the unit of $Q(\circ)$ and $g = (xx \,.\, xx)(xx \,.\, xx)$.*

Proof. The lemma follows from the proof of [4, Theorem 4.9].

Proposition 1. *Let Q be a commutative WA-quasigroup. Then Q is a WAD-quasigroup and $\varphi = \psi$ for every arithmetical form $(Q(\circ), \varphi, \psi, g)$ of Q.*

Proof. Obvious.

Proposition 2. *Every unipotent WA-quasigroup is abelian.*

Proof. Let Q be a unipotent WA-quasigroup. There is $j \in Q$ such that $aa = j$ for each $a \in Q$. Put $x \circ y = R_j^{-1}(x) \,.\, L_j^{-1}(y)$ for all $x, y \in Q$. Then $Q(\circ)$ is a loop, j is the unit of $Q(\circ)$ and $(\alpha(a) \circ a) \circ (b \circ c) = (\alpha(a) \circ b) \circ (a \circ c)$ for all $a, b, c \in Q$

and $\alpha = R_j L_j^{-1}$ (see [4, Proposition 4.8]). Further, $\alpha(a) \circ a = R_j^{-1} R_j L_j^{-1}(a)$ . . $L_j^{-1}(a) = L_j^{-1}(a) \,.\, L_j^{-1}(a) = j$ for every $a \in Q$, and hence $c = \alpha(a) \circ (a \circ c)$ for every $c \in Q$. On the other hand,

$$\alpha^2(a) \circ (\alpha(a) \circ (c \circ a)) = (\alpha^2(a) \circ j) \circ (\alpha(a) \circ (c \circ a)) =$$
$$= (\alpha^2(a) \circ \alpha(a)) \circ (c \circ a) = (\alpha^2(a) \circ c) \circ (\alpha(a) \circ a) = \alpha^2(a) \circ c \,.$$

Thus $c = \alpha(a) \circ (c \circ a) = \alpha(a) \circ (a \circ c)$ and $a \circ c = c \circ a$. We have proved that $Q(\circ)$ is commutative, and therefore $Q(\circ)$ is a commutative Moufang loop by [4, Proposition 4.1]. By Lemma 1, $\alpha(a) = a^{-1}$ is a nuclear mapping, so that $a^{-2} \in N(Q(\circ))$ for every $a \in Q$. However, since $Q(\circ)$ is a commutative Moufang loop, $a \circ a \circ a \in N(Q(\circ))$ for every $a \in Q$ ([2, pg. 128]), and so $N(Q(\circ)) = Q(\circ)$. Thus $Q(\circ)$ is an abelian group and Q is an abelian quasigroup by [4, Proposition 4.3].

Proposition 3. *Let Q be a WA-quasigroup such that the mapping $x \mapsto xx$ is a permutation. Then Q is a WAD-quasigroup.*

Proof. Let $\alpha(x) = xx$ and $a * b = \alpha^{-1}(ab)$ for all $x, a, b \in Q$. Then $(a * b) * * (a * c) = \alpha^{-1}(\alpha^{-1}(ab) \,.\, \alpha^{-1}(ac)) = (\alpha^{-2}(a) \,.\, \alpha^{-2}(b)) (\alpha^{-2}(a) \,.\, \alpha^{-2}(c)) = (\alpha^{-2}(a) \,.$ $.\, \alpha^{-2}(a)) (\alpha^{-2}(b) \,.\, \alpha^{-2}(c)) = \alpha^{-1}(a) \,.\, (\alpha^{-2}(b) \,.\, \alpha^{-2}(c)) = a * (b * c)$, since Q is a WA-quasigroup and α is an automorphism of Q. Similarly we can show $(b * a) * * (c * a) = (b * c) * a$, and hence $Q(*)$ is a distributive quasigroup. As is easy to see, α is an automorphism of $Q(*)$ and $ab \,.\, ca = (\alpha^2(a) * \alpha^2(b)) * (\alpha^2(c) * \alpha^2(a))$ for all $a, b, c \in Q$. Hence it is enough to prove that every distributive quasigroup is a D-quasigroup. However, every distributive quasigroup is triabelian, as follows from a more general theorem proved by BELOUSOV. Here we give an other direct proof of this theorem.

