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INTRODUCTION

Works concerning stability of partial differential equations began to appear in the middle of this century. They were stimulated by the efforts to generalize the well known results from the theory of stability of ordinary differential equations, as well as by technical problems connected with the motion of vibrating strings, panels, bars, rods, etc. Such problems have been investigated for example by P. C. PARKS [18], [19], R. R. NACHLINGER, C. D. FAUST [16], H. H. E. LEIPHOLZ [14]. Another group of technical problems, where the stability plays an important role, is the motion of fluids. In the recent years. the problems of stability of the equations of motion of fluids have been often investigated in connection with the bifurcation of their solutions. Such problems have been examined for example by J. SERRIN [22], G. PRODI [20], S. M. ZENKOVSKAJA [26], D. D. JOSEPH, D. H. SATTINGER [11], S. H. DAVIS, CH. KERCZEK [4], D. H. SATTINGER [21].

The equations mentioned above have been sometimes investigated in the form of abstract differential equations. The questions of stability of abstract differential equations may be divided into two groups. The first group contains problems of stability of abstract differential equations with bounded operators in Banach spaces. In this case, many results from the theory of ordinary differential equations were generalized. They are summarized for example in JU. L. DALECKIJ, M. G. KREJN [3]. Problems from the second group are concerned with the stability of abstract differential equations whose coefficients are unbounded operators in Banach spaces. In this case the theory of stability is not so developed as in the former case. Some problems from this field have been studied by Z. I. CHALILOV [9], [10], JU. I. DOMSCHLAK [7], V. I. DERGUZOV [5], [6], J. KURZWEIL [13], I. STRAŠKRABA, O. VEJVODA [25], J. P. FINK, W. S. HALL, A. R. HAUSRATH [8], H. KIELHÖFER [12], J. NEUSTUPA [17], J. BARTÁK [1], [2].

In this paper, stability of solutions of abstract differential equations with (generally) unbounded operators in Hilbert spaces is investigated.

In Part 1, the equation

$$\mathscr{L}\, u(t) \equiv \sum_{i=0}^{n} a_i(A)\, u^{(n-i)}(t) = F(t) \tag{1}$$

is studied. We assume that $a_i(A)$ are functions of a linear, selfadjoint, strictly positive operator A in a Hilbert space, $a_0(A) = Id$.

Section 1.1 is an introduction into the problems we shall deal with.

Auxiliary notions, namely the "type of operator $\mathscr{L}$" are introduced in Section 1.2. Several equivalent properties are derived there.

The correctness of the Cuachy problem given by the equation (1) and by the boundary conditions $u^{(i)}(t_0) = \varphi_i$, $\varphi_i \in \mathscr{D}(A^{(n-i)/n})$, $(i = 0, \ldots, n-1)$, $t_0 \in R^+$, is investigated in Section 1.3. The solution of the Cauchy problem is found, its uniqueness and certain correctness estimates are proved. The aim of this paper is not to study correctness of the Cauchy problem. That is why these questions are studied only in the extent that is necessary for the stability.

Three kinds of operators (stable, exponentially stable, instable) are distinguished on the basis of the type of operator in Section 1.4. It is proved there that every solution of the equation (1) is globally uniformly stable, globally uniformly exponentially stable or instable if and only if the operator $\mathscr{L}$ is stable, exponentially stable or instable, respectively.

Sufficient conditions ensuring that the operator $\mathscr{L}$ is of the type ω are given in Section 1.5. These conditions are formulated as properties of a certain algebraic equation.

Part 2 deals with the stability of solutions of the equation (1) with a right hand side F, depending on t, u and on the derivatives of u.

Section 2.1 contains several auxiliary results.

The equation with an exponentially stable operator $\mathscr{L}$ is studied in Sections 2.2, 2.3. It is proved that, roughly speaking, if the difference $F(u) - F(v)$ is "small enough" for all functions u near the solution v, then the solution v is (globally) uniformly exponentially stable and uniformly stable at constantly acting disturbances or at least (globally) uniformly stable. Further, conditions are given under which, when investigating the uniform exponential stability and stability at constantly acting disturbances, it is possible to restrict our considerations to the equation with the right hand side containing the members of the first order from the original function F only.

The equation with a stable operator is investigated in Sections 2.4, 2.5. In these sections we derive conditions ensuring that the uniform exponential stability and the stability at constantly acting disturbances are determined by the members of the first order of the function F only.

In Section 2.6, conditions that are sufficient for instability are introduced. In this case we need the operator A to satisfy rather stronger assumptions.

In Part 3, special cases of equations including in the abstract form some important equations of mathematical physics, are investigated.

Conditions ensuring stability of the zero solution of the equation

$$\mathscr{L}\, u(t) = F(t, u(t)) \tag{2}$$

are found in Section 3.1. The function F is supposed to be a linear function (with respect to the variable u) with nonconstant coefficients.

Section 3.2 deals with the equation (2), where

$$A = (-1)^p \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \right]^p .$$

Conditions necessary for the linearization of the equation (2) are verified here.

Some results on stability of the so called Timoshenko operator and of the operator of the second order are presented in Section 3.3.

The last two Sections 3.4, 3.5 are devoted to abstract differential equations with bounded operators in Hilbert spaces.

PART 1 — LINEAR EQUATIONS

1.1 FUNDAMENTAL DEFINITIONS AND NOTATION

In the whole paper, let $R_1 = (-\infty, +\infty)$, $R^+ = [0, +\infty)$. $\mathscr{D}(\cdot)$ is the domain of definition of the expression in brackets, H is a Hilbert space with a norm $\|\cdot\|$, $A : \mathscr{D}(A) \subseteq H \to H$ is a linear, selfadjoint and strictly positive operator with the spectral resolution of identity $E(s)$. We shall denote by $\sigma(A)$ the spectrum of the operator A, δ will mean the infimum of the spectrum $\sigma(A)$. The strict positivity of the operator A means that $\delta > 0$.

For a continuous function $f : \sigma(A) \to R_1$ we shall define an operator function $f(A)$ by the relation

$$f(A)\, x = \int_{\sigma(A)} f(s)\, \mathrm{d}E(s)\, x \quad \text{for} \quad x \in \mathscr{D}(f(A)) =$$
$$= \left\{ x \in H \,\Big|\, \int_{\sigma(A)} |f(s)|^2\, \mathrm{d}\|E(s)\, x\|^2 < +\infty \right\}.$$

Remark 1.1.1. In the whole paper, we shall use two types of numeration: for example (1.2.7) means the relation 7 from Section 2 of Part 1; simple numeration is used in individual proofs and does not go beyond the framework of this proof. This numeration will begin by (1) in every proof. The combined (triple) numeration will be used in all other cases: theorems, definitions, remarks etc. will be denoted in this way.

Remark 1.1.2. Constants with the superscript star (for example C^*) will have one meaning only in the whole article. On the other hand, constants without stars can have different values in different situations.

Each interval will be supposed to contain at least two different points.

Let $n \geqq 1$ be a fixed natural number. On the set

$$\mathscr{D}(\mathscr{L}) = \bigcup_{\substack{I=[a,b)\subseteq R^+ \\ I=[a,b]\subseteq R^+}} \bigcap_{i=0}^{n} \mathscr{C}^i(I, \mathscr{D}(A^{(n-i)/n})),$$

we shall define the operator $\mathscr{L}$ by the relation

$$\mathscr{L}\, u(t) = \sum_{i=0}^{n} a_i(A)\, u^{(n-i)}(t),$$

where

(1.1.1) $a_0(A) = Id$, $a_i(s) : \sigma(A) \to R_1$ are continuous functions such that there exists a constant C_0^* which satisfies the inequality $|a_i(s)| \leqq C_0^* s^{i/n}$ for $i = 1, \ldots, n$, $s \in \sigma(A)$.

Let $(u_0, u_1, \ldots, u_{n-1}) \in \mathscr{D}(A) \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{1/n})$. Then we shall define the norm $\|(u_0, u_1, \ldots, u_{n-1})\|_{\mathscr{D}(A)\times\ldots\times\mathscr{D}(A^{1/n})} = [\sum_{i=0}^{n-1} \|A^{(n-i)/n} u_i\|^2]^{1/2}$.

Let us denote $\mathscr{U} = \bigcup_{\substack{I=[a,b)\subseteq R^+ \\ I=[a,b]\subseteq R^+}} \bigcap_{i=0}^{n-1} \mathscr{C}^i(I, \mathscr{D}(A^{(n-i)/n}))$. It holds $(u(t), u'(t), \ldots$ $\ldots, u^{(n-1)}(t)) \in \mathscr{D}(A) \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{1/n})$ for $u \in \mathscr{U}$, $t \in \mathscr{D}(u)$ and so we can define the norm $\|(u(t), u'(t), \ldots, u^{(n-1)}(t))\|_{\mathscr{D}(A)\times\ldots\times\mathscr{D}(A^{1/n})}$ which we shall always denote briefly by $|||u(t)|||$.

Let $t_0 \in R^+$. We shall write $F \in \mathscr{C}([t_0, +\infty), \mathscr{D}(A^\alpha))$ for $F : [t_0, +\infty) \to H$, $\alpha \geqq 0$ if the function $\|F(t)\|_{\mathscr{D}(A^\alpha)} = \|A^\alpha F(t)\|$ is a continuous function of the variable $t \in [t_0, +\infty)$.

Let $I \subseteq R^+$ be an interval of the type $[a, b)$ or $[a, b]$ and let $F : \mathscr{D}(F) = \{(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) \mid u \in \mathscr{U}$ such that $\mathscr{D}(u) \subseteq I,\ t \in \mathscr{D}(u)\} \to H$. We shall write $F \in \mathscr{C}(\mathscr{D}(u_{/I}), \mathscr{D}(A^\alpha))$ for $\alpha \geqq 0$ if for every $u \in \mathscr{U}$ such that $\mathscr{D}(u) \subseteq I$ the function $\|F(t, u(t), u'(t), \ldots, u^{(n-1)}(t))\|_{\mathscr{D}(A^\alpha)}$ depends continuously on the variable t for $t \in \mathscr{D}(u)$.

Instead of $F \in \mathscr{C}(\mathscr{D}(u_{/R^+}), \mathscr{D}(A^\alpha))$ we shall write briefly $F \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^\alpha))$.

Let r be a positive number, let $v \in \mathscr{U}$, $R : \mathscr{D}(R) \to H$ and $\alpha \geqq 0$. We shall write $R \in \mathscr{C}(\mathscr{D}(u), \mathscr{B}(v, r, \mathscr{D}(A^\alpha))$ if $\{(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) \mid u \in \mathscr{U},\ t \in \mathscr{D}(u)$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, $|||u(t) - v(t)||| \leqq r\} \subseteq \mathscr{D}(R)$ and if for all $u \in \mathscr{U}$ such that $\mathscr{D}(u) \subseteq \subseteq \mathscr{D}(v)$, $\|R(t, u(t), u'(t), \ldots, u^{(n-1)}(t))\|_{\mathscr{D}(A^\alpha)}$ is a continuous function of the variable t for $t \in \{t \in \mathscr{D}(u) \mid |||u(t) - v(t)||| \leqq r\}$.

We shall often write $F(t, u(t))$ and $R(t, u(t))$ instead of $F(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$ and $R(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$, respectively.

We shall deal with the following equations

(1.1.2) $\mathscr{L}\, u(t) = 0$,

(1.1.3) $\mathscr{L}\, u(t) = F(t)$, where $F \in \mathscr{C}(R^+, \mathscr{D}(A^{1/n}))$ or $F \in \mathscr{C}([t_0, +\infty), \mathscr{D}(A^{1/n}))$,

(1.1.4) $\mathscr{L}\, u(t) = F(t, u(t))$, where $F \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^{1/n}))$ or $F \in \mathscr{C}(\mathscr{D}(u_{/I}), \mathscr{D}(A^{1/n}))$,

(1.1.5) $\mathscr{L}\, u(t) = F(t, u(t)) + R(t, u(t))$, where $F \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^{1/n}))$ or

$$F \in \mathscr{C}(\mathscr{D}(u_{/I}), \mathscr{D}(A^{1/n})),\ R \in \mathscr{C}(\mathscr{D}(u), \mathscr{B}(v, r, \mathscr{D}(A^{1/n}))$$

(the meaning of the function v will be clear from Definition 1.1.3).

Whenever we shall speak about the equations (1.1.3), (1.1.4), (1.1.5) without any explicit conditions on the right hand side, we shall understand that the first of the two conditions stated above (i.e. $F \in \mathscr{C}(R^+, \mathscr{D}(A^{1/n}))$, $F \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^{1/n}))$) is fulfilled.

The Cauchy problem will be defined by one of the equations (1.1.2)–(1.1.5) and by the initial conditions

(1.1.6) $\quad u^{(i)}(t_0) = \varphi_i\,,\quad t_0 \in R^+\,,\quad \varphi_i \in \mathscr{D}(A^{(n-i)/n})\quad (i = 0, \ldots, n-1)$.

Under a solution of one of the equations (1.1.2)–(1.1.5) we shall understand a function $u \in \mathscr{D}(\mathscr{L})$ fulfilling the corresponding equation for all $t \in \mathscr{D}(u)$. In the case of the equation (1.1.5), it must be moreover $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and $|||u(t) - v(t)||| \leqq r$ for $t \in \mathscr{D}(u)$.

Under the solution of the Cauchy problem we shall understand the solution of the corresponding equation for which $\mathscr{D}(u) \subseteq [t_0 \; +\infty)$, $t_0 \in \mathscr{D}(u)$ and the initial conditions (1.1.6) are fulfilled.

Definition 1.1.1. Let $v : D(v) \to H$ be a solution of the equation (1.1.4). We say that v is *stable with respect to the norm* $|||\cdot|||$ if for every $\varepsilon > 0$, $t_0 \in \mathscr{D}(v)$ there exists $\eta(\varepsilon, t_0) > 0$ such that the implication

$$|||u(t_0) - v(t_0)||| \leqq \eta(\varepsilon, t_0) \Rightarrow |||u(t) - v(t)||| \leqq \varepsilon \quad \text{for} \quad t \in \mathscr{D}(u)$$

is valid for every solution u of the equation (1.1.4) with $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$, $\mathscr{D}(u) \subseteq \mathscr{D}(v)$. If $\eta(\varepsilon, t_0)$ does not depend on $t_0 \in \mathscr{D}(v)$ we shall speak about the *uniform stability with respect to the norm* $|||\cdot|||$. We say that v is *globally stable with respect to the norm* $|||\cdot|||$ if for every $t_0 \in \mathscr{D}(v)$ there exists a constant $K(t_0)$ such that the inequality

$$|||u(t) - v(t)||| \leqq K(t_0)\, |||u(t_0) - v(t_0)|||\,,$$

holds for all solutions u of the equation (1.1.4) for which $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$, $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and for $t \in \mathscr{D}(u)$. If $K(t_0)$ does not depend on $t_0 \in \mathscr{D}(v)$ we shall speak about the *global uniform stability with respect to the norm* $|||\cdot|||$.

