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GRAPHS OF FINITE ABELIAN GROUPS

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Let G be a finite group. We define the *intersection graph* of G to be the undirected graph (without loops or multiple edges) whose vertices are in one-to-one correspondence with the non-identity subgroups of G, where two vertices are joined by an edge if and only if the corresponding subgroups intersect.

In his paper, [1], BOHDAN ZELINKA conjectured that a finite, abelian group was determined by its intersection graph. Any two non-isomorphic cyclic primary groups of the same height show that this conjecture is false. In this paper we show that the conjecture is true for finite, abelian groups with no cyclic Sylow subgroups.

Let v be a vertex of a graph. We define the star of v, denoted $\mathrm{st}(v)$, to be the set of vertices which are joined by an edge to v, together with v. A connected component of a graph is said to be complete if each of its vertices is connected to every other vertex by an edge. It is easy to see that a finite abelian group has a complete intersection graph if and only if it is cyclic of prime-power order.

A homocyclic, abelian group is an abelian group which is the direct sum of isomorphic cyclic groups. If G is an abelian p-group, we define $\Omega_k(G) = \{x \in G \mid p^k x = 0\}$. Clearly, $\Omega_k(G)$ is a characteristic subgroup of G.

Let $g(G)$ be the intersection graph of G. We define *the reduced graph* of G, $r(G)$, to be the maximal subgraph of G which has all of its connected components complete.

Lemma 1. *Let G be a finite, abelian p-group.*

(a) *The vertices of the reduced graph of G correspond to the nonidentity cyclic subgroups of G.*

(b) *The number of connected components is the number of minimal subgroups of G.*

(c) *If $1 \neq H, K \subseteq G$, then*

(i) *if* $\mathrm{st}(V_H) \subseteq \mathrm{st}(V_K)$. *then* $\mathrm{rank}\,(H) \leqq \mathrm{rank}\,(K)$, *and*

(ii) *if* $\mathrm{st}(V_H) \subset \mathrm{st}(V_K)$, *then* $\mathrm{rank}\,(H) < \mathrm{rank}\,(K)$, *where V_H, V_K are vertices of the intersection graph which correspond to H and K, respectively.*

(d) *The reduced graph is the result of applying the following procedure to the intersection graph of G:*

If the connected components of the graph are complete, terminate the procedure. Otherwise, remove from each connected component which is not complete all the vertices whose stars are maximal (with respect to inclusion). Repeat the procedure.

(e) *The procedure in part* (d) *will terminate in n steps where n is the rank of G.*

Proof. (a) A vertex in the reduced graph of G could not correspond to a non-cyclic subgroup. (Otherwise, the connected component which contained it would not be complete.) Since the reduced graph is maximal with this property, it clearly must contain a vertex corresponding to each cyclic subgroup. If not, the graph could clearly not be maximal.

(b) This is clear from part (a), as each minimal subgroup must be contained in a unique component of the reduced graph.

(c) *Part* (i). If $\mathrm{st}(V_H) \subseteq \mathrm{st}(V_K)$, K must contain all the minimal subgroups of H. Hence, rank $(K) \geqq$ rank (H).

Part (ii). If $\mathrm{st}(V_H) \subset \mathrm{st}(V_K)$, there exists a subgroup H' so that $H' \cap K \neq 1$ and $H' \cap H = 1$. Pick a minimal subgroup of $H' \cap K$. Since it is independent of H, we have rank $(K) >$ rank (H), as required.

(d) Let $n =$ rank of G. Clearly, at the first stage in the procedure we will remove all subgroups of rank n, as their stars contain everything.

At the next step we will remove all subgroups of rank $n - 1$. If the stars of any of these subgroups were not maximal, by part (c) there would be a subgroup of larger rank.

Continuing in this manner, we see that at the i^{th} step of the procedure we will remove all subgroups of rank $n - i + 1$. Hence, after n steps the only remaining vertices will correspond to cyclic groups, each connected component will be complete, and the procedure will terminate.

Since the procedure cannot remove any vertices corresponding to cyclic subgroups (if a connected component contained vertices corresponding only to cyclic subgroups, it would be complete), the result of this procedure is the reduced graph.

(e) This was proved in part (d).

Lemma 2. *If G is a finite, homocyclic, abelian p-group, then G is determined by its intersection graph.*

Proof. Let $g(G)$ be the intersection graph of G. Apply the procedure in part (d) of lemma 1, to this graph. By part (e) of lemma 1, the process of reducing the graph enables us to determine the rank of G.

Since G is homocyclic, G is isomorphic to the direct sum of m copies of the cyclic group of order p^n. Any minimal subgroup, $\langle x \rangle$, is contained in a cyclic summand C with complement D. If $p^k y = x$ with $y \in C$ and if $z \in D$ has order less than p^{k+1}, then $y + z$ will generate a cyclic subgroup of order p^{k+1} containing x. It is easy to see that all the cyclic subgroups of order p^{k+1} which contain x are obtained in this way; hence $N = \sum_{k=0}^{n-1} (p^{m-1})^k$ is the number of cyclic subgroups containing x. This is the number of vertices in the connected component of $g(G)$ which contain x. Thus, by counting the number of vertices in any connected component – they all have the same number – we can determine N. Also m is determined by the fact that $2^m - 1$ is the number of minimal subgroups, which is the same as the number of connected components. Since p is known this determines n. (It is the number of 1's in the base p expansion of N.)

