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DERIVATIVES OF HYPERGRAPHS

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In [1] the following problem of D. K. RAY-CHAUDHURI is proposed:

Let $H = (X, \mathscr{E})$, $\mathscr{E} = (E_i : i \in I)$ be a hypergraph such that $|E_i| \geqq r$, $\forall i \in I$. Let $H_r = (P_r(X), E^r)$, $E^r = (E_i^r : i \in I)$, where $P_r(X)$ is the set of r-element subsets of X and E_i^r is the set of r-element subsets of E_i, $i \in I$. H_r will be called the *r-th derivative of H*. Note that $|E_i^r| = \binom{|E_i|}{r}$. Find necessary and sufficient conditions under which a hypergraph K is the r-th derivative of a hypergraph H.

This is Problem 27 from [1]. We shall solve this problem in a special case when the intersection closure of K is intersecting. A hypergraph is called *intersecting*, if any two of its edges have a non-empty intersection.

We shall introduce some notions. The r-th derivative of a hypergraph H will be denoted by $\partial^r H$. If H is a finite hypergraph, then the intersection closure of H is the hypergraph $J(H)$ with the same vertex set as H which is the minimal hypergraph with the property that it contains all edges of H and with any two edges it contains their intersection as an edge, provided this intersection is non-empty.

Lemma 1. *Let H be a hypergraph whose edges have cardinalities greater than or equal to r, where r is a positive integer. Then*

$$\partial^r J(H) = J(\partial^r H).$$

Proof. Any edge F of $J(H)$ is of the form $\bigcap_{i=1}^{k} F_i$, where $F_1, \ldots, F_k$ are edges of H. Let $F_1^r, \ldots, F_k^r, F^r$ be the edges of $\partial^r J(H)$ corresponding to $F_1, \ldots F_k, F$ respectively. Each set of the cardinality r which is a subset of each F_i for $i = 1, \ldots, k$ is a subset of F and vice versa. Therefore $F^r = \bigcap_{i=1}^{k} F_i^r$. As this is true for each r, the assertion is proved.

The edges of $J(H)$, possibly with the empty set added form a semilattice $\mathfrak{S}(H)$ with respect to the set intersection.

A hypergraph is called *r-intersecting*, if any two of its edges have an intersection of a cardinality at least r.

Lemma 2. *Let H be a hypergraph whose edges have cardinalities greater than or equal to r, where r is a positive integer. Then $\partial^r J(H)$ is intersecting if and only if $J(H)$ is r-intersecting.*

Proof. If $J(H)$ is r-intersecting, then for any two edges F_1, F_2 of $J(H)$ we have $|F_1 \cap F_2| \geqq r$, therefore there exists at least one r-element subset of $F_1 \cap F_2$ and $F_1^r \cap F_2^r \neq \emptyset$. As F_1, F_2 were chosen arbitrarily, the graph $\partial^r J(H)$ is intersecting. If $J(H)$ is not r-intersecting, there exist edges F_1, F_2 of $J(H)$ such that $|F_1 \cap F_2| < r$. Then there exists no r-element set which is a subset of both F_1 and F_2, thus $F_2^r \cap F_1^r = \emptyset$ and $\partial^r J(H)$ is not intersecting.

Lemma 3. *Let H be an r-intersecting hypergraph. Then $\mathfrak{S}(H) \cong \mathfrak{S}(\partial^r H)$.*

Proof. Let α be a mapping of the edge set of $J(H)$ onto the edge set of $\partial^r J(H)$ such that $\alpha(F) = F^r$ for each edge F of $J(H)$; this is evidently a bijection. As we have shown, for any edges $F, F_1, \ldots, F_k$ we have $F^r = \bigcap_{i=1}^{k} F_i^r$ if and only if $F = \bigcap_{i=1}^{k} F_i$. Therefore α is an isomorphism of $\mathfrak{S}(H)$ onto $\mathfrak{S}(\partial^r H)$.

This assertion is not true in the case when H is not r-intersecting. Let H be a hypergraph with the vertex set $\{1, 2, 3, 4, 5, 6\}$ and with the edges $\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{5, 6, 1\}$, let $r = 2$. The hypergraph H is not r-intersecting. The semilattice $\mathfrak{S}(H)$ consists of the elements $\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{5, 6, 1\}$, $\{1\}$, $\{3\}$, $\{5\}$, $\emptyset$. The semilattice $\mathfrak{S}(\partial^r H)$ consists of the elements $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $\{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$, $\{\{5, 6\}, \{5, 1\}, \{6, 1\}\}$, $\emptyset$. These semilattices are not isomorphic, which is seen from the fact that they have not the same number of elements.