Theorem. [1, pg. 147]. *Let Q be a distributive quasigroup and let $a, b, c, d \in Q$ be such that $ab \,.\, cd = ac \,.\, bd$. Then the subquasigroup generated by these elements is abelian.*

Proof. The proof is divided into several lemmas. First, it is easy to observe that the group generated by all the translations L_x, R_x, $x \in Q$, is contained in the group Aut Q. If $a, b, \ldots \in Q$ then $S(a, b, \ldots)$ will denote the subquasigroup generated by $a, b, \ldots$.

Lemma. *Let $a, b, c, d \in Q$ and $ab \,.\, cd = ac \,.\, bd$. Then $ax \,.\, yd = ay \,.\, xd$ for all $x, y \in S(a, b, c, d)$.*

Proof. If $h = R_d^{-1} L_{ac}^{-1} R_{cd} L_a$ then $au \,.\, cd = ac \,.\, h(u)\, d$ for every $u \in Q$. Further, $ab \,.\, cd = ac \,.\, bd$, $aa \,.\, cd = ac \,.\, ad$, $ac \,.\, cd = ac \,.\, cd$ and $ad \,.\, cd = ac \,.\, dd$. Hence $h(a) = a$, $h(b) = b$, $h(c) = c$ and $h(d) = d$. Since h is an automorphism, the set $P = \{x \in Q \mid h(x) = x\}$ is a subquasigroup and $S(a, b, c, d) \subseteq P$. Thus $ax \,.\, cd =$

$= ac \,.\, xd$ for every $x \in S(a, b, c, d)$. By symmetry, $ax \,.\, bd = ab \,.\, xd$ for every $x \in S(a, b, c, d)$ and the result easily follows.

Lemma. *Let $a, b, c, d \in Q$ and $ab \,.\, cd = ac \,.\, bd$. Then $zx \,.\, yd = zy \,.\, xd$ and $ax \,.\, yv = ay \,.\, xv$ for all $x, y \in S(a, b, c, d)$, $z \in S(a, b, c)$ and $v \in S(b, c, d)$.*

Proof. Let $x, y \in S(a, b, c, d)$. By the preceding lemma, $ax \,.\, yd = ay \,.\, xd$. The set $P = \{u \in Q \mid ux \,.\, yd = uy \,.\, xd\}$ is a subquasigroup and $a, x, y \in P$. Hence $ux \,.\, yd = uy \,.\, xd$ for every $u \in S(a, x, y)$. Now let $y = ba$. Then $b \in S(a, x, y)$, and so $bx \,.\, (ba \,.\, d) = (b \,.\, ba) \,.\, xd$. From this we obtain the equality $bp \,.\, qd = bq \,.\, pd$ for all $p, q \in (b, x, ba, d)$. If $x = c$ then $S(b, x, ba, d) = S(a, b, c, d)$ and $bp \,.\, qd = bq \,.\, pd$ for all $p, q \in S(a, b, c, d)$. Using the symmetry, we get the equality $cp \,.\, qd = cq \,.\, pd$ for all $p, q \in S(a, b, c, d)$. The rest of the proof is now clear.

Lemma. *Q is a D-quasigroup.*

Proof. Since $aa \,.\, bc = ab \,.\, ac$, $ab \,.\, ca = ac \,.\, ba$ by the preceding lemma.

Now to the proof of the theorem itself. Let $a, b, c, d \in Q$ and $ab \,.\, cd = ac \,.\, bd$. By the preceding lemmas, $zx \,.\, yd = zy \,.\, xd$ and $dx \,.\, yd = dy \,.\, xd$ for all $x, y \in S(a, b, c, d)$ and $z \in S(a, b, c)$. Hence $ux \,.\, yd = uy \,.\, xd$ for all $u, x, y \in S(a, b, c, d)$. Similarly, $ax \,.\, yu = ay \,.\, xu$ for all $u, x, y \in S(a, b, c, d)$. In particular, $ad \,.\, bc = ab \,.\, dc$ and $ac \,.\, db = ad \,.\, cb$. Hence, as was proved above, $ux \,.\, yc = uy \,.\, xc$ and $ux \,.\, yb = uy \,.\, xb$ for all $u, x, y \in S(a, b, c, d)$. From this, $dc \,.\, ab = da \,.\, cb$, so that $ux \,.\, ya = uy \,.\, xa$ for all $u, x, y \in S(a, b, c, d)$ and the result follows easily.