Definition 1.1.2. Let $v : \mathscr{D}(v) \to H$ be a solution of the equation (1.1.4). We say that v is *exponentially stable with respect to the norm* $|||\cdot|||$ if for every $t_0 \in \mathscr{D}(v)$ there exist positive numbers $\varrho(t_0)$, $K(t_0)$, $\alpha(t_0)$ such that the implication

$$|||u(t_0) - v(t_0)||| \leqq \varrho(t_0) \Rightarrow |||u(t) - v(t)||| \leqq$$
$$\leqq K(t_0)\, e^{-\alpha(t_0)(t-t_0)} |||u(t_0) - v(t_0)|||\,,$$

is valid for all solutions u of the equation (1.1.4) for which $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$, $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and for $t \in \mathscr{D}(u)$. If $\varrho(t_0)$, $K(t_0)$, $\alpha(t_0)$ do not depend on $t_0 \in \mathscr{D}(v)$, we shall speak about the *uniform exponential stability with respect to the norm* $|||\cdot|||$. If $\varrho = +\infty$ we shall speak about the *global exponential stability with respect to the norm* $|||\cdot|||$. If $\varrho(t_0)$, $K(t_0)$, $\alpha(t_0)$ do not depend on $t_0 \in \mathscr{D}(v)$ and $\varrho = +\infty$, we shall speak about the *global uniform exponential stability with respect to the norm* $|||\cdot|||$.

Definition 1.1.3. Let $v : \mathscr{D}(v) \to H$ be a solution of the equation (1.1.4). We say that v is *uniformly stable at constantly acting disturbances with respect to the norms* $|||\cdot|||$, $\|\cdot\|_D$ if for any $\eta \in (0, r]$ there exist positive numbers η_0, η_D such that the implication $\{|||u(t_0) - v(t_0)||| \leqq \eta_0, \; \|R(t, u(t))\|_D \leqq \eta_D$ for such $t \in \mathscr{D}(u)$ for which $|||u(t) - v(t)||| \leqq \eta\} \Rightarrow \{|||u(t) - v(t)||| \leqq \eta$ for all $t \in \mathscr{D}(u)\}$ holds for every $t_0 \in \mathscr{D}(v)$ and for all solutions u of the equation (1.1.5) for which $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$, $\mathscr{D}(u) \subseteq \mathscr{D}(v)$.

Remark 1 1.3. Under the uniform stability at constantly acting disturbances with respect to the norm $|||\cdot|||$ we shall understand the uniform stability at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_{\mathscr{D}(A^{1/n})}$, in this paper.

Definition 1.1.4. Let $v : \mathscr{D}(v) \to H$ be a solution of the equation (1.1.4). We say that v is *instable with respect to the norm* $|||\cdot|||$ if v is not stable with respect to the norm $|||\cdot|||$.

Definition 1.1.5. A solution of one of the equations (1.1.2)–(1.1.5) defined on an unbounded interval will be called a *maximal solution.*

Remark 1.1.4. For the sake of simplicity, we shall suppose that the solution v from Definitions 1.1.1–1.1.4 whose stability is considered is a maximal solution. The other solutions need not be maximal.

1.2 THE TYPE OF THE OPERATOR

Let $m_i(t; t_0, s)$, $(i = 0, \ldots, n - 1)$ be solutions of the equation

(1.2.1) $$m^{(n)}(t) + a_1(s)\, m^{(n-1)}(t) + \ldots + a_n(s)\, m(t) = 0$$

fulfilling initial conditions

(1.2.2) $$m_i^{(j)}(t_0; t_0, s) = \delta_i^j \quad (i, j = 0, \ldots, n - 1)\,, \quad t_0 \in R^+\,.$$

The symbol of the derivative will always mean the derivative with respect to the variable t, $s \in \sigma(A)$ being a parameter.

The functions m_i $(i = 0, \ldots, n - 1)$ exist, are uniquely determined and $m_i^{(j)}$ depend continuously on s for $j = 0, \ldots, n$.

The function m_{n-1}, which will play an important role in the sequel, will be denoted sometimes by m only. Instead of $m_i(t; 0, s)$ we shall write also $m_i(t; s)$.

Definition 1.2.1. Let $\omega \in R_1$. We say that *the operator* $\mathscr{L}$ *is of the type* ω if there exists a constant $C(\mathscr{L})$ such that $|m^{(i)}(t; s)\, s^{(n-i-1)/n}| \leqq C(\mathscr{L})\, e^{\omega t}$ for $i \in \{0, \ldots, n - 1\}$, $t \in R^+$, $s \in \sigma(A)$.

Remark 1.2.1. If the operator $\mathscr{L}$ is of the type ω, then it is of the type ω_1 for every $\omega_1 \geqq \omega$ as well.

Lemma 1.2.1. *Let* $f : \sigma(A) \to R_1$, $g : \sigma(A) \to R_1$ *be continuous functions for which* $\mathscr{D}(f(A)) = \mathscr{D}(g(A)) = H$. *Then* $|f(s)| \leqq |g(s)|$ *for* $s \in \sigma(A)$ *if and only if* $\|f(A)\, \varphi\| \leqq \|g(A)\, \varphi\|$ *for all* $\varphi \in H$.

Proof. Let us suppose first that $|f(s)| \leqq |g(s)|$ for $s \in \sigma(A)$. Then $\|f(A)\, \varphi\|^2 = \int_{\sigma(A)} |f(s)|^2\, \mathrm{d}\|E(s)\, \varphi\|^2 \leqq \int_{\sigma(A)} |g(s)|^2\, \mathrm{d}\|E(s)\, \varphi\|^2 = \|g(A)\, \varphi\|^2$. Now let $\|f(A)\, \varphi\| \leqq \|g(A)\, \varphi\|$ for all $\varphi \in H$ and let there exist a number $s_0 \in \sigma(A)$ such that

$|f(s_0)| > |g(s_0)|$. Let us find a number $\sigma > 0$ so that $|f(s)| > |g(s)| + \frac{1}{2}(|f(s_0)| - |g(s_0)|)$ for $s \in [s_0 - \sigma, s_0 + \sigma] \cap \sigma(A) \equiv \bar{\sigma}$ and choose $\varphi_0 \in [E(s_0 + \sigma) - E(s_0 - \sigma)] H$, $\varphi_0 \neq 0$. Then $\|f(A) \varphi_0\|^2 = \int_{\bar{\sigma}} |f(s)|^2 \, d\|E(s) \varphi_0\|^2 \geqq \int_{\bar{\sigma}} (|g(s)|^2 + \frac{1}{4}(|f(s_0)| - |g(s_0)|)^2) \, d\|E(s) \varphi_0\|^2 = \|g(A) \varphi_0\|^2 + \frac{1}{4}(|f(s_0)| - |g(s_0)|)^2 \|\varphi_0\|^2 > \|g(A) \varphi_0\|^2$, which is a contradiction.

Theorem 1.2.1. *The following statements are equivalent:*

(i) *The operator $\mathscr{L}$ is of the type ω.*

(ii) *Let $t_0 \in R^+$ be a fixed number. Then there exists a constant $C(\mathscr{L})$ such that the inequality $|m^{(i)}(t; t_0, s) \, s^{(n-i-1)/n}| \leqq C(\mathscr{L}) \, e^{\omega(t-t_0)}$ is valid for $i \in \{0, \ldots, n-1\}$, $s \in \sigma(A)$, $t \geqq t_0$.*

(iii) *There exists a constant $C(\mathscr{L})$ such that the inequality $|m^{(i)}(t; t_0, s) \, s^{(n-i-1)/n}| \leqq C(\mathscr{L}) \, e^{\omega(t-t_0)}$ is valid for $i \in \{0, \ldots, n-1\}$, $s \in \sigma(A)$, $t_0 \in R^+$, $t \geqq t_0$.*

(iv) *There exists a constant $C(\mathscr{L})$ such that the inequality $\|A^{(n-i-1)/n} \, m^{(i)}(t; A) \, \varphi\| \leqq C(\mathscr{L}) \, e^{\omega t} \|\varphi\|$ is valid for $i \in \{0, \ldots, n-1\}$, $\varphi \in H$, $t \in R^+$.*

(v) *Let $t_0 \in R^+$ be a fixed number. Then there exists a constant $C(\mathscr{L})$ such that the inequality $\|A^{(n-i-1)/n} \, m^{(i)}(t; t_0, A) \, \varphi\| \leqq C(\mathscr{L}) \, e^{\omega(t-t_0)} \|\varphi\|$ is valid for $i \in \{0, \ldots, n-1\}$, $\varphi \in H$, $t \geqq t_0$.*

(vi) *There exists a constant $C(\mathscr{L})$ such that the inequality $\|A^{(n-i-1)/n} m^{(i)}(t; t_0, A) \, \varphi\| \leqq C(\mathscr{L}) \, e^{\omega(t-t_0)} \|\varphi\|$ is valid for $i \in \{0, \ldots, n-1\}$, $\varphi \in H$, $t_0 \in R^+$, $t \geqq t_0$.*

Proof. The theorem follows directly from Lemma 1.2.1 in virtue of the independence of the operator $\mathscr{L}$ on t.

Remark 1.2.2. Using Lemma 1.2.1 we can reformulate the condition (1.1.1) as follows: $a_0(A) = Id$, $a_i(s) : \sigma(A) \to R_1$ are continuous functions such that there exists a constant C_0^* so that $\|a_i(A) \, A^{-i/n} \varphi\| \leqq C_0^* \|\varphi\|$ is valid for $i \in \{1, \ldots, n\}$, $\varphi \in H$.

1.3 CORRECTNESS OF THE CAUCHY PROBLEM

Lemma 1.3.1. *If $k \in \{0, \ldots, n-1\}$ then*

$$m_k(t; s) = \sum_{i=1}^{n-k} a_{i-1}(s) \, m^{(n-k-i)}(t; s) .$$

Proof. Obviously, the functions m_k are solutions of the equation (1.2.1). So it suffices to prove

$$m_k^{(j)}(0; s) = \delta_k^j \quad \text{for} \quad j = 0, \ldots, n-1 . \tag{1}$$

Clearly

$$(2) \qquad m_k^{(j)}(0; s) = \sum_{i=1}^{n-k} a_{i-1}(s)\, m^{(n-k-i+j)}(0; s) .$$

Let us distinguish the following three cases

$$(3) \qquad -k - 1 + j < -1 ,$$

$$(4) \qquad -k - 1 + j = -1 ,$$

$$(5) \qquad -k - 1 + j > -1 .$$

Then in the case (3), $m_k^{(j)}(0; s) = 0$ according to (2); $\delta_k^j = 0$ because $j < k$. In the case (4), $m_k^{(j)}(0; s) = a_0(s)\, m^{(n-1)}(0; s) = 1$ according to (2); $\delta_k^j = 1$. In the case (5),

$$m_k^{(j)}(0; s) = a_0(s)\, m^{(n-k-1+j)}(0; s) + \sum_{i=2}^{n-k} a_{i-1}(s)\, m^{(n-k-i+j)}(0; s) =$$

$$= - \sum_{i=1}^{n} a_i(s)\, m^{(n-i-k-1+j)}(0; s) + \sum_{i=1}^{n-k-1} a_i(s)\, m^{(n-i-k-1+j)}(0; s) =$$

$$= - \sum_{i=n-k}^{n} a_i(s)\, m^{(n-i-k-1+j)}(0; s) = 0$$

because of $n - i - k - 1 + j \leqq n - 2$; $\delta_k^j = 0$ because $j > k$. The relation (1) is proved.

Lemma 1.3.2. *Suppose that the operator $\mathscr{L}$ is of the type ω. Then there exists a constant C_1^* such that*

$$(1.3.1) \qquad \left|m^{(i)}(t; s)\, s^{(n-i-1)/n}\right| \leqq C_1^* e^{\omega t}$$

for $i \in \{0, \ldots, 2n - 1\}$, $t \geqq 0$, $s \in \sigma(A)$.

Proof. The inequality (1.3.1) is fulfilled for $i = 0, \ldots, n - 1$ by Definition 1.2.1 (with $C_1^* = C(\mathscr{L})$). As m solves the equation (1.2.1) it is

$$m^{(n)}(t; s) = - \sum_{i=1}^{n} a_i(s)\, m^{(n-i)}(t; s) ;$$

therefore

$$\left|m^{(n)}(t; s)\right| \leqq \sum_{i=1}^{n} \left|a_i(s)\right| \left|m^{(n-i)}(t; s)\right| \leqq C_0^* C(\mathscr{L}) \sum_{i=1}^{n} s^{i/n} s^{(1-i)/n} e^{\omega t} \leqq \tilde{C}_0 s^{1/n} e^{\omega t} ,$$

where $\tilde{C}_0 = \max(C_0^* C(\mathscr{L})\, n, C(\mathscr{L}))$. So the inequality (1.3.1) is valid for $i = 0, \ldots, n$ if we put $C_1^* = \tilde{C}_0$. Now we shall proceed by induction. So we shall suppose that there exists a constant $\tilde{C}_k$ such that the inequality (1.3.1) is fulfilled for $i = 0, \ldots, n + k$ ($k \geqq 0$). Using the induction assumption we get from the equation (1.2.1)

$$\left|m^{(n+k+1)}(t; s)\right| \leqq \sum_{i=1}^{n} |a_i(s)| \left|m^{(n-i+k+1)}(t; s)\right| \leqq C_0^* \tilde{C}_k \sum_{i=1}^{n} s^{i/n} s^{(k-i+2)/n} e^{\omega t} \leqq$$

$$\leqq \tilde{C}_{k+1} s^{(k+2)/n} e^{\omega t}, \quad \text{where} \quad \tilde{C}_{k+1} = \max\left(C_0^* \tilde{C}_k n, \tilde{C}_k\right).$$

We have proved the validity of the inequality (1.3.1) for $i = 0, \ldots, n + k + 1$ with $C_1^* = \tilde{C}_{k+1}$. Now, it is clear that it suffices to put $C_1^* = \tilde{C}_{n-1} = \max\left(C_0^* \tilde{C}_{n-2} n, \tilde{C}_{n-2}\right)$ in order to guarantee the validity of the inequality (1.3.1) for all $i \in \{0, \ldots \ldots, 2n - 1\}$.

Lemma 1.3.3. *Let the operator $\mathscr{L}$ be of the type ω. Then if we put $C_2^* = nC_0^* C_1^*$, the inequality $\left|m_k^{(j)}(t; s)\right| \leqq C_2^* s^{(j-k)/n} e^{\omega t}$ holds for $k = 0, \ldots, n - 1$, $j = 0, \ldots, n$, $s \in \sigma(A)$, $t \geqq 0$.*

Proof. By Lemma 1.3.1,

$$m_k^{(j)}(t; s) = \sum_{i=1}^{n-k} a_{i-1}(s)\, m^{(n-k-i+j)}(t; s).$$

Further, it is $0 \leqq n - k - i + j \leqq 2n - 1$ and hence by Lemma 1.3.2

$$\left|m_k^{(j)}(t; s)\right| \leqq C_0^* C_1^* \sum_{i=1}^{n-k} s^{(i-1)/n} s^{(j+1-k-i)/n} e^{\omega t} = C_2^* s^{(j-k)/n} e^{\omega t}.$$

Remark 1.3.1. As the operator $\mathscr{L}$ does not depend on t, it holds: *If the operator $\mathscr{L}$ is of the type ω then*

$$\left|m^{(i)}(t; t_0, s)\right| \leqq C_1^* s^{(1-n+i)/n} e^{\omega(t-t_0)},$$

$$\left|m_k^{(j)}(t; t_0, s)\right| \leqq C_2^* s^{(j-k)/n} e^{\omega(t-t_0)}$$

for $i = 0, \ldots, 2n - 1$, $k = 0, \ldots, n - 1$, $j = 0, \ldots, n$, $s \in \sigma(A)$, $t_0 \in R^+$, $t \geqq t_0$.

