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GENERALIZATION OF SOME KNOWN PROPERTIES OF CANTOR SET

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Introduction. It is known [5], [6], [7] that, for any $d\,(0 \leqq d \leqq 1)$ there exist Cantor points x and y such that

$$y - x = d\,. \tag{1}$$

It has subsequently been shown by UTZ [8] from geometrical considerations that if $0 \leqq d \leqq 1$ and $\frac{1}{3} \leqq |m| \leqq 3$ then there exists at least one pair of Cantor points x and y such that

$$y = mx + d\,. \tag{2}$$

By putting $m = 1$, we see that (1) is a particular case of (2). Using the properties of Kinney's functions [4], it has been shown recently by M. DASGUPTA (in Czechoslovak Mathematical Journal, [2]) that given any d there exists at least one pair of Cantor points x, y such that $d = (2y + x)/3$ (i.e. every d is a point of trisection of a segment whose end points are Cantor points).

We now propose to give a more general theorem with the help of Utz's theorem [8] from which results due to M. Dasgupta and H. STEINHAUS (and others) follow.

Theorem 1. *Given two positive real numbers μ and ν where $\mu \leqq \nu$ and $\frac{1}{3} \leqq \mu/\nu \leqq 1$, each point d, in $0 < d < 1$, divides a segment $[x, y] \subseteq [0, 1]$ in the ratio $\nu : \mu$ whose end points x and y are Cantor points.*

Proof. Let d be any point in $0 < d \leqq \nu/(\mu + \nu)$. We now choose d' such that $d' = [(\mu + \nu)/\nu]\, d$.

Hence $d = \nu d'/(\mu + \nu)$. Since $0 < d \leqq \nu/(\mu + \nu)$, we have $0 < d' \leqq 1$. We now choose $m = -\mu/\nu$ in Utz result (2).

Therefore $\frac{1}{3} \leqq |m| \leqq 1(<3)$ and thus $y = -(\mu/\nu)\, x + d'$ where $x \in C$ and $y \in C$ where C represents Cantor middle third set based on the unit interval $[0, 1]$. Therefore

$$\frac{\nu y + \mu x}{\nu} = d' \quad \text{or} \quad \frac{\nu y + \mu x}{\nu} = \frac{\mu + \nu}{\nu}\, d \quad \text{or} \quad d = \frac{\nu y + \mu x}{\mu + \nu}\,. \tag{3}$$

Taking $\nu/(\mu + \nu) < d < 1$, we get

$$1 - \frac{\nu}{\mu + \nu} > 1 - d > 0 \quad \text{or} \quad 0 < 1 - d < \frac{\mu}{\mu + \nu}\left(\leqq \frac{\nu}{\mu + \nu}\right).$$

Hence by previous argument, we get $x \in C$, $y \in C$, and

$$1 - d = \frac{\nu y + \mu x}{\mu + \nu} \quad \text{or} \quad \mu + \nu - d(\mu + \nu) = \nu y + \mu x$$

or

$$(\mu + \nu)\, d = \mu(1 - x) + \nu(1 - y) = \mu x' + \nu y'$$

where $x'(= 1 - x) \in C$, and $y'(= 1 - y) \in C$. Thus

$$d = \frac{\mu x' + \nu y'}{\mu + \nu} \tag{4}$$

where

$$\frac{\nu}{\mu + \nu} < d < 1 .$$

Taking (3) and (4) together, the required result follows.

Note 1. This theorem is trivially true for $d = 0, 1$.

M. Dasgupta [2] gave the following theorem:

Each point d in $(0 < x < 1)$ is a point of trisection on a segment of the interval $0 \leqq x \leqq 1$, the two end points of which are Cantor points.

This theorem follows as a corollary of Theorem 1 when $\mu = 1$, $\nu = 2$.

Randolph [5] and Bose Majumdar [1] have shown that each point of $[0, 1]$ is the middle point of a pair of Cantor points.

Taking $\mu = \nu = 1$, we observe that Randolph's abd Bose Majumdar's result is a particular case of Theorem 1.