Let Q be a quasigroup. A mapping g (h) of Q into Q is called *left* (*right*) *regular* if there exists a mapping g^* (h^*) such that $g(xy) = g^*(x) \,.\, y$ ($h(xy) = x \,.\, h^*(y)$). A mapping k is called *middle regular* if there is a mapping k^* such that $k(x) \,.\, y = x \,.\, k^*(y)$. By L_Q we shall denote the set of all the left regular mappings and L_Q^* will be the set of all the corresponding mappings g^*. Similarly we define R_Q, R_Q^*, F_Q and F_Q^*. As is easy to see, mappings from L_Q, L_Q^*, R_Q, R_Q^*, F_Q and F_Q^* are permutations and all these sets are subgroups in S_Q.

Lemma 3. *Let Q be a WAD-quasigroup and let $(Q(\circ), \varphi, \psi, g)$ be an arithmetical form of Q. Then*

(i) $L_Q = L_Q^* = R_Q = R_Q^* = F_Q = F_Q^*$,

(ii) *if k is a mapping of Q into Q then $k \in L_Q$ iff there is $a \in N(Q(\circ))$ such that $k(x) = x \circ a$ for every $x \in Q$.*

Proof. Let $k \in L_Q$. Then $k((\varphi(a) \circ \psi(b)) \circ g) = (\varphi\, k^*(a) \circ \psi(b)) \circ g$ for all $a, b \in Q$. Substituting $\psi^{-1}(g^{-1})$ for b, we obtain the equality $k\, \varphi(a) = \varphi\, k^*(a)$. Hence $k((a \circ b) \circ g) = (k(a) \circ b) \circ g$ for all $a, b \in Q$, so that $k(a \circ g) = k(a) \circ g$ and $k(a \circ b) = k(a) \circ b$. Thus $k(b) = k(j) \circ b$ and the equality $k(j) \circ (a \circ b) = (k(j) \circ a) \circ b$ yields $k(j) \in N(Q(\circ))$. The rest is clear.

Let Q be a commutative Moufang loop. We shall say that Q is 3-elementary if $x^3 = j$ for every $x \in Q$, where j is the unit of Q.

Proposition 4. *Let Q be a commutative WA-quasigroup. The following conditions are equivalent:*

(i) $aa \,.\, ax = bb \,.\, bx$ *for all* $a, b, x \in Q$.

(ii) $aa \,.\, ax = xx \,.\, xx$ *for all* $a, x \in Q$.

(iii) *Q is isotopic to a commutative 3-elementary Moufang loop.*

(iv) *Every commutative Moufang loop isotopic to Q is 3-elementary.*

Proof. The implication (i) implies (ii) is trivial.

(ii) implies (iii). Let $(Q(\circ), \varphi, \varphi, g)$ be an arithmetical form of Q (see Proposition 1) and j the unit of $Q(\circ)$. Then

$$(((\varphi^2(a) \circ \varphi^2(a)) \circ \varphi^2(a)) \circ (\varphi(g) \circ \varphi(g))) \circ g = aa \,.\, aj = jj \,.\, jj =$$
$$= (\varphi(g) \circ \varphi(g)) \circ g \,.$$

Hence $\varphi^2(a) \circ \varphi^2(a) \circ \varphi^2(a) = j$, so that $a \circ a \circ a = j$.

(iii) implies (iv). As is well known, isotopic commutative Moufang loops are isomorphic.

(iv) implies (i). Let $(Q(\circ), \varphi, \varphi, g)$ be an arithmetical form of Q. Then $aa \,.\, ax = ((((\varphi^2(a) \circ \varphi^2(a)) \circ (\varphi^2(a) \circ \varphi^2(x))) \circ (\varphi(g) \circ \varphi(g))) \circ g = (\varphi^2(x) \circ (\varphi(g) \circ \varphi(g))) \circ g$ by (iv) and with respect to the diassociativity of $Q(\circ)$. Thus $aa \,.\, ax = bb \,.\, bx$.