Lemma 1.3.4. *There exists at most one maximal solution of the Cauchy problem* (1.1.3), (1.1.6).

Proof. As to the linearity of the operator $\mathscr{L}$ it suffices to prove that if $\mathscr{L}\, u(t) = 0$, $u^{(i)}(t_0) = 0$ $(i = 0, \ldots, n - 1)$ then u is identically zero. Suppose on the contrary that

(1) there exists a point $\tau > t_0$ such that $u(\tau) \neq 0$.

This implies that for a suitable $\alpha \geqq \delta$, it holds

(2) $$E(\alpha)\, u(\tau) \neq 0.$$

Let us put $P = E(\alpha)$, $u_0(t) = P\, u(t)$. Obviously u_0 solves the equation $\mathscr{L}\, u(t) = 0$ and fulfils the initial conditions $u_0^{(i)}(t_0) = 0$ $(i = 0, \ldots, n - 1)$. (Cf. M. Sova [23] pp. 217–222.)

As to the relation $\mathscr{L} u_0(t) = \mathscr{L}P\, u(t) = (P\, u(t))^{(n)} + a_1(A)P\,(P\, u(t))^{(n-1)} + \ldots$ $\ldots + a_n(A)\, PP\, u(t) = 0$ and to the boundedness of the operators $a_i(A)P$ $(i = 1, \ldots, n)$, Theorem 7.2 from M. Sova [24] p. 38 yields $P\, u(t) \equiv 0$, which contradicts (2).

Lemma 1.3.5. *Let the operator $\mathscr{L}$ be of the type ω. Then the function*

$$u(t) = \sum_{i=0}^{n-1} m_i(t;\, t_0, A)\, \varphi_i + \int_{t_0}^{t} m(t + t_0 - \tau;\, t_0, A)\, F(\tau)\, d\tau =$$

$$= \sum_{i=0}^{n-1} \int_{\sigma(A)} m_i(t;\, t_0, s)\, dE(s)\, \varphi_i + \int_{t_0}^{t} \int_{\sigma(A)} m(t + t_0 - \tau;\, t_0, s)\, dE(s)\, F(\tau)\, d\tau$$

is a maximal solution of the Cauchy problem (1.1.3), (1.1.6).

Proof. Lemma follows directly from Lemma 1.3.3 and Remark 1.3.1 (See also I. Straškraba, O. Vejvoda [25] pp. 638–639, Propositions 1.1.2–1.1.5.)

Lemma 1.3.6. *Let the operator $\mathscr{L}$ be of the type ω. Then*

$$\left|\left|\left| \sum_{i=0}^{n-1} m_i(t;\, t_0, A)\, \varphi_i \right|\right|\right| \leqq n C_2^* \left(\sum_{i=0}^{n-1} \| A^{(n-i)/n} \varphi_i \|^2 \right)^{1/2} e^{\omega(t-t_0)}$$

for $\varphi_i \in \mathscr{D}(A^{(n-i)/n})$, $(i = 0, \ldots, n-1)$, $t_0 \in R^+$, $t \geqq t_0$.

Proof: By Lemma 1.3.3 and Remark 1.3.1 it holds

$$\| A^{(n-j)/n} \sum_{i=0}^{n-1} m_i^{(j)}(t;\, t_0, A)\, \varphi_i \|^2 \leqq n \sum_{i=0}^{n-1} \int_{\sigma(A)} s^{2(n-j)/n} |m_i^{(j)}(t;\, t_0, s)|^2\, d\| E(s)\, \varphi_i \|^2 \leqq$$

$$\leqq n C_2^{*2} e^{2\omega(t-t_0)} \sum_{i=0}^{n-1} \int_{\sigma(A)} s^{2(n-j)/n} s^{2(j-i)/n}\, d\| E(s)\, \varphi_i \|^2 =$$

$$= n C_2^{*2} e^{2\omega(t-t_0)} \sum_{i=0}^{n-1} \| A^{(n-i)/n} \varphi_i \|^2 \quad \text{for} \quad j = 0, \ldots, n-1\,.$$

Summing this inequalities from $j = 0$ to $j = n - 1$ we get

$$\sum_{j=0}^{n-1} \| A^{(n-j)/n} \sum_{i=0}^{n-1} m_i^{(j)}(t;\, t_0, A)\, \varphi_i \|^2 \leqq n^2 C_2^{*2} \sum_{i=0}^{n-1} \| A^{(n-i)/n} \varphi_i \|^2\, e^{2\omega(t-t_0)}\,.$$

Lemma 1.3.7. *Let the operator $\mathscr{L}$ be of the type ω, $F \in \mathscr{C}([t_0 + \infty), \mathscr{D}(A^{1/n}))$. Then*

$$\left|\left|\left| \int_{t_0}^{t} m(t + t_0 - \tau;\, t_0, A)\, F(\tau)\, d\tau \right|\right|\right| \leqq n\, C(\mathscr{L}) \int_{t_0}^{t} e^{\omega(t-\tau)} \| A^{1/n} F(\tau) \|\, d\tau$$

for $t_0 \in R^+$, $t \geqq t_0$.

Proof. This lemma is a special case of Lemma 2.1.1. That is why we omit the proof here.

From Lemmas 1.3.4, 1.3.5, 1.3.6, 1.3.7 we obtain easily

Theorem 1.3.1. (Correctness Theorem) *Let the operator* $\mathscr{L}$ *be of the type* ω, $F \in \mathscr{C}([t_0, +\infty), \mathscr{D}(A^{1/n}))$. *Then the maximal solution* u *of the Cauchy problem* (1.1.3), (1.1.6)

(i) *is determined uniquely,*

(ii) *has the form* $u(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i + \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F(\tau)\, d\tau$ *for* $t \geqq t_0$,

(iii) *fulfils the estimate* $|||u(t)||| \leqq C_3^* |||u(t_0)|||\, e^{\omega(t-t_0)} + C_4^* \int_{t_0}^{t} e^{\omega(t-\tau)} \|A^{1/n} F(\tau)\|\, d\tau$ *for* $t \geqq t_0$, *where* $C_3^* = nC_2^*$, $C_4^* = n\, C(\mathscr{L})$.

1.4 STABILITY, EXPONENTIAL STABILITY, INSTABILITY OF THE OPERATOR

Definition 1.4.1. We say that *the operator* $\mathscr{L}$ *is stable* (*exponentially stable, instable*) if $\mathscr{L}$ is of the type 0 ($\mathscr{L}$ is of the type $\omega < 0$, $\mathscr{L}$ is not of the type 0, respectively).

Remark 1.4.1. Obviously, an exponentially stable operator is stable, too.

Let I be an interval of the type $[a, b) \subseteq R^+$, or $[a, b] \subseteq R^+$. Under the symbol $O_{/I}$ we shall understand the function $O_{/I} \in \mathscr{D}(\mathscr{L})$, for which $\mathscr{D}(O_{/I}) = I$, $O_{/I} \equiv 0$.

Lemma 1.4.1. *If the operator* $\mathscr{L}$ *is stable* (*exponentially stable*) *then the zero solution* $O_{/[0,+\infty)}$ *of the equation* (1.1.2) *is globally uniformly stable* (*globally uniformly exponentially stable*) *with respect to the norm* $|||\cdot|||$.

Proof. Lemma follows immediately from Theorem 1.3.1.

Lemma 1.4.2. *Let us suppose that the zero solution* $O_{/[0,+\infty)}$ *of the equation* (1.1.2) *is globally uniformly stable* (*globally uniformly exponentially stable*) *with respect to the norm* $|||\cdot|||$. *Then every maximal solution* v *of the equation* (1.1.3) *is globally uniformly stable* (*globally uniformly exponentially stable*) *with respect to the norm* $|||\cdot|||$.

Proof. Obviously, if the zero solution $O_{/[0,+\infty)}$ is globally uniformly stable (globally uniformly exponentially stable), then for every $t_0 \in R^+$ the solution $O_{/[t_0,+\infty)}$ is globally uniformly stable (globally uniformly exponentially stable).

Now, let v be a maximal solution of the equation (1.1.3). Let us write all other solutions u of the equation (1.1.3) for which $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ in the form $u = v_{/\mathscr{D}(u)} + w$. Then w fulfils the equation (1.1.2) on $\mathscr{D}(u)$. Lemma follows directly from the relation $w = u - v_{/\mathscr{D}(u)}$.

Lemma 1.4.3. *Let the operator $\mathscr{L}$ be stable (exponentially stable). Then every maximal solution v of the equation (1.1.3) is globally uniformly stable (globally uniformly exponentially stable) with respect to the norm $|||\cdot|||$.*

Proof. Lemma is an easy consequence of Lemmas 1.4.1, 1.4.2.

Lemma 1.4.4. *If the operator $\mathscr{L}$ is instable, then for every $t_0 \in R^+$ the zero solution $O_{/[t_0,+\infty)}$ of the equation (1.1.2) is instable with respect to the norm $|||\cdot|||$.*

Proof. We shall prove the instability of the solution $O_{/[t_0,+\infty)}$ of the equation (1.1.2). Taking into account the assumption of the lemma we obtain by the relation (*ii*) of Theorem 1.2.1:

(1) For an arbitrary natural number k there exist $s_k \in \sigma(A)$, $t_k > t_0$, $i_k \in \{0, \dots, n-1\}$ such that $|m^{(i_k)}(t_k; t_0, s_k)\, s_k^{(n-i_k-1)/n}| > k$.

Without loss of generality we may suppose $i_k = i \in \{0, \dots, n-1\}$.

Let us find a number $\sigma_k > 0$ so that

(2) $$|m^{(i)}(t_k; t_0, s)\, s^{(n-i-1)/n}| > k \quad \text{for} \quad s \in [s_k - \sigma_k, s_k + \sigma_k] \cap \sigma(A).$$

Further let us choose $\varphi_k \in [E(s_k + \sigma_k) - E(s_k - \sigma_k)]\, H$, $\varphi_k \neq 0$ and put $\psi_k = A^{-1/n}\varphi_k$. Evidently $\psi_k \in \mathscr{D}(A^{1/n})$. Let us put $u_k(t) = m(t; t_0, A)\, \psi_k/(k\|A^{1/n}\psi_k\|)$. Then:

(3) The function u_k is a maximal solution of the equation (1.1.2) such that
$$|||u_k(t_0)||| = k^{-1}.$$

By (2) it holds

(4) $$|||u_k(t_k)||| \geqq \|A^{(n-i)/n}\, m^{(i)}(t_k; t_0, A)\, \psi_k\|\, (k\|A^{1/n}\psi_k\|)^{-1} > k\|\varphi_k\|\, (k\|A^{1/n}\psi_k\|)^{-1} = 1.$$

Lemma follows at once from (3), (4).

Lemma 1.4.5. *If for every $t_0 \in R^+$ the zero solution $O_{/[t_0,+\infty)}$ of the equation (1.1.2) is instable with respect to the norm $|||\cdot|||$, then every maximal solution v of the equation (1.1.3) is instable with respect to the norm $|||\cdot|||$.*

Proof. Let v be a maximal solution of the equation (1.1.3). Let us write all other solutions u of the equation (1.1.3) for which $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ in the form $u = v_{/\mathscr{D}(u)} + w$. Then the function w solves the equation (1.1.2) on $\mathscr{D}(u)$ and the statement of the lemma follows directly from the relation $w = u - v_{/\mathscr{D}(u)}$.

Lemma 1.4.6. *Let the operator $\mathscr{L}$ be instable. Then every maximal solution v of the equation (1.1.3) is instable with respect to the norm $|||\cdot|||$.*

Proof. Lemma is an easy consequence of Lemmas 1.4.4, 1.4.5.

Theorem 1.4.1. *The following statements are true:*

(i) *The operator $\mathscr{L}$ is stable if and only if every maximal solution of the equation (1.1.3) is globally uniformly stable with respect to the norm $|||\cdot|||$.*

(ii) *The operator $\mathscr{L}$ is exponentially stable if and only if every maximal solution of the equation (1.1.3) is globally uniformly exponentially stable with respect to the norm $|||\cdot|||$.*

(iii) *The operator $\mathscr{L}$ is instable if and only if every maximal solution of the equation (1.1.3) is instable with respect to the norm $|||\cdot|||$.*

Proof. The statements (i), (iii) are easy consequences of Lemmas 1.4.3, 1.4.6 and Remark 1.4.1. According to Lemma 1.4.3, in order to prove (ii) it suffices to prove:

(1) If the operator $\mathscr{L}$ is not exponentially stable then there exists a maximal solution of the equation (1.1.3) that is not globally uniformly exponentially stable with respect to the norm $|||\cdot|||$.

So let us suppose that the operator $\mathscr{L}$ is not exponentially stable. Then by (iv) of Theorem 1.2.1 it holds:

(2) For an arbitrary $\omega < 0$ and k a natural number there exist $t_{(\omega,k)} \geqq 0$, $i_{(\omega,k)} \in \{0, \ldots, n-1\}$, $\varphi_{(\omega,k)} \in H$ in such a way that $\|A^{(n-i_{(\omega,k)}-1)/n}\, m^{(i_{(\omega,k)})}(t_{(\omega,k)}; A)\,. \,\varphi_{(\omega,k)}\| > k e^{\omega t_{(\omega,k)}} \|\varphi_{(\omega,k)}\|$.

Without loss of generality we can suppose $i_{(\omega,k)} = i \in \{0, \ldots, n-1\}$. Putting $u_{(\omega,k)}(t) = m(t; A)\, \psi_{(\omega,k)}$, where $\psi_{(\omega,k)} = A^{-1/n}\varphi_{(\omega,k)}$, we get

(3) $$|||u_{(\omega,k)}(t_{(\omega,k)})||| \geqq \|A^{(n-i)/n}\, m^{(i)}(t_{(\omega,k)}; A)\, \psi_{(\omega,k)}\| > \\ > k e^{\omega t_{(\omega,k)}} \|\varphi_{(\omega,k)}\| = k e^{\omega t_{(\omega,k)}} |||u_{(\omega,k)}(0)|||$$

and by Theorem 1.3.1

(4) $u_{(\omega,k)}$ is a maximal solution of the equation (1.1.2).

Using (2), (3), (4) we obtain:

(5) The zero solution $O_{/[0,+\infty)}$ of the equation (1.1.2) is not globally uniformly exponentially stable with respect to the norm $|||\cdot|||$.

According to Theorem 1.3.1 it holds:

(6) There exists a solution $v : [0, +\infty) \to H$ of the equation (1.1.3).

Obviously,

(7) if the solution v is globally uniformly exponentially stable with respect to the norm $|||\cdot|||$, then the solution $O_{/[0,+\infty)}$ of the equation (1.1.2) is globally uniformly exponentially stable with respect to the norm $|||\cdot|||$.

Now, (1) follows directly from (5), (6), (7).

The aim of this section is to find sufficient conditions ensuring the operator $\mathscr{L}$ to be (or not to be) of the type ω.

If $+\infty$ is a limit point of $\sigma(A)$ we shall suppose in addition:

(1.5.1) There exist numbers a_i^* such that $\lim_{\substack{s\to+\infty \\ s\in\sigma(A)}} a_i(s)\, s^{-i/n} = a_i^*, \quad (i = 1, \ldots, n)$,

in this section.