Theorem 2. *Given two positive real numbers μ and ν where $\mu \leqq \nu$ and $\frac{1}{3} \leqq \mu/\nu \leqq 1$ and any point d in $[0, 1]$, the aggregate of pairs of points (x, y), $0 \leqq x, y \leqq 1$ such that $d = (\mu f(y) + \nu f(x))/(\mu + \nu)$ is either finite or has the power c, where $f(x)$ is the Kinney's function.*

Proof. We shall prove the result by the help of Kinney's function.

Kinney [4] defined two functions $f(x)$ and $V(x)$ as follows:

Let x be any point in $0 \leqq x \leqq 1$. Then we write,

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \quad \text{where} \quad x_i = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{for every} \quad i .$$

Moreover, we shall have x uniquely represented by replacing any final "1" in it, with a chain of 2's.

We write

$$f(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{3^i} \quad \text{and} \quad V(x) = \sum_{i=1}^{\infty} \frac{V_i(x)}{3^i}$$

where $f_i(x) = 2\delta(x_i, 2)$ and $v_i(x) = 2\delta(x_i, 1)$ with the property $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ if $a \neq b$. Since $f_i(x)$ and $V_i(x)$ for any $x \in [0, 1]$ are, each, either 0 or 2, it follows that $f(x)$ and $V(x)$ are both points of the Cantor middle third set C. It has been shown by D. K. GANGULY and Bose Majumdar [3] that each of Kinney's function $f(x)$ and $V(x)$ map the unit interval $[0, 1]$ onto the Cantor middle third set C and excepting for an enumerable subset of C, every point of C is the image of continuum number of points $x \in [0, 1]$ under the mapping by each of the Kinney's function $f(x)$ and $V(x)$.

[Sketch of the proof. Let us take any $p \in C$, where

$$p = \sum_{i=1}^{\infty} \frac{c_i}{3^i}, \quad c_i = \binom{0}{2},$$

for every i. Then we find a corresponding $x \in [0, 1]$ by choosing $x = \sum_{i=1}^{\infty} (x_i/3^i)$ where $x_i = 2$ if $c_i = 2$ and $x_i = 0$ or 1 if $c_i = 0$ and thus $f(x) = p = \sum_{i=1}^{\infty} (c_i/3^i) = \sum_{i=1}^{\infty} (f_i(x)/3^i)$; the power of the set $\{x\}$ with $f(x) = p$ is obviously finite or $2^{\aleph_0}$ $(=c)$ [9].]

We have seen in the Theorem 1 above, each point d in $[0, 1]$ is such that there corresponds a segment $[q, p] \subseteq [0, 1]$ satisfying $d = (\mu q + \nu p)/(\mu + \nu)$ with $p \in C$ and $q \in C$. Now p and q being fixed, satisfying conditions as stated in the Theorem 1, let us assume that [as shown in [3]] the set $\{x\} = E \subset [0, 1]$ is such that $p = f(x) = f(x') = f(x'') = \dots$ where $x, x', x'', \dots \in E$ and we also assume that the set $\{y\} = F \subset [0, 1]$ is such that $q = f(y) = f(y') = f(y'') = \dots$ where $y, y', y'', \dots \in F$. Here $d = (\mu q + \nu p)/(\mu + \nu) = [\mu f(y) + \nu f(x)]/(\mu + \nu)$ for any $x \in E$ and any $y \in E$.

Since $\overline{\overline{E}}$ and $\overline{\overline{F}}$ (power of the sets E and F respectively) are either each finite or c [3], it follows that the power $\overline{\overline{E \times F}}$ is c or c^2 $(= c)$ [9] or a finite number.

Hence the power of the set $\{(p, q)\}$ corresponding to a given $d \in [0, 1]$ and a number μ/ν, where $\frac{1}{3} \leqq \mu/\nu \leqq 1$, satisfying $d = (\mu q + \nu p)/(\mu + \nu)$, $p \in C$, $q \in C$, is either c or a finite number.

Note 2. We could have proved this theorem in similar manner by taking Kinney's function $V(x)$.

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