A commutative WA-quasigroup satisfying the equivalent conditions of the preceding proposition will be called primitive.

Proposition 5. *Let Q be a WAD-quasigroup. Define a binary relation r on Q by $a \mathrm{r} b$ iff $a = k(b)$ for some $k \in L_Q$. Then*

(i) *if $(Q(\circ), \varphi, \psi, g)$ is an arithmetical form of Q and $a, b \in Q$ then $a \mathrm{r} b$ iff $b = a \circ x$ for some $x \in N(Q(\circ))$,*

(ii) *r is a normal congruence relation of Q,*

(iii) *the factorquasigroup Q/r is a primitive commutative WA-quasigroup,*

(iv) *if a class A of r is a subquasigroup then A is an abelian quasigroup,*

(v) *if $a \mathrm{r} aa$ for an $a \in Q$ then the class $A = \{x \in Q \mid x \mathrm{r} a\}$ is an abelian subquasigroup of Q.*

Proof. (i) is obvious from Lemma 3.

(ii) Let $(Q(\circ), \varphi, \psi, g)$ be an arithmetical form of Q and $a, b, c \in Q$. If $a \mathrm{r} b$ then $b = a \circ x$ for an $x \in N(Q(\circ))$ and $bc = (\varphi(a \circ x) \circ \psi(c)) \circ g = ((\varphi(a) \circ \psi(c)) \circ g) \circ \circ \varphi(x) = ac \circ \varphi(x)$, since $\varphi(x) \in N(Q(\circ))$. The rest can be proved similarly.

(iii) Let $(Q(\circ), \varphi, \psi, g)$ be an arithmetical form of Q and $k(a) = \psi(a^{-1} \circ \varphi\psi^{-1}(a))$ for every $a \in Q$. Since $\varphi\psi^{-1}$ is a nuclear mapping, $k(a) \in N(Q(\circ))$. On the other hand, $ba \circ k(a) = ((\varphi(b) \circ \psi(a)) \circ g) \circ k(a) = (\varphi(b) \circ (\psi(a) \circ k(a))) \circ g = (\varphi(b) \circ \varphi(a)) \circ g =$

$= ab \circ k(b)$. Hence ab r ba for all $a, b \in Q$ and Q/r is commutative. Finally, $x \circ x \circ \circ x \in N(Q(\circ))$ for each $x \in Q$, r is a normal congruence of $Q(\circ)$, $Q(\circ)/r$ is 3-elementary and $Q(\circ)/r$ is isotopic to Q/r. According to Proposition 4, Q/r is primitive.

(iv) Let $x \in A$ and $j = xx \,.\, xx$. As A is a subquasigroup, $j \in A$. Consider $(Q(\circ), \varphi, \psi, g)$, the arithmetical form corresponding to j in the sense of Lemma 2. Then j is the unit of $Q(\circ)$ and (i) yields the equality $A = N(Q(\circ))$. However, $g = jj \in A$, and so $(A(\circ), \varphi/A, \psi/A, g)$ is an arithmetical form of A. Since $A(\circ)$ is an abelian group, A is an abelian quasigroup. Now the proof is complete, because (v) is a straightforward consequence of (iv).

Corollary 1. (i) *Every simple WAD-quasigroup is either abelian or commutative and primitive.*

(ii) *Every finite simple WA-quasigroup is a WAD-quasigroup.*

Proof. (i) follows immediately from Proposition 5.

(ii) Let Q be a finite simple WA-quasigroup and $k(x) = xx$ for every $x \in Q$. Since k is an endomorphism of Q and Q is simple, k is one-to-one or $k(x) = k(y)$ for all $x, y \in Q$. In the first case, $k(Q) = Q$ (because of the finiteness of Q) and Q is a WAD-quasigroup by Proposition 3. In the second case, Q is unipotent and hence abelian by Proposition 2.

Theorem 2. *Let Q be a WA-quasigroup. Then it is a WAD-quasigroup, provided at least one of the following conditions holds:*

(i) *Q is commutative.*

(ii) *Q is unipotent.*

(iii) *The mapping $x \mapsto xx$ is biunique.*

(iv) *Q is finite and simple.*

(v) *Q is idempotent.*

Proof. Apply Propositions 1, 2, 3 and Corollary 1.