Let $\lambda_i = \lambda_i(s)$ $(i = 1, \ldots, n)$ be the roots of the equation

$$\lambda^n(s) + a_1(s)\,\lambda^{n-1}(s) + \ldots + a_n(s) = 0\,. \tag{1.5.2}$$

Further, let Λ_i $(i = 1, \ldots, n)$ be the roots of the equation

$$\Lambda^n + a_1^*\Lambda^{n-1} + \ldots + a_n^* = 0\,. \tag{1.5.3}$$

With help of Laplace transformation we obtain

Lemma 1.5.1. *It holds*

$$m(t; s) = \int_0^t \int_0^{\tau_1} \ldots \int_0^{\tau_{n-2}} e^{\lambda_1(s)(t-\tau_1)} e^{\lambda_2(s)(\tau_1-\tau_2)} \ldots e^{\lambda_{n-1}(s)(\tau_{n-2}-\tau_{n-1})} e^{\lambda_n(s)\tau_{n-1}}\, d\tau_{n-1} \ldots d\tau_1$$

for $s \in \sigma(A)$. Besides, for such $s \in \sigma(A)$ for which $\lambda_i(s) \neq \lambda_j(s)$ for $i \neq j$ it holds

$$m(t; s) = \sum_{i=1}^{n} \frac{e^{\lambda_i(s)t}}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (\lambda_i(s) - \lambda_j(s))}\,.$$

Lemma 1.5.2. *Let $\bar{\omega} < \omega$ be constants and let either* $\operatorname{Re} \lambda_i(s) \leqq \bar{\omega}$ *for $i = 1, \ldots, n$, $s \in \sigma(A) \cap [\delta, S_0]$ or* $\operatorname{Re} \lambda_i(s) \leqq \omega$, $\{i \neq j \Rightarrow \lambda_i(s) \neq \lambda_j(s)\}$ $(i, j = 1, \ldots, n)$, $s \in \sigma(A) \cap [\delta, S_0]$. *Then there exists a constant K (depending on S_0 and $\omega - \bar{\omega}$) such that $|m^{(i)}(t; s)| \leqq Ke^{\omega t}$ for $i = 0, \ldots, n-1$, $s \in \sigma(A) \cap [\delta, S_0]$, $t \geqq 0$.*

Lemma 1.5.3. *There exists a constant K such that $|\lambda_i(s)| \leqq Ks^{1/n}$ for $i = 1, \ldots, n$, $s \in \sigma(A)$.*

Proof. Put $\bar{\lambda}(s) = \lambda(s)\, s^{-1/n}$. Then $\bar{\lambda}^n(s) + a_1(s)\, s^{-1/n}\, \bar{\lambda}^{n-1}(s) + \ldots + a_n(s)\, s^{-1} = 0$. As the coefficients at the powers of $\bar{\lambda}(s)$ are bounded for $s \in \sigma(A)$ we can find a constant K fulfilling the inequality $|\bar{\lambda}_i(s)| \leqq K$. This yields $|\lambda_i(s)| \leqq Ks^{1/n}$ for $i = 1, \ldots, n$, $s \in \sigma(A)$.

Lemma 1.5.4. *Let $+\infty$ be a limit point of $\sigma(A)$. Suppose that the equation (1.5.3) has simple roots only. Then for certain constants $S_0 \geqq \delta$, $K > 0$ the relation $\lambda_i(s) - \lambda_j(s)| \geqq Ks^{1/n}$ holds for $i \neq j$, $(i, j = 1, \ldots, n)$, $s \in \sigma(A) \cap [S_0, +\infty)$.*

Proof. Denoting suitably the roots we have $\lim_{\substack{s\to+\infty \\ s\in\sigma(A)}} \lambda_i(s)\, s^{-1/n} = \Lambda_i$. The statement of the lemma follows easily from $\min_{\substack{i,j=1,\dots,n \\ i\neq j}} |\Lambda_i - \Lambda_j| > 0$.

Theorem 1.5.1. *Let* $\bar{\omega} < \omega$, $S_0 \in [\delta, +\infty]$. *Suppose that* $\operatorname{Re} \lambda_i(s) \leqq \bar{\omega}$ *for* $i = 1, \dots, n$, $s \in \sigma(A) \cap [\delta, S_0)$ *and* $\operatorname{Re} \lambda_i(s) \leqq \omega$, $\{i \neq j \Rightarrow \lambda_i(s) \neq \lambda_j(s)\}$ *for* $i, j = 1, \dots, n$, $s \in \sigma(A) \cap [S_0, +\infty)$. *Besides, if* $+\infty$ *is a limit point of* $\sigma(A)$ *we shall suppose that the equation* (1.5.3) *has simple roots only. Then the operator* $\mathscr{L}$ *is of the type* ω.

Proof. Lemmas 1.5.1, 1.5.2, 1.5.3, 1.5.4 yield the existence of a constant C for which $|m^{(i)}(t; s)| \leqq C s^{(i+1-n)/n} e^{\omega t}$ for $i = 0, \dots, n-1$, $s \in \sigma(A)$. This proves the theorem.

Theorem 1.5.2. *If the relation* $\operatorname{Re} \lambda_{i_0}(s_0) > \omega$ *is valid for some numbers* $s_0 \in \sigma(A)$, $i_0 \in \{1, \dots, n\}$, *then the operator* $\mathscr{L}$ *is not of the type* ω.

Proof. We shall investigate the equation (1.2.1) at the point s_0, i.e. the equation

$$m^{(n)}(t) + a_1(s_0)\, m^{(n-1)}(t) + \dots + a_n(s_0)\, m(t) = 0\,. \tag{1}$$

Instead of $m_k(t; s_0)$ we shall write $m_k(t)$ only. The system of functions m_k ($k = 0, \dots, n-1$) forms a fundamental system of solutions of the equation (1). Thus, it is possible to write every solution of the equation (1) in the form $\sum_{k=0}^{n-1} \beta_k m_k$ for a suitable choice of constants β_k.

Let us introduce the following condition:

$(2)_k$ There exists a constant C_k such that $|m_k(t)| \leqq C_k e^{\omega t}$ for $t \geqq 0$.

Suppose that the condition $(2)_k$ is fulfilled for all $k \in \{0, \dots, n-1\}$. Then, because the function $\exp(\lambda_{i_0}(s_0)\, t)$ solves the equation (1), it holds

$$|\exp(\lambda_{i_0}(s_0)\, t)| = \Big|\sum_{k=0}^{n-1} \beta_k\, m_k(t)\Big| \leqq C e^{\omega t} \tag{3}$$

for a certain constant C and every $t \geqq 0$.

Because $|\exp(\lambda_{i_0}(s_0)\, t)| = \exp(\operatorname{Re} \lambda_{i_0}(s_0)\, t)$, (3) yields $\exp((\operatorname{Re} \lambda_{i_0}(s_0) - \omega)\, t) \leqq C$, which is not possible. So we have proved

(4) There exists $k_0 \in \{0, \dots, n-1\}$ for which the condition $(2)_{k_0}$ is not fulfilled.

Let us introduce the another condition:

$(5)_k$ There exists a constant C_k such that $|m^{(k)}(t)| \leqq C_k e^{\omega t}$ for $t \geqq 0$.

By Lemma 1.3.1, $m_{k_0}(t) = \sum_{i=1}^{n-k_0} a_{i-1}(s_0)\, m^{(n-k_0-i)}(t)$. Supposing that $(5)_k$ is fulfilled for all $k \in \{0, \dots, n-1\}$ we get $|m_{k_0}(t)| \leqq Ce^{\omega t}$ for a certain constant C and all $t \geqq 0$. But this contradicts (4). Thus the relation $(5)_{k_0}$ is not valid for at least one $k_0 \in \{0, \dots, n-1\}$. This proves the theorem.

PART 2 — NONLINEAR EQUATIONS

2.1 INTRODUCTION

We shall deal with the equations (1.1.4), (1.1.5) in this part.

Lemma 2.1.1. *Let the operator $\mathscr{L}$ be of the type ω, $F \in \mathscr{C}(\mathscr{D}(u_{/I}), \mathscr{D}(A^{1/n}))$. Then*

$$\left|\left|\left| \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F(\tau, u(\tau))\, d\tau \right|\right|\right| \leqq n\, C(\mathscr{L}) \int_{t_0}^{t} e^{\omega(t-\tau)}\, \|A^{1/n} F(\tau, u(\tau))\|\, d\tau$$

for every $u \in \mathscr{U}$ such that $[t_0, t] \subseteq \mathscr{D}(u) \subseteq I$.

Proof. If $i \in \{0, \dots, n-1\}$, $\tau \in [t_0, t]$, then

$$\|A^{(n-i)/n}\, m^{(i)}(t + t_0 - \tau; t_0, A)\, F(\tau, u(\tau))\|^2 =$$

$$= \int_{\sigma(A)} s^{2(n-i)/n} |m^{(i)}(t + t_0 - \tau; t_0, s)|^2\, d\|E(s)\, F(\tau, u(\tau))\|^2 \leqq$$

$$\leqq C^2(\mathscr{L})\, e^{2\omega(t-\tau)} \int_{\sigma(A)} s^{2(n-i)/n} s^{2(i-n+1)/n}\, d\|E(s)\, F(\tau, u(\tau))\|^2 =$$

$$= [C(\mathscr{L})\, e^{\omega(t-\tau)} \|A^{1/n}\, F(\tau, u(\tau))\|]^2 .$$

Thus

$$\left|\left|\left| \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F(\tau, u(\tau))\, d\tau \right|\right|\right| =$$

$$= \left[\sum_{i=0}^{n-1} \left\| A^{(n-i)/n} \frac{\partial^i}{\partial t^i} \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F(\tau, u(\tau))\, d\tau \right\|^2 \right]^{1/2} \leqq$$

$$\leqq \sum_{i=0}^{n-1} \left\| A^{(n-i)/n} \frac{\partial^i}{\partial t^i} \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F(\tau, u(\tau))\, d\tau \right\| \leqq$$

$$\leqq \int_{t_0}^{t} \sum_{i=0}^{n-1} \|A^{(n-i)/n}\, m^{(i)}(t + t_0 - \tau; t_0, A)\, F(\tau, u(\tau))\|\, d\tau \leqq$$

$$\leqq n\, C(\mathscr{L}) \int_{t_0}^{t} e^{\omega(t-\tau)} \|A^{1/n}\, F(\tau, u(\tau))\|\, d\tau .$$

As an easy consequence of Lemmas 1.3.3, 1.3.6, 2.1.1 and Remark 1.3.1 we obtain

Theorem 2.1.1. *Let the operator* $\mathscr{L}$ *be of the type* ω, $F \in \mathscr{C}(\mathscr{D}(u_{/I}), \mathscr{D}(A^{1/n}))$, $\mathscr{D}(u) \subseteq [t_0, +\infty) \subseteq I$, $t_0 \in \mathscr{D}(u)$, *let* $u : \mathscr{D}(u) \to H$ *be the solution of the equation* (1.1.4) *fulfilling the initial conditions* (1.1.6). *Then*

(i) $$u(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i + \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F(\tau, u(\tau))\, d\tau\,,$$

(ii) $$\left|\left|\left| \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i \right|\right|\right| \leqq C_3^* |||u(t_0)|||\, e^{\omega(t-t_0)}\,,$$

$$\left|\left|\left| \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F(\tau, u(\tau))\, d\tau \right|\right|\right| \leqq C_4^* \int_{t_0}^{t} e^{\omega(t-\tau)} \|A^{1/n} F(\tau, u(\tau))\|\, d\tau$$

for $t \in \mathscr{D}(u)$.

Theorem 2.1.2. *Let* $v : \mathscr{D}(v) \to H$ *be a maximal solution of the equation* (1.1.4), $F \in \mathscr{C}(\mathscr{D}(u_{/\mathscr{D}(v)}), \mathscr{D}(A^{1/n}))$. *Then the zero solution* $O_{/\mathscr{D}(v)}$ *of the equation*

$$\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t)) \tag{2.1.1}$$

is stable (globally stable, globally uniformly stable, uniformly stable, exponentially stable, uniformly exponentially stable, globally uniformly exponentially stable, uniformly stable at constantly acting disturbances, instable) with respect to the norm $|||\cdot|||$ *if and only if the solution* v *of the equation* (1.1.4) *is stable (globally stable, globally uniformly stable, uniformly stable, exponentially stable, uniformly exponentially stable, globally uniformly exponentially stable, uniformly stable at constantly acting disturbances, instable, respectively) with respect to the norm* $|||\cdot|||$.

Proof. Let us write all other solutions u of the equation (1.1.4), satisfying the condition $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, in the form $u = v_{/\mathscr{D}(u)} + w$. As v solves the equation (1.1.4), w solves the equation (2.1.1) on $\mathscr{D}(u)$. The statements of theorem follow directly from the relation $w = u - v_{/\mathscr{D}(u)}$.

Theorem 2.1.3. (Ju. L. Daleckij, M. G. Krejn [3], p. 155). *Let*

$$\varphi(t) \leqq \alpha e^{-\nu(t-t_0)} + \beta \int_{t_0}^{t} e^{-\nu(t-\tau)}\, p(\tau)\, \varphi(\tau)\, d\tau\,,$$

where $p(t)$ *is a nonnegative continuous function,* α, β, ν *are constants,* $t \geqq t_0$. *Then*

$$\varphi(t) \leqq \alpha e^{-\nu(t-t_0) + \beta \int_{t_0}^{t} p(\tau) d(\tau)} \quad \textit{for} \quad t \geqq t_0\,.$$

In this section we shall suppose that the operator $\mathscr{L}$ is of the type $\omega < 0$, $F \in \in \mathscr{C}(\mathscr{D}(u_{/\mathscr{D}(v)}), \mathscr{D}(A^{1/n}))$, and $v : \mathscr{D}(v) \to H$ is a maximal solution of the equation (1.1.4).

Theorem 2.2.1. *Suppose*

(2.2.1) *There exist numbers K, $R > 0$ such that $\|A^{1/n}[F(t, v(t) + u(t)) - - F(t, v(t))]\| \leqq K|||u(t)|||$ for $u \in \mathscr{U}$ satisfying $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, and for $t \in \mathscr{D}(u)$ such that $|||u(t)||| \leqq R$.*

Then if $\omega + KC_4^* < 0$ or $\omega + KC_4^* \leqq 0$, the solution v of the equation (1.1.4) is uniformly exponentially stable (if $R = +\infty$ moreover, then the solution v is globally uniformly exponentially stable) or uniformly stable, respectively (if, moreover, $R = +\infty$, then the solution v is globally uniformly stable), with respect to the norm $|||\cdot|||$.

Proof. According to Theorem 2.1.2 it suffices to prove the (global) uniform exponential stability, resp. the (global) uniform stability of the zero solution $O_{/\mathscr{D}(v)}$ of the equation (2.1.1).

Let $t_0 \in \mathscr{D}(v)$, $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$, $u : \mathscr{D}(u) \to H$ be such a solution of the problem (2.1.1), (1.1.6) that

$$|||u(t_0)||| = [\sum_{i=0}^{n-1} \|A^{(n-i)/n}\varphi_i\|^2]^{1/2} \leqq \min\left(\frac{R}{2}, \frac{R}{2C_3^* + 1}\right).$$

In the case $R < +\infty$ let us suppose

(1) There exists a number $h > 0$ such that $[t_0, t_0 + h] \subseteq \mathscr{D}(u)$, $|||u(\tau)||| < R$ for $\tau \in [t_0\ t_0 + h)$, $|||u(t_0 + h)||| = R$.