The multiplication in a semilattice will be denoted by $\circ$ or $\prod$. Let $\mathfrak{S}$ be a finite semilattice, let φ be a mapping of $\mathfrak{S}$ into the set N of all non-negative integers. Let A be a non-empty subset of $\mathfrak{S}$, $|A| = m$. We define an operator $INEX\ (\varphi; A)$ as follows.

For each positive integer j such that $1 \leqq j \leqq m$ let $\mathscr{A}_j$ be the set of all j-element subsets of A. Then

$$INEX\ (\varphi; A) = \sum_{j=1}^{m} (-1)^{j+1} \sum_{B \in \mathscr{A}_j} \varphi\Big(\prod_{x \in B} x\Big)\,.$$

If $\mathfrak{S}$ is a finite semilattice whose elements are sets and in which the multiplication is the intersection of sets and if $\varphi(a)$ denotes the cardinality of the element a of $\mathfrak{S}$, then $INEX\ (\varphi; A)$ is the cardinality of the set union of all elements of A. This is the well-known Inclusion-Exclusion Principle and this is also the reason of the notation $INEX\ (\varphi; A)$.

Lemma 4. *Let $\mathfrak{S}$ be a finite semilattice, let φ be a mapping of $\mathfrak{S}$ into the set N of all non-negative integers. Let A be a subset of $\mathfrak{S}$ with at least two elements, let $a \in A$. Then*

$$INEX(\varphi; A) = INEX(\varphi; A - \{a\}) + \varphi(a) - INEX(\varphi; a \circ (A - \{a\})).$$

Remark. By $a \circ (A - \{a\})$ we denote the set of all elements of $\mathfrak{S}$ of the form $a \circ x$, where $x \in A - \{a\}$.

Proof. Let j be an integer, $1 \leqq j \leqq m$, where m is the cardinality of A. Let $\mathscr{A}_j$ (or $\mathscr{A}'_j$) be the set of all j-element subsets of A (or $A - \{a\}$ respectively). Denote $A' = A - \{a\}$. Further, let $\mathscr{A}''_j = \mathscr{A}_j - \mathscr{A}'_j$. Each element of $\mathscr{A}''_j$ for $j \geqq 2$ is obtained from an element of A'_{j-1} by adding a for $j = 1$ we have $\mathscr{A}''_1 = \{\{a\}\}$. Thus for $j \geqq 2$ we have

$$\sum_{B \in \mathscr{A}_j} \varphi(\prod_{x \in B} x) = \sum_{B \in \mathscr{A}_j'} \varphi(\prod_{x \in B} x) + \sum_{C \in \mathscr{A}_j''} \varphi(\prod_{x \in C} x) = \sum_{B \in \mathscr{A}_j'} \varphi(\prod_{x \in B} x) + \sum_{C \in \mathscr{A}'_{j-1}} \varphi(\prod_{x \in C} (a \circ x)).$$

For $j = 1$ we have

$$\sum_{B \in \mathscr{A}_1} \varphi(\prod_{x \in B} x) = \sum_{x \in A} \varphi(x) = \sum_{x \in A - \{a\}} \varphi(x) + \varphi(a).$$

Thus

$$INEX(\varphi, A) = \sum_{j=1}^{m} (-1)^{j+1} \sum_{B \in \mathscr{A}_j} \varphi(\prod_{x \in B} x) =$$

$$= \sum_{j=1}^{m} (-1)^{j+1} (\sum_{B \in \mathscr{A}_j'} \varphi(\prod_{x \in B} x) + \sum_{C \in \mathscr{A}'_{j-1}} \varphi(\prod_{x \in C} (a \circ x))) =$$

$$= \sum_{j=1}^{m} (-1)^{j+1} \sum_{B \in \mathscr{A}_j'} \varphi(\prod_{x \in B} x) + \sum_{j=2}^{m} (-1)^{j+1} \sum_{C \in \mathscr{A}'_{j-1}} \varphi(\prod_{x \in C} (a \circ x)) + \varphi(a) =$$

$$= INEX(\varphi, A') + \sum_{j=1}^{m-1} (-1)^{j} \sum_{C \in \mathscr{A}'_{j-1}} \varphi(\prod_{x \in C} (a \circ x)) + \varphi(a) =$$

$$= INEX(\varphi; A') - \sum_{j=1}^{m-1} (-1)^{j+1} \sum_{C \in \mathfrak{A}_j'} \varphi(\prod_{x \in C} (a \circ x)) + \varphi(a).$$

