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GENERALIZATION OF SOME KNOWN PROPERTIES OF CANTOR SET

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Introduction. It is known [5], [6], [7] that, for any \(d (0 \leq d \leq 1)\) there exist Cantor points \(x\) and \(y\) such that

\[ y - x = d. \]

(1)

It has subsequently been shown by Utz [8] from geometrical considerations that if \(0 \leq d \leq 1\) and \(\frac{1}{2} \leq |m| \leq 3\) then there exists at least one pair of Cantor points \(x\) and \(y\) such that

\[ y = mx + d. \]

(2)

By putting \(m = 1\), we see that (1) is a particular case of (2). Using the properties of Kinney’s functions [4], it has been shown recently by M. Das Gupta in Czechoslovak Mathematical Journal, [2]) that given any \(d\) there exists at least one pair of Cantor points \(x, y\) such that \(d = (2y + x)/3\) (i.e. every \(d\) is a point of trisection of a segment whose end points are Cantor points).

We now propose to give a more general theorem with the help of Utz’s theorem [8] from which results due to M. Das Gupta and H. Steinhaus (and others) follow.

Theorem 1. Given two positive real numbers \(\mu\) and \(v\) where \(\mu \leq v\) and \(\frac{1}{2} \leq \mu/v \leq 1\), each point \(d\), in \(0 < d < 1\), divides a segment \([x, y]\) \(\subseteq [0, 1]\) in the ratio \(v:\mu\) whose end points \(x\) and \(y\) are Cantor points.

Proof. Let \(d\) be any point in \(0 < d \leq v/(\mu + v)\). We now choose \(d'\) such that

\[ d' = \left[(\mu + v)/v\right]d. \]

Hence \(d = v d'/ (\mu + v)\). Since \(0 < d \leq v/(\mu + v)\), we have \(0 < d' \leq 1\). We now choose \(m = -\mu/v\) in Utz result (2).

Therefore \(\frac{1}{2} \leq |m| \leq 1(\leq 3)\) and thus \(y = -(\mu/v)x + d'\) where \(x \in C\) and \(y \in C\) where \(C\) represents Cantor middle third set based on the unit interval \([0, 1]\). Therefore

\[ \frac{vy + \mu x}{v} = d' \quad \text{or} \quad \frac{vy + \mu x}{v} = \frac{\mu + v}{v} d \quad \text{or} \quad d = \frac{vy + \mu x}{\mu + v}. \]

(3)
Taking \(\frac{v}{\mu + v} < d < 1\), we get

\[
1 - \frac{v}{\mu + v} > 1 - d > 0 \quad \text{or} \quad 0 < 1 - d < \frac{\mu}{\mu + v} \left( \leq \frac{v}{\mu + v} \right).
\]

Hence by previous argument, we get \(x \in C\), \(y \in C\), and

\[
1 - d = \frac{vy + \mu x}{\mu + v} \quad \text{or} \quad \mu + v - d(\mu + v) = vy + \mu x
\]
or

\[
(\mu + v) d = \mu(1 - x) + v(1 - y) = \mu x' + vy'
\]

where \(x' = 1 - x \in C\), and \(y' = 1 - y \in C\). Thus

\[
d = \frac{\mu x' + vy'}{\mu + v}
\]

where

\[
\frac{v}{\mu + v} < d < 1.
\]

Taking (3) and (4) together, the required result follows.

**Note 1.** This theorem is trivially true for \(d = 0, 1\).

M. Dasgupta [2] gave the following theorem:

*Each point \(d\) in \((0 < x < 1)\) is a point of trisection on a segment of the interval \(0 \leq x \leq 1\), the two end points of which are Cantor points.*

This theorem follows as a corollary of Theorem 1 when \(\mu = 1, v = 2\).

Randolph [5] and Bose Majumdar [1] have shown that each point of \([0, 1]\) is the middle point of a pair of Cantor points.

Taking \(\mu = v = 1\), we observe that Randolph’s and Bose Majumdar’s result is a particular case of Theorem 1.