Then with help of Theorem 2.1.1 we get

$$|||u(t)||| \leqq C_3^*|||u(t_0)|||\, e^{\omega(t-t_0)} + \\ + C_4^* \int_{t_0}^t e^{\omega(t-\tau)}\|A^{1/n}[F(\tau, v(\tau) + u(\tau)) - F(\tau, v(\tau))]\|\, d\tau \leqq \\ \leqq C_3^*|||u(t_0)|||\, e^{\omega(t-t_0)} + KC_4^* \int_{t_0}^t e^{\omega(t-\tau)}|||u(\tau)|||\, d\tau \quad \text{for} \quad t \in [t_0, t_0 + h]$$

and so by Theorem 2.1.3,

$$|||u(t)||| \leqq C_3^*|||u(t_0)|||\, e^{(\omega + KC_4^*)(t-t_0)} \leqq C_3^*|||u(t_0)||| \leqq \\ \leqq C_3^* \frac{R}{2C_3^* + 1} < R \quad \text{for} \quad t \in [t_0, t_0 + h].$$

But the last inequality is a contradiction with (1). Thus, if $t \in \mathscr{D}(u)$ then $|||u(t)||| \leqq \leqq C_3^* |||u(t_0)||| \, e^{(\omega + KC_4^*)(t - t_0)}$. This proves the theorem in the case $R < +\infty$. In the case $R = +\infty$ we can proceed similarly.

Theorem 2.2.2. *Let us suppose that*

(2.2.2) $F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t))$ *for* $u \in \mathscr{U}$ *such that* $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, $t \in \mathscr{D}(u)$ *where* $F(t, v(t)) \in \mathscr{C}(\mathscr{D}(v), \mathscr{D}(A^{1/n}))$, $F_L, F_N \in \in \mathscr{C}(\mathscr{D}(u_{/\mathscr{D}(v)}), \mathscr{D}(A^{1/n}))$,

(2.2.3) *there exist numbers* $K_1, K_2, R_1 > 0$, $\nu > 0$ *such that if* $u \in \mathscr{U}$ *fulfils* $\mathscr{D}(u) \subseteq \subseteq \mathscr{D}(v)$ *and* $t \in \mathscr{D}(u)$ *is such that* $|||u(t)||| \leqq R_1$ *then*

$$\|A^{1/n} F_L(t, u(t))\| \leqq K_1 |||u(t)|||\,,$$
$$\|A^{1/n} F_N(t, u(t))\| \leqq K_2 |||u(t)|||^{1+\nu}\,.$$

Then if $\omega + K_1 C_4^* < 0$, *the solution* v *of the equation* (1.1.4) *is uniformly exponentially stable with respect to the norm* $|||\cdot|||$.

Proof. Let us choose a number $R \in (0, R_1]$ so small that

(1) $$\omega + (K_1 + K_2 R^\nu)\, C_4^* < 0\,.$$

Then if $u \in \mathscr{U}$ fulfils $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and $t \in \mathscr{D}(u)$ fulfils $|||u(t)||| \leqq R$, it holds

(2) $$\|A^{1/n}[F(t, v(t) + u(t)) - F(t, v(t))]\| = = \|A^{1/n}[F_L(t, u(t)) + F_N(t, u(t))]\| \leqq \|A^{1/n} F_L(t, u(t))\| + + \|A^{1/n} F_N(t, u(t))\| \leqq (K_1 + K_2 R^\nu)\, |||u(t)|||\,.$$

The assertion of the theorem follows easily from (1), (2) with help of Theorem 2.2.1.

2.3 STABILITY AT CONSTANTLY ACTING DISTURBANCES IN THE CASE OF THE EXPONENTIALLY STABLE OPERATOR

In this section we shall suppose that the operator $\mathscr{L}$ is of the type $\omega < 0$, $F \in \in \mathscr{C}(\mathscr{D}(u_{/\mathscr{D}(v)}), \mathscr{D}(A^{1/n}))$ and $v : \mathscr{D}(v) \to H$ is a maximal solution of the equation (1.1.4).

Theorem 2.3.1. *Let* F *fulfil the condition* (2.2.1) *and let* $\omega + KC_4^* < 0$. *Then the solution* v *of the equation* (1.1.4) *is uniformly stable at constantly acting disturbances with respect to the norm* $|||\cdot|||$.

Proof. It follows from Theorem 2.1.2 that it suffices to prove the uniform stability at constantly acting disturbances of the zero solution $O_{/\mathscr{D}(v)}$ of the equation $\mathscr{L}\, u(t) =$

$= F(t, v(t) + u(t)) - F(t, v(t))$. Let $\eta \in (0, r]$ be given. Without loss of generality we may assume $\eta \leqq R$. Let us choose a number $h > 0$ such that $C_3^* e^{(\omega + KC_4^*)h} < 1$. Further, let us find numbers $\eta_0 \in (0, \frac{1}{2}\eta]$, $\eta_D > 0$ to this h so that

$$C_3^* \eta_0 + C_4^* \eta_D h e^{-\omega h} \leqq \frac{\eta}{2}, \tag{1}$$

$$(C_3^* \eta_0 + C_4^* \eta_D h e^{-\omega h})\, e^{(\omega + KC_4^*)h} \leqq \eta_0 . \tag{2}$$

Let $t_0 \in \mathscr{D}(v)$, $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$, let $u : \mathscr{D}(u) \to H$ be a solution of the equation $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t)) + R(t, u(t))$. By Theorem 2.1.1 it holds

$$|||u(t)||| \leqq C_3^* |||u(t_0)|||\, e^{\omega(t - t_0)} + C_4^* \int_{t_0}^t e^{\omega(t - \tau)} \|A^{1/n}[F(\tau, v(\tau) + u(\tau)) - F(\tau, v(\tau))]\| \,\mathrm{d}\tau + C_4^* \int_{t_0}^t e^{\omega(t-\tau)} \|A^{1/n} R(\tau, u(\tau))\| \,\mathrm{d}\tau \quad \text{for} \quad t \in \mathscr{D}(u) . \tag{3}$$

Surely it suffices to prove the validity of the following implication:

$$\{|||u(t_0)||| \leqq \eta_0 ,\ \|A^{1/n} R(t, u(t))\| \leqq \eta_D \ \text{for} \ t \in \mathscr{D}(u) \ \text{such that} \ |||u(t)||| \leqq \eta\} \Rightarrow \{|||u(t)||| \leqq \eta \ \text{for} \ t \in \mathscr{D}(u)\} . \tag{4}$$

Suppose

(5) There exists a number $\tilde{h} < h$ such that $[t_0, t_0 + \tilde{h}] \subseteq \mathscr{D}(u)$, $|||u(\tau)||| < \eta$ for $\tau \in [t_0, t_0 + \tilde{h})$, $|||u(t_0 + \tilde{h})||| = \eta$.

Using (3) we get

$$|||u(t)||| \leqq (C_3^* \eta_0 + C_4^* \eta_D \tilde{h} e^{-\omega \tilde{h}})\, e^{\omega(t - t_0)} + KC_4^* \int_{t_0}^t e^{\omega(t-\tau)} |||u(\tau)||| \,\mathrm{d}\tau \leqq$$

$$\leqq (C_3^* \eta_0 + C_4^* \eta_D h e^{-\omega h})\, e^{\omega(t - t_0)} + KC_4^* \int_{t_0}^t e^{\omega(t-\tau)} |||u(\tau)||| \,\mathrm{d}\tau \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}]$$

and thus by Theorem 2.1.3

$$|||u(t)||| \leqq (C_3^* \eta_0 + C_4^* \eta_D h e^{-\omega h})\, e^{(\omega + KC_4^*)(t - t_0)} \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] . \tag{6}$$

We obtain from (1), (6)

$$|||u(t_0 + \tilde{h})||| \leqq C_3^* \eta_0 + C_4^* \eta_D h e^{-\omega h} \leqq \frac{\eta}{2} < \eta ,$$

which is a contradiction with (5). So we have proved

$$|||u(t)||| \leqq \eta \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u) . \tag{7}$$

If $t_0 + h \in \mathscr{D}(u)$ then in virtue of (2), (6), (7)

(8) $$|||u(t_0 + h)||| \leqq \eta_0 \, .$$

Using k-times the relation (8) we obtain

(9) $|||u(t_0 + kh)||| \leqq \eta_0$ for every natural number for which $t_0 + kh \in \mathscr{D}(u)$.

Now, let us find an integer k and a number $s \in [0, h)$ to $t \in \mathscr{D}(u)$ in such a way that $t = t_0 + kh + s$. Then taking (7), (9) into account we see that $|||u(t)||| = |||u(t_0 + kh + s)||| \leqq \eta$. This proves the theorem.

Similarly as Theorem 2.2.2 we can prove

Theorem 2.3.2. *Let F fulfil (2.2.2), (2.2.3) and $\omega + K_1 C_4^* < 0$. Then the solution v of the equation (1.1.4) is uniformly stable at constantly acting disturbances with respect to the norm $|||\cdot|||$.*

2.4 THE CASE OF THE STABLE OPERATOR

In this section we shall suppose that the operator $\mathscr{L}$ is stable (i.e. $\mathscr{L}$ is of the type 0), $F \in \mathscr{C}(\mathscr{D}(u_{/\mathscr{D}(v)}), \mathscr{D}(A^{1/n}))$ and $v : \mathscr{D}(v) \to H$ is a maximal solution of the equation (1.1.4).

For the function F that fulfils the condition (2.2.2) we introduce the following conditions:

(2.4.1) There exists a number $\varkappa > 0$ such that if $\varphi_i \in \mathscr{D}(A^{(n-i)/n})$, $i = 0, \ldots, n - 1$, $[\sum_{i=0}^{n-1} \|A^{(n-i)/n}\varphi_i\|^2]^{1/2} \leqq \varkappa$ then for any $t_0 \in R^+$ there exists a maximal solution of the equation $\mathscr{L}\, u(t) = F_L(t, u(t))$, satisfying the initial conditions (1.1.6).

(2.4.2) There exist constants $K, R > 0$, $\nu > 0$ such that

(i) if $\mathscr{D}(u_i) \subseteq \mathscr{D}(v)$ for $u_i \in \mathscr{U}$ and $t \in \mathscr{D}(u_1) \cap \mathscr{D}(u_2)$ is such that $|||u_i(t)||| \leqq R$ $(i = 1, 2)$ then $\|A^{1/n}[F_L(t, u_1(t)) - F_L(t, u_2(t))]\| \leqq K|||u_1(t) - u_2(t)|||$,

(ii) if $u \in \mathscr{U}$ fulfils $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and $t \in \mathscr{D}(u)$ is such that $|||u(t)||| \leqq R$ then $\|A^{1/n} F_N(t, u(t))\| \leqq K|||u(t)|||^{1+\nu}$.

Theorem 2.4.1. *Let (2.2.2), (2.4.1), (2.4.2) be fulfilled for the function F. Further, let $F_L(t, O_{/I}) = 0$ for every $I \subseteq \mathscr{D}(v)$ and let the zero solution $O_{/D(v)}$ of the equation*

(2.4.3) $$\mathscr{L}\, u(t) = F_L(t, u(t))$$

be uniformly exponentially stable with respect to the norm $|||\cdot|||$. Then the solution v

of the equation (1.1.4) *is uniformly exponentially stable with respect to the norm* $|||\cdot|||$ *as well.*

Proof. According to Theorem 2.1.2, it suffices to prove the uniform exponential stability of the zero solution $O_{/D(v)}$ of the equation

$$\mathscr{L}\, u(t) = F_L(t, u(t)) + F_N(t, u(t)) . \tag{1}$$

Let $t_0 \in \mathscr{D}(v)$. We restrict ourselves to such $\varphi_i \in \mathscr{D}(A^{(n-i)/n})$ for which $[\sum_{i=0}^{n-1} \|A^{(n-i)/n}\varphi_i\|^2]^{1/2} \leqq \varkappa$. In virtue of (2.4.1) there exists a maximal solution u_L of the equation (2.4.3) satisfying the initial conditions (1.1.6) for such φ_i. By Theorem 2.1.1

$$u_L(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i + \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F_L(\tau, u_L(\tau))\, d\tau . \tag{2}$$

By the assumptions of the theorem we conclude:

(3) There exist positive numbers C, α, ϱ (independent of the choice of $t_0 \in \mathscr{D}(v)$) such that $|||u_L(t_0)||| \leqq \varrho \Rightarrow |||u_L(t)||| \leqq Ce^{-\alpha(t-t_0)}\, |||u_L(t_0)|||$ for $t \geqq t_0$.

If u_N is a solution of the equation (1) that fulfils the initial conditions (1.1.6), then

$$u_N(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i + \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F_L(\tau, u_N(\tau))\, d\tau + \tag{4}$$

$$+ \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F_N(\tau, u_N(\tau))\, d\tau , \quad \text{for} \quad t \in \mathscr{D}(u_N) .$$

Let $0 < \alpha_1 < \alpha$, $C_1 > C$. Let us find a number $h > 0$ so that

$$Ce^{-(\alpha-\alpha_1)h} < 1 ,$$

and choose a number $R_1 \in (0, \min(R, \varrho, \varkappa))$ to this fixed h in such a way that

$$C + C_3^* C_4^* K R_1^\nu h e^{[C_4^* K(2+R_1^\nu)+\alpha_1]h} \leqq C_1 , \tag{5}$$

$$Ce^{-(\alpha-\alpha_1)h} + C_3^* C_4^* K R_1^\nu h e^{[C_4^* K(2+R_1^\nu)+\alpha_1]h} \leqq 1 . \tag{6}$$

Finally, let us fix a number $\eta \in (0, R_1)$ so that $C_1\eta < R_1$ and make a restriction to such initial conditions for which

$$|||u_L(t_0)||| = |||u_N(t_0)||| = [\sum_{i=0}^{n-1} \|A^{(n-i)/n}\varphi_i\|^2]^{1/2} \leqq \eta .$$

It follows from (3) that

$$|||u_L(t)||| \leqq C\eta < C_1\eta < R_1 \leqq R \quad \text{for} \quad t \geqq t_0 .$$

Suppose

(7) There exists a number $\tilde{h} \leqq h$ so that $[t_0, t_0 + \tilde{h}] \subseteq \mathscr{D}(u_N)$, $|||u_N(\tau)||| < R_1$ for $\tau \in [t_0, t_0 + \tilde{h})$, $|||u_N(t_0 + \tilde{h})||| = R_1$.