If $C \in \mathfrak{A}'_j$, then there may exist $x \in C$, $y \in C$ such that $x \neq y$, $a \circ x = a \circ y$. Nonetheless, we shall prove that in spite of it

$$\sum_{j=1}^{m-1} (-1)^{j+1} \sum_{C \in \mathfrak{A}_j'} \varphi(\prod_{x \in C} (a \circ x)) = INEX(\varphi; a \circ A').$$

Let the cardinality of A' be p, let the cardinality of $a \circ A'$ be q. Let $a \circ A' = \{c_1, \ldots, c_q\}$. Let $D_i = \{x \in A' \mid a \circ x = c_i\}$ for $i = 1, \ldots, q$. The elements of D_i will be denoted by $b_1^{(i)}, \ldots, b_{r(i)}^{(i)}$ for each $i = 1, \ldots, q$. Let $\{c_{s(1)}, \ldots, c_{s(t)}\}$ be a subset of $a \circ A'$. There are $u = \prod_{l=1}^{t} (2^{r(s(l))} - 1)$ subsets C of A' such that $\{a \circ x \mid x \in C\} =$

$= \{c_{s(1)}, \ldots, c_{s(t)}\}$. But, as well-known, $\frac{1}{2}(u + 1)$ of them have cardinalities congruent with t modulo 2 and $\frac{1}{2}(u - 1)$ have cardinalities congruent with $t + 1$ modulo 2. As the expressions $\varphi(\prod_{x \in C} (a \circ x))$ occur in the expression $\sum_{j=1}^{m-1} (-1)^{j+1} \sum_{C \in \mathfrak{A}_j'} \varphi(\prod_{x \in C} (a \circ x))$ with the plus (or minus) sign if C has an odd (or even respectively) cardinality, we see that really

$$\sum_{j=1}^{m-1} (-1)^{j+1} \sum_{C \in \mathfrak{A}_j'} \varphi(\prod_{x \in C} (a \circ x)) = INEX\,(\varphi, a \circ A')$$

and thus the assertion is proved.

Now let a finite semilattice $\mathfrak{S}$ and a mapping φ of $\mathfrak{S}$ into the set N be given. In the semilattice $\mathfrak{S}$ we put $a \prec b$ for $a \in \mathfrak{S}$, $b \in \mathfrak{S}$ if and only if $a \neq b$, $a \circ b = a$. We say that the semilattice $\mathfrak{S}$ with the mapping φ is representable by a system of subsets of a set M, if there exists a system of subsets of M which forms a semilattice isomorphic to $\mathfrak{S}$ with respect to the set inclusion and the cardinality of the set from this system which corresponds in the isomorphism to the element $x \in \mathfrak{S}$ is equal to $\varphi(x)$.

Lemma 5. *Let $\mathfrak{S}$ be a finite semilattice, let φ be a mapping of $\mathfrak{S}$ into the set N of all non-negative integers, let q be a positive integer. The following two assertions are equivalent:*

(i) *$INEX\,(\varphi; \mathfrak{S}) \leqq n$, $INEX\,(\varphi; \mathfrak{S}(x)) \leqq \varphi(x)$ for each $x \in \mathfrak{S}$ and $\varphi(x) < \varphi(y)$ for $x \in \mathfrak{S}$, $y \in \mathfrak{S}$, $x \prec y$ (here $\mathfrak{S}(x)$ denotes the subsemilattice of $\mathfrak{S}$ consisting of the elements $y \prec x$.*

(ii) *There exists a set M of the cardinality n such that the semilattice $\mathfrak{S}$ with the mapping φ is representable by a system of subsets of M.*

Proof. (i) $\Rightarrow$ (ii). We shall proceed according to the number of elements of $\mathfrak{S}$. If this number is 1, the assertion holds trivially. Let $k \geqq 2$; suppose that the assertion holds for each $\mathfrak{S}$ with at most $k - 1$ elements. Let $\mathfrak{S}$ have k elements. Choose a maximal element a of $\mathfrak{S}$; as $\mathfrak{S}$ is finite, such an element exists. The set $\mathfrak{S}' = \mathfrak{S} - \{a\}$ is a subsemilattice of $\mathfrak{S}$, because a, being maximal, cannot be a product of two elements different from a. From Lemma 4 we have

$$INEX\,(\varphi; \mathfrak{S}) = INEX\,(\varphi; \mathfrak{S}') - INEX\,(\varphi; a \circ \mathfrak{S}') + \varphi(a)\,.$$