Proposition 6. *Let Q be a WAD-quasigroup with an idempotent $j \in Q$ and let $(Q(\circ), \varphi, \psi, j)$ be the arithmetical form corresponding to j in the sense of Lemma 2. For all $a \in Q$ let $k(a) = \psi(a^{-1} \circ \varphi\psi^{-1}(a))$. Then*

(i) *k is an endomorphism of $Q(\circ)$ and $k(a) \in N(Q(\circ))$ for every $a \in Q$,*

(ii) *k is an endomorphism of Q and $k(Q)$ is an abelian quasigroup,*

(iii) *$k(a) = k(b)$ iff $ab = ba$,*

(iv) *the set $A = \{a \in Q \mid aj = ja\}$ is a normal commutative subquasigroup of Q.*

Proof. (i) Clearly, $\varphi(a) = \psi(a) \circ k(a)$ for each $a \in Q$. Hence $(\psi(a) \circ \psi(b)) \circ \circ k(a \circ b) = \psi(a \circ b) \circ k(a \circ b) = \varphi(a \circ b) = \varphi(a) \circ \varphi(b) = (\psi(a) \circ k(a)) \circ (\psi(b) \circ \circ k(b)) = (\psi(a) \circ \psi(b)) \circ (k(a) \circ k(b))$, since both $k(a)$ and $k(b)$ belong to $N(Q(\circ))$. Thus $k((a \circ b) = k(a) \circ k(b)$.

(ii) Let $a \in Q$. Then $k\varphi = \varphi k$, as follows from the definition of k, and consequently

$$\psi\, k(a) \circ k^2(a) = \varphi\, k(a) = k\, \varphi(a) = k\, \psi(a) \circ k^2(a)\,.$$

Thus $\psi k = k\psi$ and $k(ab) = k(\varphi(a) \circ \psi(b)) = \varphi\, k(a) \circ \psi\, k(b) = k(a)\,.\,k(b)$. Let $B = k(Q)$. As k is an endomorphism of both $Q(\circ)$ and Q, $B(\circ)$ is a subloop and B is a subquasigroup. However, $B(\circ) \subseteq N(Q(\circ))$ and $\varphi(B) \subseteq B$, $\psi(B) \subseteq B$. Now it is obvious that $(B(\circ), \varphi \mid B, \psi \mid B, j)$ is an arithmetical form of B and that B is abelian.

(iii) If $ab = ba$ then

$$(\psi(a) \circ \psi(b)) \circ k(a) = \varphi(a) \circ \psi(b) = \varphi(b) \circ \psi(a) = (\psi(b) \circ \psi(a)) \circ k(b)\,,$$

so that $k(a) = k(b)$. Conversely, if $k(a) = k(b)$ then the equality $ab \circ k(b) = ba \circ \circ k(a)$ yields $ab = ba$.

(iv) This is obvious from (ii) and (iii).

A quasigroup Q is called *anticommutative* if $ab \neq ba$, whenever $a, b \in Q$ and $a \neq b$.

Corollary 2. *Every anticommutative WAD-quasigroup is abelian.*

Proof. Let Q be an anticommutative WAD-quasigroup, $x \in Q$ and $a * b = L_x^{-1}(a)\,.\,L_x^{-1}(b)$ for all $a, b \in Q$. Then $Q(*)$ is a WAD-quasigroup with a left unit and $(a * b) * (c * d) = L_{xx}^{-1} L_{xx.xx}^{-1}(ab\,.\,cd)$ for all $a, b, c, d \in Q$. As is easy to see, $Q(*)$ is anticommutative, $Q(*)$ has an idempotent element and $Q(*)$ is abelian iff Q is so. Hence we can assume that Q contains at least one idempotent element. Let k be the endomorphism of Q defined in Proposition 6. Then $k(a) = k(b)$ iff $ab = ba$ and $k(Q)$ is an abelian quasigroup. Since Q is anticommutative, k is one-to-one, and therefore Q is isomorphic to $k(Q)$.