Then using (4), the condition (2.4.2), Theorem 2.1.1 and the equality $F_L(t, O_{/[t_0, +\infty)}) = 0$, we obtain

$$|||u_N(t)||| \leqq C_3^*|||u_N(t_0)||| + C_4^* \int_{t_0}^{t} [\|A^{1/n} F_L(\tau, u_N(\tau))\| + \|A^{1/n} F_N(\tau, u_N(\tau))\|] \, d\tau \leqq$$

$$\leqq C_3^*|||u_N(t_0)||| + C_4^* K(1 + R_1^\nu) \int_{t_0}^{t} |||u_N(\tau)||| \, d\tau \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] \, .$$

This yields with help of Theorem 2.1.3

(8) $$|||u_N(t)||| \leqq C_3^*|||u_N(t_0)||| \, e^{C_4^*K(1+R_1^\nu)\tilde{h}} \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] \, .$$

Subtracting (2) from (4) and using Theorem 2.1.1, the estimate (8) and the condition (2.4.2) we obtain

$$|||u_N(t) - u_L(t)||| \leqq C_4^* K \int_{t_0}^{t} |||u_N(\tau) - u_L(\tau)||| \, d\tau + C_4^* K \int_{t_0}^{t} |||u_N(\tau)|||^{1+\nu} \, d\tau \leqq$$

$$\leqq C_4^* K \int_{t_0}^{t} |||u_N(\tau) - u_L(\tau)||| \, d\tau + C_3^* C_4^* K R_1^\nu \tilde{h} e^{C_4^*K(1+R_1^\nu)\tilde{h}} |||u_N(t_0)|||$$

$$\text{for} \quad t \in [t_0, t_0 + \tilde{h}] \, .$$

Thus it follows from Theorem 2.1.3 that

(9) $$|||u_N(t) - u_L(t)||| \leqq C_3^* C_4^* K R_1^\nu \tilde{h} e^{C_4^*K(2+R_1^\nu)\tilde{h}} |||u_N(t_0)||| \leqq$$

$$\leqq C_3^* C_4^* K R_1^\nu \tilde{h} e^{[C_4^*K(2+R_1^\nu)+\alpha_1]\tilde{h}} e^{-\alpha_1(t-t_0)} |||u_N(t_0)||| \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] \, .$$

From (3), (9) we obtain

(10) $$|||u_N(t)||| \leqq |||u_L(t)||| + |||u_N(t) - u_L(t)||| \leqq$$

$$\leqq [C + C_3^* C_4^* K R_1^\nu \tilde{h} e^{[C_4^*K(2+R_1^\nu)+\alpha_1]\tilde{h}}] \, e^{-\alpha_1(t-t_0)} |||u_N(t_0)||| \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] \, .$$

As $\tilde{h} \leqq h$ we get from (5), (10) that

$$|||u_N(t)||| \leqq [C + C_3^* C_4^* K R_1^\nu h e^{[C_4^*K(2+R_1^\nu)+\alpha_1]h}] \, \eta \leqq C_1 \eta < R_1 \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] \, .$$

However, this is a contradiction with (7). Thus

(11) $$|||u_N(t)||| < R_1 \leqq R \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u_N)$$

and the relations (9), (10) will be valid for $t \in [t_0, t_0 + h] \cap \mathscr{D}(u_N)$ if we write h instead of $\tilde{h}$.

If $t_0 + h \in \mathscr{D}(u_N)$ we get by virtue of (3), (9), (11)

$$(12)\quad |||u_N(t_0 + h)||| \leqq [Ce^{-(\alpha-\alpha_1)h} + C_3^* C_4^* K R_1^\nu h e^{[C_4^* K(2+R_1^\nu)+\alpha_1]h}]\, e^{-\alpha_1 h} |||u_N(t_0)||| \,.$$

From (5), (6), (10), (12) we conclude

$$(13)\qquad |||u_N(t)||| \leqq C_1 e^{-\alpha_1(t-t_0)} |||u_N(t_0)||| \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u_N)\,,$$

$$(14)\qquad |||u_N(t_0 + h)||| \leqq e^{-\alpha_1 h} |||u_N(t_0)||| \quad \text{if} \quad t_0 + h \in \mathscr{D}(u_N)\,.$$

The uniform exponential stability of the zero solution $O_{/\mathscr{D}(v)}$ of the equation (1) is an easy consequence of the inequalities (13), (14). By (14) the inequality

$$|||u_N(t_0 + kh)||| \leqq e^{-\alpha_1 kh} |||u_N(t_0)|||$$

holds for an arbitrary natural number k provided $t_0 + kh \in \mathscr{D}(u_N)$. Now let $t \in \mathscr{D}(u_N)$. Let us find a natural number k and a number $s \in [0, h)$ so that $t = t_0 + kh + s$. Then

$$|||u_N(t)||| = |||u_N(t_0 + kh + s)||| \leqq C_1 e^{-\alpha_1 s} |||u_N(t_0 + kh)||| \leqq$$
$$\leqq C_1 e^{-\alpha_1(s+kh)} |||u_N(t_0)||| = C_1 e^{-\alpha_1(t-t_0)} |||u_N(t_0)||| \,.$$

This proves the theorem.

Moreover, it is clear that we have proved

Corollary 2.4.1. *Let the assumptions of Theorem* 2.4.1 *be fulfilled and let the implication*

$$|||u_L(t_0)||| \leqq \varrho \Rightarrow |||u_L(t)||| \leqq Ce^{-\alpha(t-t_0)} |||u_L(t_0)|||$$

hold for all $t_0 \in \mathscr{D}(v)$, $t \geqq t_0$ *and for solutions* u_L *of the equation* (2.4.3). *Let* $C_1 > C$, $0 < \alpha_1 < \alpha$. *Then there exists a number* $\eta > 0$ *such that the implication*

$$|||u_N(t_0)||| \leqq \eta \Rightarrow |||u_N(t)||| \leqq C_1 e^{-\alpha_1(t-t_0)} |||u_N(t_0)|||$$

holds for $t_0 \in \mathscr{D}(v)$, *for solutions* u_N *of the equation* $\mathscr{L}\, u(t) = F_L(t, u(t)) + F_N(t, u(t))$ *and for* $t \in \mathscr{D}(u_N)$.

2.5 STABILITY AT CONSTANTLY ACTING DISTURBANCES IN THE CASE OF THE STABLE OPERATOR

In this section we shall suppose that the operator $\mathscr{L}$ is stable (i.e. $\mathscr{L}$ is of the type 0), $F \in \mathscr{C}(\mathscr{D}(u_{/\mathscr{D}(v)}), \mathscr{D}(A^{1/n}))$ and $v : \mathscr{D}(v) \to H$ is a solution of the equation (1.1.4).

Theorem 2.5.1. *Let* (2.2.2), (2.4.1), (2.4.2) *be satisfied for the function* F. *Further, let* $F_L(t, O_{/I}) = 0$ *for every* $I \subseteq \mathscr{D}(v)$ *and let the zero solution* $O_{/\mathscr{D}(v)}$ *of the equation*

$$(2.5.1)\qquad \mathscr{L}\, u(t) = F_L(t, u(t))$$

be uniformly exponentially stable with respect to the norm $|||\cdot|||$. *Then the solution* v *of the equation* (1.1.4) *is uniformly stable at constantly acting disturbances with respect to the norm* $|||\cdot|||$.

Proof. According to Theorem 2.1.2 it suffices to prove the uniform stability at constantly acting disturbances of the zero solution $O_{/\mathscr{D}(v)}$ of the equation

$$\mathscr{L}\, u(t) = F_L(t, u(t)) + F_N(t, u(t)) \,. \tag{1}$$

Let $t_0 \in \mathscr{D}(v)$. Remember that if u_L is a maximal solution of the equation (2.5.1) fulfilling the initial conditions (1.1.6) then

$$u_L(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i + \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F_L(\tau, u_L(\tau))\, \mathrm{d}\tau \,; \tag{2}$$

(3) there exist positive constants C, α, ϱ (independent of the choice of $t_0 \in \mathscr{D}(v)$) such that $|||u_L(t_0)||| \leqq \varrho \Rightarrow |||u_L(t)||| \leqq C e^{-\alpha(t-t_0)} |||u_L(t_0)|||$ for $t \geqq t_0$.

If u_D solves the equation

$$\mathscr{L}\, u(t) = F_L(t, u(t)) + F_N(t, u(t)) + R(t, u(t)) \tag{4}$$

and fulfils the initial conditions (1.1.6) it is

$$\begin{aligned} u_D(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i + \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F_L(\tau, u_D(\tau))\, \mathrm{d}\tau \,+ \\ + \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, F_N(\tau, u_D(\tau))\, \mathrm{d}\tau \,+ \\ + \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, R(\tau, u_D(\tau))\, \mathrm{d}\tau \,, \quad \text{for} \quad t \in \mathscr{D}(u_D) \,. \end{aligned} \tag{5}$$

By Theorem 2.1.1

$$\begin{aligned} \|A^{1/n}\, R(\tau, u(\tau))\| \leqq \eta_D \quad \text{for all} \quad \tau \in [t_0, t] \Rightarrow \\ \Rightarrow \left|\left|\left| \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A)\, R(\tau, u(\tau))\, \mathrm{d}\tau \right|\right|\right| \leqq C_4^*\, \eta_D(t - t_0) \,. \end{aligned} \tag{6}$$

Let now $\eta \in (0, r]$ be given. Without loss of generality we may suppose $\eta \leqq \leqq \min(R, \varrho, \varkappa)$. Consider a number $h > 0$ such that $Ce^{-\alpha h} < 1$. Further, let us choose a number $R_1 \in (0, \eta]$ for which

$$Ce^{-\alpha h} + C_3^* C_4^* K h R_1^{\nu} e^{C_4^* K(2 + \eta^{\nu})h} < 1 \,.$$

Let us find a number $\eta_0 \in (0, R_1)$ such that

$$C_3^* \eta_0 e^{C_4^* K(1 + \eta^{\nu})h} < R_1 \,, \quad C\eta_0 \leqq R \,. \tag{7}$$

Finally, let us choose a number $\eta_D > 0$ satisfying

(8) $$(C_3^*\eta_0 + C_4^*\eta_D h)\, e^{C_4*K(1+\eta^\nu)h} < R_1\,,$$

(9) $$[Ce^{-\alpha h} + C_3^* C_4^* K h R_1^\nu e^{C_4*K(2+\eta^\nu)h}]\,\eta_0 + $$
$$+ [C_4^{*2} K h^2 R_1^\nu e^{C_4*K(2+\eta^\nu)h} + C_4^* h e^{C_4*Kh}]\,\eta_D \leqq \eta_0\,.$$

Now, we shall verify the implication

(10) $\{|||u_D(t_0)||| = [\sum_{i=0}^{n-1} \|A^{(n-i)/n}\varphi_i\|^2]^{1/2} \leqq \eta_0,\ \|A^{1/n} R(t, u_D(t))\| \leqq \eta_D$ for such $t \in \mathscr{D}(u_D)$ which satisfy $|||u_D(t)||| \leqq \eta\} \Rightarrow \{|||u_D(t)||| \leqq \eta$ for $t \in \mathscr{D}(u_D)\}$.

By (2.4.1) there exists a maximal solution u_L of the equation (2.5.1) that fulfils the same initial conditions as u_D. According to (3), (7)

(11) $$|||u_L(t)||| \leqq C\eta_0 \leqq R \quad \text{for} \quad t \geqq t_0\,.$$

Suppose

(12) There exists a number $\tilde{h} \leqq h$ such that $[t_0, t_0 + \tilde{h}] \subseteq \mathscr{D}(u_D)$, $|||u_D(\tau)||| < R_1$ for $\tau \in [t_0, t_0 + \tilde{h})$, $|||u_D(t_0 + \tilde{h})||| = R_1$.

Because $R_1 \leqq \eta \leqq R$ we obtain from (5) using Theorem 2.1.1 and the relations (2.4.2), (6), $F_L(t, O_{/[t_0,+\infty)}) = 0$ the inequality

$$|||u_D(t)||| \leqq C_3^*|||u_D(t_0)||| + C_4^*K\int_{t_0}^t |||u_D(\tau)|||\,\mathrm{d}\tau + C_4^*K\int_{t_0}^t |||u_D(\tau)|||^{1+\nu}\,\mathrm{d}\tau +$$
$$+ C_4^*\eta_D(t - t_0) \leqq C_3^*\eta_0 + C_4^*K(1 + \eta^\nu)\int_{t_0}^t |||u_D(\tau)|||\,\mathrm{d}\tau + C_4^*\eta_D\tilde{h}$$
$$\text{for} \quad t \in [t_0, t_0 + \tilde{h}]\,.$$

Hence with help of Theorem 2.1.3 and the relation (8) we conclude

(13) $$|||u_D(t)||| \leqq (C_3^*\eta_0 + C_4^*\eta_D\tilde{h})\, e^{C_4*K(1+\eta^\nu)(t-t_0)} \leqq$$
$$\leqq (C_3^*\eta_0 + C_4^*\eta_D h)\, e^{C_4*K(1+\eta^\nu)h} < R_1 \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}]\,.$$

But this is a contradiction with (12). So we have proved

(14) $$|||u_D(t)||| < R_1 \leqq \eta \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u_D)\,.$$

Subtracting (2) from (5) and using Theorem 2.1.1 and the relations $\eta \leqq R$, (2.4.2), (6), (11), (13), (14), we get

$$|||u_D(t) - u_L(t)||| \leqq C_4^*K\int_{t_0}^t |||u_D(\tau) - u_L(\tau)|||\,\mathrm{d}\tau + C_4^*K\int_{t_0}^t |||u_D(\tau)|||^{1+\nu}\,\mathrm{d}\tau +$$

$$+ C_4^* \eta_D(t - t_0) \leqq C_4^* K \int_{t_0}^{t} |||u_D(\tau) - u_L(\tau)||| \, d\tau +$$

$$+ C_4^* K h R_1^\nu (C_3^* \eta_0 + C_4^* \eta_D h) \, e^{C_4 * K(1 + \eta^\nu)h} + C_4^* \eta_D h \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u_D) .$$

This yields

$$(15) \qquad |||u_D(t) - u_L(t)||| \leqq \{C_3^* C_4^* K h R_1^\nu e^{C_4 * K(1 + \eta^\nu)h} \eta_0 +$$

$$+ [C_4^{*2} K h^2 R_1^\nu e^{C_4 * K(1 + \eta^\nu)h} + C_4^* h] \, \eta_D\} \, e^{C_4 * K h} \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u_D) .$$

It follows from (3), (15) that

$$(16) \qquad |||u_D(t_0 + h)||| \leqq |||u_L(t_0 + h)||| + |||u_D(t_0 + h) - u_L(t_0 + h)||| \leqq$$

$$\leqq [C e^{-\alpha h} + C_3^* C_4^* K h R_1^\nu e^{C_4 * K(2 + \eta^\nu)h}] \, \eta_0 +$$

$$+ [C_4^{*2} K h^2 R_1^\nu e^{C_4 * K(2 + \eta^\nu)h} + C_4^* h e^{C_4 * K h}] \, \eta_D , \quad \text{if} \quad t_0 + h \in \mathscr{D}(u_D) .$$

Further, we conclude from (9), (16) that

$$(17) \qquad |||u_D(t_0 + h)||| \leqq \eta_0 \quad \text{if} \quad t_0 + h \in \mathscr{D}(u_D) .$$

The implication (10) is now an easy consequence of (14), (17). Let us find an integer k and a number $s \in [0, h)$ to $t \in \mathscr{D}(u_D)$ in such a way that $t = t_0 + kh + s$. Then using k-times the relation (17) we can see that $|||u_D(t_0 + kh)||| \leqq \eta_0$ and by virtue of (14), $|||u_D(t)||| = |||u_D(t_0 + kh + s)||| \leqq \eta$.

This proves the implication (10) and hence also the theorem.

2.6 INSTABILITY

In this section we shall suppose that $F \in \mathscr{C}(\mathscr{D}(u_{/[t_0, +\infty)}), \mathscr{D}(A^{1/n}))$, and $v : \mathscr{D}(v) = [t_0. +\infty) \to H$ is a maximal solution of the equation (1.1.4).