Notice that in the above proof, we have seen that every connected component of the reduced graph of a homocyclic, abelian p-group contains the same number of vertices and this number determines the group.

Lemma 3. *Let G be a finite, abelian p-group. Then G is determined by its intersection graph.*

Proof. Suppose $G = A_1 \oplus \ldots \oplus A_r$, where the A_i are the homocyclic components of G. We will show how to determine the A_i from the intersection graph of G. We assume without loss of generality that

$$A_j = \bigoplus_{k=1}^{m_j} Z_{p^{n_j}} \quad \text{with} \quad n_1 < n_2 < \ldots < n_r \quad \text{and} \quad j = 1, \ldots, r\,.$$

Note that if $n = \text{rank}\,(G)$, then $n = \sum_{i=1}^{r} m_i$.

Let $r(G)$ be the reduced graph of G. The process of reducing the graph has enabled us to determine the rank of the rank of G. If $\langle x \rangle$ is a cyclic subgroup, we denote by $(x)_*^G$ the set of vertices in the connected component of $r(G)$ which contains $V_{\langle x \rangle}$.

The proof will now proceed in the following steps.

(1) Let $x \in G$, $x = \sum_{i=1}^{r} a_i$ with $a_i \in A_i$ such that $\langle x \rangle$ is a minimal subgroup of G. Then if l is the smallest integer such that $a_l \neq 0$, we have $|(x)_*^G| = |(a_l)_*^G|$.

Proof. $H = \Omega_{n_l}(\bigoplus_{i=l}^{r} A_i)$ is a homocyclic group. By the proof of lemma 2, $|(x)_*^H| = |(a_l)_*^H|$. We will set up a 1-1 correspondence between $(x)_*^G$ and $(a_l)_*^G$. Let ψ be a 1-1 correspondence between $(x)_*^G$ and $(a_l)_*^G$ which preserves orders. Suppose $Z = U + V$ where $U \in \bigoplus_{j=1}^{l-1} A_j$ and $V \in \bigoplus_{j=l}^{r} A_j$. Then, clearly, $Z \in (x)_*^G$ if and only if $Z' = U + \psi(V) \in (a_l)_*^G$.

Let $S = \{|(x)_*^G| \mid x \text{ is a cyclic subgroup of } G\}$. Let $s_1 < s_2 < \ldots < s_{r'}$ be the distinct elements of S. Furthermore, let t_i be the number of connected components of $r(G)$ which have s_i vertices, for $i = 1, \ldots, r'$.

(2) $r = r'$ and the m_i, $1 \leqq i \leqq r$ are determined.

Proof. As in part (1), let $\langle x \rangle$ be a minimal subgroup, where $x = \sum_{i=1}^{r} a_i$, $a_i \in A$, then $|(x)_*^G| = |(a_l)_*^G|$ where l is the smallest integer such that $a_l \neq 0$. Thus, $r' \leqq r$. However, if $\langle a_i \rangle$ and $\langle a_{i+1} \rangle$ are minimal subgroups of A_i and A_{i+1} respectively, then $|(a_i)_*^G| < |(a_{i+1})_*^G|$. (This follows since if $H = \Omega_{n_i}(G)$, by part (1) $|(a_i)_*^H| = |(a_{i+1})_*^H|$, and $(a_i)_*^H = (a_i)_*^G$, while there is a larger cyclic subgroup containing a_{i+1}.) Hence, $r \leqq r'$, and thus $r = r'$.

From part (1), the number of connected components which have s_r vertices is the number of minimal subgroups which are contained in A_r. This number must then be $p^{m_r - 1} + \ldots + p + 1$. Thus, m_r is determined.

Suppose by induction that $m_1, \ldots, m_{r-i+1}$ has been determined. An easy calculation shows that the number of minimal subgroups contained in A_{r-i} is $t_{r-i}/p^{m_r + \ldots + m_{r-i+1}}$. As above this number determines m_{r-i}, the rank of A_{r-i}. Thus, all the m_j's are determined.

(3) All the A_i's (and hence G) are determined.

Proof. All that remains is to determine the n_i's. Now s_1 is the same as the number of vertices in a connected components of $\Omega_{n_1}(G)$, a homocyclic group. By lemma 2, n_1 is determined.

Suppose by induction that $n_1, \ldots, n_i$ are determined. An easy calculation shows that s_{i+1} must be of the form

$$s_i + p^{(m_1 n_1 + \ldots + m_i n_i)} \sum_{j=n_i}^{n_{i+1}-1} (p^j)^{n - (m_1 + \ldots + m_i) - 1}$$

this clearly determines n_{i+1}.

Now since all the n_i's are determined, all the A_i's are determined and hence G is determined. This completes the proof.

Theorem. *If G is a finite, abelian group with no cyclic sylow subgroups, then G is determined by its intersection graph.*

Proof. By the theorem in [1], from the intersection graph of G, we can determine the intersection graph of each of the sylow subgroups of G.

By lemma 1 (e), the rank of each sylow subgroup is determined. Similarly, the number of minimal subgroups is determined. This number must be of the form $p^{n-1} + \ldots + p + 1$ where n is the rank. As $n > 1$, this clearly determines p. This establishes the result.

Bibliography

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