If the condition (i) holds for $\mathfrak{S}$, it holds also for $\mathfrak{S}'$. As $a \succ a \circ x$ for each $x \in \mathfrak{S}'$, we have $\mathfrak{S}(a) = a \circ \mathfrak{S}'$ and $INEX\,(\varphi; a \circ \mathfrak{S}') \leqq \varphi(a)$. Thus $INEX\,(\varphi; \mathfrak{S}) \geqq$ $\geqq INEX\,(\varphi; \mathfrak{S}')$. As $INEX\,(\varphi; \mathfrak{S}) \leqq n$, we have $INEX\,(\varphi; \mathfrak{S}') \leqq n + INEX\,.$ $.\,(\varphi; a \circ \mathfrak{S}') - \varphi(a)$; denote it by n'. Obviously $\mathfrak{S}'$ and $a \circ \mathfrak{S}'$ satisfy the other conditions from (i), because they are subsemilattices of $\mathfrak{S}$. According to the induction assumption there exists a representation of $\mathfrak{S}'$ by a system of subsets of a set M_1

of the cardinality n' (where the corresponding mapping is the restriction of φ onto $\mathfrak{S}'$). Also there exists a representation of $a \circ \mathfrak{S}'$ by a system of subsets of a set M_2 of the cardinality $\varphi(a)$. Take both these representations and identify the elements of M_1 and M_2 so that the sets which represent the same element from $a \circ \mathfrak{S}'$ coincide. The result is the required representation of $\mathfrak{S}$ by subsets of a set M with the cardinality $n' + \varphi(a) - INEX(\varphi; a \circ \mathfrak{S}') = n$.

(ii) $\Rightarrow$ (i). This follows from the Inclusion-Exclusion Principle.

Theorem. *Let r be a positive integer, let N be the set of all positive integers. Let $N(r)$ be the set of all positive integers which can be written in the form $\binom{m}{r}$, where $m \in N$. Let ψ be defined so that $\psi\left(\binom{m}{r}\right) = m$ for each $m \in N$. Let K be an intersecting hypergraph. Then $K \cong \partial^r H$ for a hypergraph H, if and only if:*

(α) *the number n_0 of vertices of K is in $N(r)$;*

(β) *the cardinality of each edge of $J(K)$ is in $N(r)$;*

(γ) *if $\varphi(F) = \psi(|F|)$ for each edge F of $J(K)$, then $INEX(\varphi; \mathfrak{S}(K)) \leqq n_0$, $INEX(\varphi; \mathfrak{S}(K)(F)) \leqq \varphi(F)$ for each edge F of $J(K)$ and $\varphi(F_1) < \varphi(F_2)$ for $F_1 \subset F_2$, $F_1 \neq F_2$.*

If these conditions are fulfilled, the hypergraph H is r-intersecting and is determined by K up to isomorphism.

Remark. The symbol $\mathfrak{S}(K)(F)$ has an analogous meaning as $\mathfrak{S}(x)$ in Lemma 5.

Proof. Necessity. Suppose that there exists H such that $\partial^r H \cong K$. As $J(K)$ is intersecting, $J(H)$ must be r-intersecting according to Lemma 2. The condition (α) follows from the definition of the r-th derivative, the condition (β) follows from Lemma 1. Now $\mathfrak{S}(H) \cong \mathfrak{S}(K)$, thus H is a representation of the semilattice $\mathfrak{S}(K)$ with the mapping φ by a system of subsets of a set with n vertices, where n is such an integer that $\binom{n}{r}$ is the number of vertices of K. Thus (γ) follows from Lemma 5.

Sufficiency. It follows from Lemma 5.

The hypergraph H is determined up to isomorphism, because from the proof of Lemma 5 we see that the construction of the representation of $\mathfrak{S}$ with φ by sets gives a unique result up to isomorphism.

In the case when K is not intersecting, the hypergraph H is not determined uniquely up to isomorphism. This is caused by the fact that if two edges of $J(K)$ are disjoint, we cannot determine uniquely the cardinality of the intersection of the corresponding edges of $J(H)$; it may be equal to an arbitrary integer between zero and $r - 1$.

Let H_1 be a graph with the vertex set $\{1, 2, 3, 4\}$ and with the edges $\{1, 2\}$ $\{3, 4\}$, let H_2 be a graph with the same vertex set and with the edges $\{1, 2\}$, $\{1, 3\}$. We see that H_1 non $\cong H_2$, $\partial^2 H_1 \cong \partial^2 H_2$.

This fact and the fact that for H which is not intersecting the semilattices $S(H)$ and $S(\partial^r H)$ need not be isomorphic complicate considerably the situation and thus the problem for such graphs remains open.

Reference

[1] Hypergraph Seminar. Ohio State University 1972. Ed. by C. Berge and D. K. Ray-Chaudhuri. Springer Verlag Berlin—Heidelberg—New York 1974.

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