Theorem 2.6.1. *If $+\infty$ is a limit point of $\sigma(A)$ then we shall suppose that the equation* (1.5.3) *has simple roots only. Further, let us assume*:

(2.6.1) *There exist numbers $K > 0$, $R > 0$, $\nu > 0$ so that $\|A^{1/n}[F(t, v(t) + u(t)) - F(t, v(t))]\| \leqq K |||u(t)|||^{1+\nu}$ for such $u \in \mathscr{U}$ that $\mathscr{D}(u) = [t_0, +\infty)$ and such $t \in \mathscr{D}(u)$ that $|||u(t)||| \leqq R$.*

(2.6.2) *If $\lambda_i(s)$ are solutions of the equation* (1.5.2) *then $\lambda_i(s) \neq \lambda_j(s)$ for $i \neq j$ $(i, j = 1, \ldots, n)$, $s \in \sigma(A)$.*

(2.6.3) *There exists an eigenvalue s_0 of the operator A and an index $i_0 \in \{1, \ldots, n\}$ such that* $\operatorname{Re} \lambda_{i_0}(s_0) > 0$ *and* $\max_{i=1,\ldots,n} \sup_{s \in \sigma(A)} \operatorname{Re} \lambda_i(s) \leqq \operatorname{Re} \lambda_{i_0}(s_0)$.

(2.6.4) *There exists a number* $\varkappa > 0$ *such that if* $\varphi_i \in \mathscr{D}(A^{(n-i)/n})$ $(i = 0, \ldots, n-1)$, $[\sum_{i=0}^{n-1} \|A^{(n-i)/n}\varphi_i\|^2]^{1/2} \leqq \varkappa$ *then there exists a maximal solution of the equation* $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))$ *fulfilling the initial conditions* (1.1.6).

Then the solution v *of the equation* (1.1.4) *is instable with respect to the norm* $|||\cdot|||$.

Proof. According to Theorem 2.1.2 it is sufficient to show that the zero solution $O_{/[t_0, +\infty)}$ of the equation

(1) $$\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))$$

is instable with respect to the norm $|||\cdot|||$. Theorem 1.5.1 yields:

(2) The operator $\mathscr{L}$ is of the type $\omega = \operatorname{Re} \lambda_{i_0}(s_0)$.

Choose such a number C that

(3) $$C \in \left(1, \min\left(2, 1 + \frac{C_4^* K}{\omega \nu} R^\nu\right)\right).$$

We can easily verify that there exists a number $\eta_0 \in (0, \varkappa]$ such that if $\eta \in (0, \eta_0]$ the equation

(4) $$e^{\omega h} = \frac{1}{\eta C}\left[\frac{(C-1)\,\omega\nu}{C_4^* K C}\right]^{1/\nu}$$

has a unique solution $h = h(\eta) > 0$.

Denote by φ_0 such an element of the space H that

(5) $$A\varphi_0 = s_0 \varphi_0\,,$$
$$\|(\varphi_0, \lambda_{i_0}(A)\,\varphi_0, \ldots, \lambda_{i_0}^{n-1}(A)\,\varphi_0)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{(n-1)/n})\times\ldots\times\mathscr{D}(A^{1/n})} = 1\,.$$

Further, let u_0 be a maximal solution of the equation (1), satisfying the initial conditions $u_0^{(j)}(t_0) = \eta \lambda_{i_0}^j(A)\,\varphi_0$, $(j = 0, \ldots, n-1)$, $\eta \in (0, \eta_0]$. Then it holds

(6) $$u_0(t) = \eta \exp(\lambda_{i_0}(A)(t - t_0))\,\varphi_0 + \\ + \int_{t_0}^t m(t + t_0 - \tau; t_0, A)\,[F(\tau, v(\tau) + u_0(\tau)) - F(\tau, v(\tau))]\,d\tau\,.$$

Using (5) we get

(7) $$|||\eta \exp(\lambda_{i_0}(A)(t - t_0))\,\varphi_0||| = \\ = \eta[\sum_{i=0}^{n-1} \|A^{(n-i)/n} \lambda_{i_0}^i(A) \exp(\lambda_{i_0}(A)(t - t_0))\,\varphi_0\|^2]^{1/2} = \\ = \eta \exp(\omega(t - t_0))\,, \quad \text{in particular,} \quad |||u_0(t_0)||| = \eta\,.$$

Let us suppose:

(8) There exists a number $\tilde{h} \in (0, h)$ such that

$$|||u_0(t)||| < \eta C e^{\omega(t-t_0)} \text{ for } t \in [t_0, t_0 + \tilde{h}),$$
$$|||u_0(t_0 + \tilde{h})||| = \eta C e^{\omega\tilde{h}}.$$

Then from (3), (4), (8) we obtain

$$|||u_0(t)||| \leqq \eta C e^{\omega\tilde{h}} < \eta C e^{\omega h} = \left[\frac{(C-1)\,\omega\nu}{C_4^*KC}\right]^{1/\nu} \leqq \left[\frac{C_4^*K}{\omega\nu} R^\nu \frac{\omega\nu}{C_4^*K}\right]^{1/\nu} = R$$
$$\text{for} \quad t \in [t_0, t_0 + \tilde{h}],$$

and so with respect to (2), (4), (6), (7), (8), (2.6.1) and Theorem 2.1.1 we can conclude

$$|||u_0(t_0 + \tilde{h})||| \leqq \eta e^{\omega\tilde{h}} + C_4^*K \int_{t_0}^{t_0+\tilde{h}} e^{\omega(t_0+\tilde{h}-\tau)} |||u_0(\tau)|||^{1+\nu} \,\mathrm{d}\tau \leqq$$
$$\leqq \eta e^{\omega\tilde{h}} + C_4^*K(\eta C)^{1+\nu} \int_{t_0}^{t_0+\tilde{h}} e^{\omega(t_0+\tilde{h}-\tau)} e^{\omega(\tau-t_0)(1+\nu)} \,\mathrm{d}\tau \leqq$$
$$\leqq \eta e^{\omega\tilde{h}} + \frac{C_4^*K}{\omega\nu} (\eta C)^{1+\nu} e^{\omega(\nu+1)\tilde{h}} = \eta e^{\omega\tilde{h}}\left[1 + \frac{C_4^*K}{\omega\nu} \eta^\nu C^{1+\nu} e^{\omega\nu\tilde{h}}\right] <$$
$$< \eta e^{\omega\tilde{h}}\left[1 + \frac{C_4^*K}{\omega\nu} \eta^\nu C^{1+\nu} e^{\omega\nu h}\right] = \eta C e^{\omega\tilde{h}},$$

which is a contradiction with (8). Thus

(9) $$|||u_0(t)||| \leqq \eta C e^{\omega(t-t_0)} \leqq R \quad \text{for} \quad t \in [t_0, t_0 + h].$$

Now, it follows from (3), (4), (6), (7), (9), (2.6.1) and Theorem 2.1.1 that

$$|||u_0(t_0 + h)||| \geqq \eta e^{\omega h} - C_4^*K \int_{t_0}^{t_0+h} e^{\omega(t_0+h-\tau)} |||u_0(\tau)|||^{1+\nu} \,\mathrm{d}\tau \geqq$$
$$\geqq \eta e^{\omega h}\left[1 - \frac{C_4^*K}{\omega\nu} \eta^\nu C^{1+\nu} e^{\omega\nu h}\right] = \frac{1}{C}\left[\frac{(C-1)\,\omega\nu}{C_4^*KC}\right]^{1/\nu} (2 - C) = C_0 > 0\,.$$

As $|||u_0(t_0)||| = \eta$, which is a consequence of (7), the last relations prove the theorem.

3.1 THE LINEAR EQUATION WITH VARIABLE COEFFICIENTS

In this section, we shall deal with the equation

$$\mathscr{L}\, u(t) = F(t, u(t)), \quad F \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^{1/n})).$$

We shall suppose the function F to be linear with respect to the variable u. Further, we shall suppose that H is a real Hilbert space of real vector functions $h = h(x) = (h_1(x), \dots, h_k(x))$ $(k \geqq 1)$ that are defined on a subset Ω of a Euclidean space E_N.

Let $q_i \geqq 0$ $(i = 0, \dots, n-1)$ be natural numbers f_{ij} $(i = 0, \dots, n-1,\ j = 1, \dots, q_i)$ functions fulfilling the condition

(3.1.1) $f_{ij} : \sigma(A) \to R_1$ are continuous functions and there exists a constant F^* such that $|f_{ij}(s)| \leqq F^* s^{(n-i-1)/n}$ for $s \in \sigma(A)$, $i = 0, \dots, n-1$, $j = 1, \dots, q_i$.

Let a_{ij} be mapping defined on $R^+ \times \bar{\Omega}$ such that $a_{ij}(t, x) f_{ij}(A)\, u^{(i)}(t, x) \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^{1/n}))$ $(i = 0, \dots, n-1, j = 1, \dots, q_i)$.

Remark 3.1.1. In accordance with the notation from the previous sections we shall not always write $u(t, x)$ but sometimes only $u(t)$. Similarly in the case of a_{ij}. On the contrary, $g(x)$ will mean a function independent of t.

Theorem 3.1.1. *Let us suppose:*

(3.1.2) *There exist real constants $\bar{a}_{ij}$, K_{ij} $(i = 0, \dots, n-1,\ j = 1, \dots, q_i)$ such that $\|A^{1/n}[(a_{ij}(t) - \bar{a}_{ij}) f_{ij}(A)\, u^{(i)}(t)]\| \leqq K_{ij} |||u(t)|||$ for $u \in \mathscr{U}$, $t \in \mathscr{D}(u)$, $i = 0, \dots, n-1$, $j = 1, \dots, q_i$.*

Further, let the operator

$$\bar{\mathscr{L}}\, u(t) = \mathscr{L}\, u(t) - \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \bar{a}_{ij} f_{ij}(A)\, u^{(i)}(t)$$

be of the type ω and

$$\omega + nC(\bar{\mathscr{L}}) \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} K_{ij} < 0$$

(resp. $\leqq 0$). Then the zero solution $O_{/[0,+\infty)}$ of the equation

$$\mathscr{L}\, u(t) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} a_{ij}(t) f_{ij}(A)\, u^{(i)}(t) \tag{3.1.3}$$

is globally uniformly exponentially stable with respect to the norm $|||\cdot|||$ and uniformly stable at constantly acting disturbances with respect to the norm $|||\cdot|||$ (resp. globally uniformly stable with respect to the norm $|||\cdot|||$).

Proof. If we write the equation (3.1.3) in the form

$$\mathcal{L}\, u(t) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} [a_{ij}(t) - \bar{a}_{ij}]\, f_{ij}(A)\, u^{(i)}(t)\,,$$

the assertion of the theorem follows easily from Theorems 2.2.1, 2.3.1.

3.2 THE GENERAL NONLINEAR EQUATION WITH $A = (-1)^p \Delta^p$

We shall deal with the equation

$$\text{(3.2.1)} \qquad \mathcal{L}\, u(t) = F(t, u(t))\,, \quad F \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^{1/n}))$$

in this section. We shall use the notation and conventions from Section 3.1 here.

Let $\Omega = (0, \pi c_1) \times (0, \pi c_2) \times \ldots \times (0, \pi c_N)$, $c_i > 0$ $(i = 1, \ldots, N)$ be a subset of the Euclidean space E_N. We shall suppose that $H = L_2(\Omega)$ is a real Hilbert space.

Let $p \geqq 1$ be a natural number such that $2p/n$ is an integer.

Remark 3.2.1. By the symbol $\sum_k$ we shall denote $\sum_{\substack{1 \leq k_i < +\infty \\ i = 1,\ldots,N}}$.

The operator A will be defined as follows:

$$\text{(3.2.2)} \quad A\, v(x) = (-1)^p [\sum_{i=1}^{N} D_i^2]^p\, v(x) \text{ for } v \in \mathscr{D}(A) = \Big\{ u(x) \in L_2(\Omega \mid u(x) =$$

$$= \sum_k u_k \sin \frac{k_1 x_1}{c_1} \ldots \sin \frac{k_N x_N}{c_N}\,, \quad k = (k_1, \ldots, k_N),$$

$$\sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_N}{c_N}\right)^2 \right]^{2p} u_k^2 < +\infty \Big\}\,, \quad D_i = \frac{\partial}{\partial x_i}$$

(in the sense of distributions).

Now we shall introduce some properties of the right hand side F that will be used in the following. Suppose that

$$\text{(3.2.3)} \quad F(t, u(t)) = f(t, x, f_{01}(A)\, u(t), \ldots, f_{0q_0}(A)\, u(t), f_{11}(A)\, u'(t), \ldots$$
$$\ldots, f_{1q_1}(A)\, u'(t), \ldots, f_{n-1\,1}(A)\, u^{(n-1)}(t), \ldots, f_{n-1\,q_{n-1}}(A)\, u^{(n-1)}(t)),$$

(3.2.4) the functions f_{ij} fulfil the condition (3.1.1),

(3.2.5) $F(t + T, u(t)) = F(t, u(t))$ for $u \in \mathscr{U}$, $t \in \mathscr{D}(u)$, $T > 0$.

Suppose that there exists a T-periodic solution $v : R^+ \to H$ of the equation (3.2.1).

The following Lemmas 3.2.1–3.2.6 can be proved in a similar way as Lemmas 5.1–5.6 from J. Barták [1], pp. 428–431. That is why we omit their proofs here.

Lemma 3.2.1. *The operator A is selfadjoint.*

Lemma 3.2.2. *The spectrum of the operator A is a point spectrum*

$$\sigma(A) = \left\{\lambda_k = \left[\sum_{i=1}^{N}\left(\frac{k_i}{c_i}\right)^2\right]^p \middle|\ k = (k_1, \ldots, k_N),\quad 1 \leqq k_i < +\infty, \right.$$
$$\left. k_i \text{ integers}, (i = 1, \ldots, N)\right\}$$

and the eigenfunctions corresponding to an eigenvalue λ_k *are* $\sin(k_1 x_1/c_1) \ldots \ldots \sin(k_N x_N/c_N)$. *Further,* $\delta = [\sum_{i=1}^{N}(1/c_i)^2]^p$.

Lemma 3.2.3. *Let* $u \in \mathscr{D}(A)$ *have the form*

$$u(x) = \sum_k u_k \sin\frac{k_1 x_1}{c_1} \ldots \sin\frac{k_N x_N}{c_N}. \tag{3.2.6}$$

Then

$$A\,u(x) = \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_N}{c_N}\right)^2\right]^p u_k \sin\frac{k_1 x_1}{c_1} \ldots \sin\frac{k_N x_N}{c_N}.$$

If

$$v \in \mathscr{D}(A^{1/n}) = \left\{u(x) \in L_2(\Omega) \mid u(x) = \sum_k u_k \sin\frac{k_1 x_1}{c_1} \ldots \sin\frac{k_N x_N}{c_N}, \right.$$
$$\left. k = (k_1, \ldots, k_N),\quad \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_N}{c_N}\right)^2\right]^{2p/n} u_k^2 < +\infty\right\}$$

has the form (3.2.6), *where we write* v_k *instead of* u_k, *then*

$$A^{1/n} v(x) = \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_N}{c_N}\right)^2\right]^{p/n} v_k \sin\frac{k_1 x_1}{c_1} \ldots \sin\frac{k_N x_N}{c_N}.$$

Lemma 3.2.4. *There exists a constant* K_0^* *such that if* $u \in \mathscr{D}(A^{1/n})$ *then*

$$N^{-1/2}\left\|\sum_{i=1}^{N} D_i^{2p/n} u(x)\right\| \leqq \|A^{1/n} u(x)\| \leqq K_0^*\left\|\sum_{i=1}^{N} D_i^{2p/n} u(x)\right\|.$$

Lemma 3.2.5. *There exists a constant* K_1^* *so that if* $u \in \mathscr{D}(A^{1/n})$ *then*

$$\|u\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* \|A^{1/n} u\|.$$

Put $s^* = (N+1)/2$ if N is odd, $s^* = (N+2)/2$ if N is even.