Proposition 7. *Let Q be a WAD-quasigroup with an idempotent element j, $A = \{x \in Q \mid ax\,.\,bc = ab\,.\,jc$ for some $a, b, c \in Q\}$ and let P be the subquasigroup of Q generated by A. Then*

(i) *P is a normal subquasigroup of Q,*
(ii) *the factorquasigroup Q/P is abelian,*
(iii) *P is a primitive commutative WAD-quasigroup.*

Proof. The proof is similar to that of [1, Theorem 8.7]. Let $(Q(\circ), \varphi, \psi, j)$ be the arithmetical form of Q corresponding to j. If $a, b, c \in Q$ then there is a uniquely determined element $h(a, b, c) \in Q$ such that $(a \circ b) \circ c = (a \circ h(a, b, c)) \circ (b \circ c)$. Let $B(\circ)$ be the subloop of $Q(\circ)$ generated by all the elements $h(a, b, c)$, $a, b, c \in Q$. Since $k(h(a, b, c)) = h(k(a), k(b), k(c))$ for every endomorphism k of $Q(\circ)$, $B(\circ)$ is a normal subloop in $Q(\circ)$ and B is a normal subquasigroup in Q. The factorloop $Q(\circ)/B(\circ)$ is clearly an abelian group, and hence the factorquasigroup Q/B is abelian.

Further, according to [1, Lemma 8.6],

$$\varphi h(a, b, c) = h(\varphi(a), \varphi(b), \varphi(c)) =$$
$$= h(\psi(a) \circ k(a), \psi(b) \circ k(b), \psi(c) \circ k(c)) = \psi h(a, b, c)$$

(k is the endomorphism defined in Proposition 6) and $x \circ x \circ x = j$ for every $x \in B$. Hence B is a commutative primitive WAD-quasigroup. Now it remains to prove that $B = P$. To this purpose it suffices to show that $ab \,.\, jc = ah(a, b, c) \,.\, bc$ for all $a, b, c \in Q$. Indeed, let $ax \,.\, bc = ab \,.\, jc$. Then

$$(\varphi^2(a) \circ \varphi\psi(b)) \circ \psi^2(c) = (\varphi^2(a) \circ \varphi\psi(x)) \circ (\varphi\psi(b) \circ \psi^2(c)) \,.$$

However, $\varphi^2(a) = \varphi(\psi(a) \circ k(a)) = \varphi\psi(a) \circ \varphi\, k(a)$, $\psi^2(c) = \varphi\psi(c) \circ \psi\, k(c^{-1})$ and $\varphi\, k(a)$, $\psi\, k(c^{-1})$ belong to $N(Q(\circ))$. Thus $(a \circ x) \circ (b \circ c) = (a \circ b) \circ c$.

If Q is a quasigroup then the multiplication group $A(Q)$ of Q is the subgroup of S_Q generated by all the translations L_x, R_x, $x \in Q$. In [3], it is proved that $A(Q)$ is a solvable group, if Q is a finite distributive quasigroup. The following proposition is a generalization of this result.

Proposition 8. *Let Q be a finite WAD-quasigroup. Then $A(Q)$ is a solvable group.*

Proof. Let $(Q(\circ), \varphi, \psi, g)$ be an arithmetical form of Q, $G = A(Q(\circ))$ and let H be the subgroup of S_Q generated by $G \cup \{\varphi, \psi\}$. Since φ, ψ are automorphisms of $Q(\circ)$ and $\varphi\psi = \psi\varphi$, G is a normal subgroup in H and H/G is an abelian group. On the other hand, the multiplication group of a finite commutative Moufang loop is nilpotent (see [2, pg. 106]), and consequently H is solvable. Finally, as is easy to see, $A(Q) \subseteq H$ and the proof is complete.

References

[1] *В. Д. Белоусов:* Основы теории квазигрупп и луп, Москва 1967.

[2] *R. H. Bruck:* A survey of binary systems, Springer Verlag, Berlin—Heidelberg—Göttingen.

[3] *B. Fischer:* Distributive Quasigruppen endlicher Ordnung, Math. Zeit. *83*, 1964, 267—303.

[4] *T. Kepka:* Quasigroups which satisfy certain generalized forms of the abelian identity, Čas. Pěst. Mat. *100*, 1975, 46—60.

[5] *H. Orlik - Pflugfelder:* A special class of Moufang loops, Proc. Amer. Math. Soc. *26*, 1970, 583—586.

Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta KU).