**Theorem 2.** Given two positive real numbers \(\mu\) and \(v\) where \(\mu \leq v\) and \(\frac{1}{2} \leq \mu/v \leq 1\) and any point \(d\) in \([0, 1]\), the aggregate of pairs of points \((x, y)\), \(0 \leq x, y \leq 1\) such that \(d = (\mu f(y) + v f(x))/(\mu + v)\) is either finite or has the power \(c\), where \(f(x)\) is the Kinney’s function.

**Proof.** We shall prove the result by the help of Kinney’s function.

Kinney [4] defined two functions \(f(x)\) and \(V(x)\) as follows:

Let \(x\) be any point in \(0 \leq x \leq 1\). Then we write,

\[
x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \quad \text{where} \quad x_i = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{for every} \quad i.
\]
Moreover, we shall have \( x \) uniquely represented by replacing any final "1" in it, with a chain of 2's.

We write

\[
f(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{3^i} \quad \text{and} \quad V(x) = \sum_{i=1}^{\infty} \frac{V(x)}{3^i}
\]

where \( f_i(x) = 2\delta(x_i, 2) \) and \( v_i(x) = 2\delta(x_i, 1) \) with the property \( \delta(a, b) = 1 \) if \( a = b \) and \( \delta(a, b) = 0 \) if \( a \neq b \). Since \( f(x) \) and \( V(x) \) for any \( x \in [0, 1] \) are, each, either 0 or 2, it follows that \( f(x) \) and \( V(x) \) are both points of the Cantor middle third set \( C \). It has been shown by D. K. Ganguly and Bose Majumdar [3] that each of Kinney's function \( f(x) \) and \( V(x) \) map the unit interval \([0, 1]\) onto the Cantor middle third set \( C \) and excepting for an enumerable subset of \( C \), every point of \( C \) is the image of continuum number of points \( x \in [0, 1] \) under the mapping by each of the Kinney's function \( f(x) \) and \( V(x) \).

**[Sketch of the proof.** Let us take any \( p \in C \), where

\[
p = \sum_{i=1}^{\infty} \frac{c_i}{3^i}, \quad c_i = \binom{0}{2},
\]

for every \( i \). Then we find a corresponding \( x \in [0, 1] \) by choosing \( x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \) where \( x_i = 2 \) if \( c_i = 2 \) and \( x_i = 0 \) or 1 if \( c_i = 0 \) and thus \( f(x) = p = \sum_{i=1}^{\infty} \frac{c_i}{3^i} = \sum_{i=1}^{\infty} \frac{f_i(x)}{3^i} \); the power of the set \( \{x\} \) with \( f(x) = p \) is obviously finite or \( 2^{\aleph_0} (=c) \) [9].]

We have seen in the Theorem 1 above, each point \( d \in [0, 1] \) is such that there corresponds a segment \([q, p] \subseteq [0, 1]\) satisfying \( d = (\mu q + v p)/(\mu + v) \) with \( p \in C \) and \( q \in C \). Now \( p \) and \( q \) being fixed, satisfying conditions as stated in the Theorem 1, let us assume that [as shown in [3]] the set \( \{x\} = E \subset [0, 1] \) is such that \( p = f(x) = f(x') = f(x'') = \ldots \) where \( x, x', x'', \ldots \in E \) and we also assume that the set \( \{y\} = F \subset [0, 1] \) is such that \( q = f(y) = f(y') = f(y'') = \ldots \) where \( y, y', y'', \ldots \in F \). Here \( d = (\mu q + v p)/(\mu + v) = (\mu f(y) + v f(x))/(\mu + v) \) for any \( x \in E \) and any \( y \in F \).

Since \( E \) and \( F \) (power of the sets \( E \) and \( F \) respectively) are either each finite or \( c \) [3], it follows that the power \( E \times F \) is \( c \) or \( c^2 (=c) \) [9] or a finite number.

Hence the power of the set \( \{(p, q)\} \) corresponding to a given \( d \in [0, 1] \) and a number \( \mu/v \), where \( \frac{1}{2} \leq \mu/v \leq 1 \), satisfying \( d = (\mu q + v p)/(\mu + v) \), \( p \in C \), \( q \in C \), is either \( c \) or a finite number.

**Note 2.** We could have proved this theorem in similar manner by taking Kinney's function \( V(x) \).
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References


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