Lemma 3.2.6. (The Sobolev Embedding Theorem.) *There exists a constant* K_2^* *such that to each* $u \in W_2^{s^*}(\Omega)$ *there exists a continuous representant of this element satisfying* $\|u\|_{C(\bar\Omega)} = \max_{x \in \bar\Omega} |u(x)| \leqq K_2^* \|u\|_{W_2^{s^*}(\Omega)}$.

In the rest of this section, we shall consider only such solutions that $f_{ij}(A)\, u^{(i)}(t)$ $(i = 0, \ldots, n-1,\ j = 1, \ldots, q_i)$ are continuous for $t \in \mathscr{D}(u)$, $x \in \bar{\Omega}$. We shall call them continuous representants (of solutions).

Suppose

(3.2.7) There exist continuous **G**-derivatives of the function f with respect to the variables $f_{ij}(A)\, u^{(i)}$ $(i = 0, \ldots, n-1, j = 1, \ldots, q_i)$ up to the second order.

Then

(3.2.8) $F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t))$, where

$$F_L(t, u(t)) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} a_{ij}(t)\, f_{ij}(A)\, u^{(i)}(t),$$

$$F_N(t, u(t)) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \sum_{k=0}^{n-1} \sum_{l=1}^{q_k} r_{ijkl}(t, u(t))\, f_{ij}(A)\, u^{(i)}(t)\, f_{kl}(A)\, u^{(k)}(t),$$

$$a_{ij}(t) = \frac{\partial f}{\partial f_{ij}}(t, x, f_{01}(A)\, v(t), \ldots, f_{n-1\, q_{n-1}}(A)\, v^{(n-1)}(t)),$$

$$r_{ijkl}(t, u(t)) = \int_0^1 \int_0^1 \tilde{r}_{ijkl}(t, v(t) + \vartheta\sigma\, u(t))\, \sigma\, \mathrm{d}\vartheta\, \mathrm{d}\sigma,$$

$$\tilde{r}_{ijkl}(t, u(t)) = \frac{\partial^2 f}{\partial f_{ij}\, \partial f_{kl}}(t, x, f_{01}(A)\, u(t), \ldots, f_{n-1\, q_{n-1}}(A)\, u^{(n-1)}(t))$$

$(i, k = 0, \ldots, n-1,\ j = 1, \ldots, q_i,\ l = 1, \ldots, q_k)$.

If $g = g(t, x, f_{01}, \ldots, f_{0q_0}, \ldots, f_{n-1\,1}, \ldots, f_{n-1\,q_{n-1}})$ we denote by $\mathfrak{M}_k(g)$ the system of all derivatives of the type

$$\frac{\partial^\Gamma g}{\partial x^{\gamma_{-1}}\, \partial^{\gamma_{01}} f_{01} \ldots \partial^{\gamma_{n-1 q_{n-1}}} f_{n-1\, q_{n-1}}}, \quad \Gamma = \gamma_{-1} + \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \gamma_{ij}, \quad 0 \leqq \Gamma \leqq k.$$

Further, let us denote

$$\varrho^* = \{\sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \sup_{t \in R^+, x \in \bar{\Omega}}{}^2 |f_{ij}(A)\, v^{(i)}(t, x)|\}^{1/2},$$

$$K_{\varrho^*} = \{y \in E_{\boldsymbol{q}} \mid \|y\|_{E_{\boldsymbol{q}}} \leqq \varrho^*\}, \quad \text{where} \quad \boldsymbol{q} = q_0 + q_1 + \ldots + q_{n-1}.$$

Theorem 3.2.1. *Let $v : R^+ \to H$ be a T-periodic solution of the equation* (3.2.1). *Let F satisfy the conditions* (3.2.3)–(3.2.5), (3.2.7) *and let the operator A be defined by the relation* (3.2.2). *Let there exist a closed sphere $K_\varrho \subseteq E_{\boldsymbol{q}}$, $\boldsymbol{q} = q_0 + \ldots + q_{n-1}$, $K_\varrho \neq \emptyset$, $\operatorname{Int} K_\varrho \supseteq K_{\varrho^*}$ so that all derivatives from the systems $\mathfrak{M}_{2p/n}(a_{ik})$, $\mathfrak{M}_{2p/n}(\tilde{r}_{ijkl})$ $(i, k = 0, \ldots, n-1,\ j = 1, \ldots, q_i,\ l = 1, \ldots, q_k)$ exist and are continuous on $[0, T] \times \bar{\Omega} \times K_\varrho$. Further, let $2s^* \leqq 2p/n + 1$ and let F_L, F_N be defined by the relation* (3.2.8).

Then $F_L(t, O_{/I}) = 0$ *for every* $I \subseteq \mathscr{D}(v)$, *the conditions* (2.2.1), (2.2.2), (2.2.3), (2.4.2) *and* (3.1.2) *being fulfilled with some constants* K, K_1, K_2, K_{ij}, $R > 0$, $R_1 > 0$, $v = 1$, $\bar{a}_{ij}$.

If, moreover, the operator $\mathscr{L}$ *is stable, the operator*

$$\bar{\mathscr{L}}\, u(t) = \mathscr{L}\, u(t) - \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \bar{a}_{ij} f_{ij}(A)\, u^{(i)}(t)$$

is of the type ω,

$$\omega + nC(\bar{\mathscr{L}}) \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} K_{ij} < 0$$

and (2.4.1) *holds then the solution* v *is uniformly exponentially stable with respect to the norm* $|||\cdot|||$ *and uniformly stable at constantly acting disturbances with respect to the norm* $|||\cdot|||$.

Proof. Let $u \in \mathscr{U}$. Then according to Lemmas 3.2.5, 3.2.6 and to the relations $2s^* \leqq 2p/n + 1$, (3.2.4), the following inequalities hold:

(1)
$$\|f_{ij}(A)\, u^{(i)}(t)\|_{C(\bar{\Omega})} \leqq K_2^* \|f_{ij}(A)\, u^{(i)}(t)\|_{W_2^{s^*}(\Omega)} \leqq$$
$$\leqq K_2^* \|f_{ij}(A)\, u^{(i)}(t)\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* K_2^* \|A^{1/n} f_{ij}(A)\, u^{(i)}(t)\| \leqq$$
$$\leqq K_1^* K_2^* F^* \|A^{1/n} A^{(n-i-1)/n}\, u^{(i)}(t)\| \leqq K_1^* K_2^* F^* |||u(t)|||$$

and so to every $u \in \mathscr{U}$ there exits its continuous representant. Moreover, this implies that ϱ^* is a finite number.

I) We shall prove (3.1.2).

Let $\bar{a}_{ij}$ $(i = 0, \ldots, n-1,\ j = 1, \ldots, q_i)$ be constants. Under the expression $(a(t) - \bar{a})\, V(t)$ we shall understand some of the terms $(a_{ij}(t) - \bar{a}_{ij}) f_{ij}(A)\, u^{(i)}(t)$ $(i = 0, \ldots, n-1, j = 1, \ldots, q_i)$. Using Lemma 3.2.4 we get

$$\|A^{1/n}[(a(t) - \bar{a})\, V(t)]\| \leqq K_0^* \| \sum_{j=1}^{N} D_j^{2p/n}[(a(t) - \bar{a})\, V(t)]\| \leqq$$
$$\leqq K_0^* \sum_{j=1}^{N} \sum_{i=0}^{2p/n} \binom{2p/n}{i} D_j^i(a(t) - \bar{a})\, D_j^{2p/n-i}\, V(t)\,.$$

Thus it suffices to prove

(2)
$$\|D^i(a(t) - \bar{a})\, D^{2p/n-i}\, V(t)\| \leqq C_1 |||u(t)|||$$

for $i = 0, \ldots, 2p/n$, where D means some of the derivatives D_i $(i = 1, \ldots, N)$; C_i are constants in this proof.

We can easily find that $D^i(a(t) - \bar{a})$ is a linear combination of members of the form

(3)
$$\mathfrak{m}_i(a - \bar{a}) \prod_{j=0}^{n-1} \prod_{k=1}^{q_j} \prod_{l=1}^{\delta_{jk}} (D^l f_{jk}(A)\, v^{(j)}(t))^{\beta_{jkl}}\,,$$

where $\mathfrak{m}_i(a - \bar{a})$ is a function from the system $\mathfrak{M}_i(a - \bar{a})$ and

(4) $$\beta_{jk\delta_{jk}} = 0 \Rightarrow \delta_{jk} = 0 \quad \text{for} \quad j = 0, \ldots, n-1\,, \quad k = 1, \ldots, q_j\,,$$
$$\sum_{j=0}^{n-1} \sum_{k=1}^{q_j} \sum_{l=1}^{\delta_{jk}} l\beta_{jkl} \leqq i\,.$$

Using the assumptions on functions from the system $\mathfrak{M}_j(a - \bar{a})$ we can obtain

(5) $$\sup_{t \in R^+} \|\mathfrak{m}_i(a(t) - \bar{a})\|_{C(\bar{\Omega})} \leqq C_2$$

for $\mathfrak{m}_i \in \mathfrak{M}_i$, $(i = 0, \ldots, 2p/n)$.

First we shall prove (2) for $0 \leqq i \leqq 2p/n - s^*$. The condition (4) implies that all derivatives D^l appearing in (3) are of the order $l \leqq 2p/n - s^*$ and so (see the proof of inequalities (1)) we get

(6) $$\|D^l f_{jk}(A)\, v^{(j)}(t)\|_{C(\bar{\Omega})} \leqq K_1^* K_2^* F^* |||v(t)|||\,.$$

Further,

(7) $$\|D^{2p/n-i}\, V(t)\| \leqq \|V(t)\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* \|A^{1/n}\, V(t)\| \leqq K_1^* F^* |||u(t)|||\,.$$

Now the relations (3), (5), (6), (7) prove (2) for $0 \leqq i \leqq 2p/n - s^*$. It remains to show that (2) holds for $2p/n - s^* + 1 \leqq i \leqq 2p/n$. In this case, we can obtain (similarly as (1))

(8)
$$\|D^{2p/n-i}\, V(t)\|_{C(\bar{\Omega})} \leqq K_2^* \|V(t)\|_{W_2^{2p/n-i+s^*}(\Omega)} \leqq K_2^* \|V(t)\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* K_2^* F^* |||u(t)|||\,.$$

Hence in order to prove (2), it suffices to show that

(9) $$\|D^i(a(t) - \bar{a})\| \leqq C_3\,.$$

Using (4) we can show that there is no more than one derivative D^ν of an order $\nu > 2p/n - s^*$ in (3) (see also J. Barták [1], pp. 433–434). Obviously $\nu \leqq 2p/n$ and thus

(10) $$\|D^\nu f_{jk}(A)\, v^{(j)}(t)\| \leqq \|f_{jk}(A)\, v^{(j)}(t)\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* F^* |||v(t)|||\,.$$

The other derivatives in (3) are of an order $l \leqq 2p/n - s^*$ and so

(11) $$\|D^l f_{jk}(A)\, v^{(j)}(t)\|_{C(\bar{\Omega})} \leqq K_2^* \|f_{jk}(A)\, v^{(j)}(t)\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* K_2^* F^* |||v(t)|||\,.$$

The relations (5), (10), (11) prove (9). So we have proved (3.1.2).

II) Now we shall prove (ii) of (2.4.2). Let us choose a number $R > 0$ so small that

$$(f_{01}(A)\,(v(t, x) + \vartheta\sigma\, u(t, x)), \ldots, f_{n-1\,q_{n-1}}(A)\,(v^{(n-1)}(t, x) + \vartheta\sigma\, u^{(n-1)}(t, x))) \in K_\varrho$$

for $\vartheta, \sigma \in [0, 1]$, $u \in \mathscr{U}$, $x \in \bar{\Omega}$ and such $t \in \mathscr{D}(u)$ that $|||u(t)||| \leqq R$. (This is possible according to (1).)

Further, let $r(t, u(t))$ be some of the functions $r_{ijkl}(t, u(t))$, $V(t)$, $W(t)$ some of the functions $f_{ij}(A)\, u^{(i)}(t)$ $(i, k = 0, \ldots, n-1,\ j = 1, \ldots, q_i,\ l = 1, \ldots, q_k)$.

To prove (ii) of (2.4.2) it suffices to show that

$$\|A^{1/n}[r(t, u(t))\, V(t)\, W(t)]\| \leqq C_4 |||u(t)|||^2 .$$

Using Lemma 3.2.4 we get

$$\|A^{1/n}[r(t, u(t))\, V(t)\, W(t)]\| \leqq K_0^* \| \sum_{j=1}^{N} D_j^{2p/n}[r(t, u(t))\, V(t)\, W(t)]\| \leqq$$

$$\leqq K_0^* \sum_{j=1}^{N} \sum_{i=0}^{2p/n} \sum_{k=0}^{2p/n-i} \binom{2p/n}{i} \binom{2p/n - i}{k} \|D_j^i r(t, u(t))\, D_j^{2p/n-i-k}\, V(t)\, D_j^k\, W(t)\| .$$

So we have to prove

$$\|D^i r(t, u(t))\, D^{2p/n-i-k}\, V(t)\, D^k\, W(t)\| \leqq C_5 |||u(t)|||^2 \tag{12}$$

for $i = 0, \ldots, 2p/n$, $k = 0, \ldots, 2p/n - i$; D means some of the D_j's, $j = 1, \ldots, N$.

Similarly as in J. Barták [1], p. 435, we find that $D^i r(t, u(t))$ is a linear combination of members of the form

$$\int_0^1 \int_0^1 \mathfrak{m}_i(\tilde{r}) \prod_{j=0}^{n-1} \prod_{k=1}^{q_j} \prod_{l=1}^{\delta_{jk}} (D^l f_{jk}(A)\, (v^{(j)}(t) + \vartheta\sigma\, u^{(j)}(t)))^{\beta_{jkl}} \sigma\, d\vartheta\, d\sigma , \tag{13}$$

where $\mathfrak{m}_i(\tilde{r})$ is a function from the system $\mathfrak{M}_i(\tilde{r})$ and

$$\beta_{jk\delta_{jk}} = 0 \Rightarrow \delta_{jk} = 0 \quad \text{for } j = 0, \ldots, n-1 , \quad k = 1, \ldots, q_j , \tag{14}$$

$$\sum_{j=0}^{n-1} \sum_{k=1}^{q_j} \sum_{l=1}^{\delta_{jk}} l \beta_{jkl} \leqq i .$$

The proof of (12) is quite analogous to the preceding one. That is why we shall sketch it only briefly. Firstly, let $0 \leqq i \leqq 2p/n - s^*$. Then (14) yields that all derivatives D^l in (13) are of an order $l \leqq 2p/n - s^*$ and thus

$$\|D^l f_{jk}(A)\, (v^{(j)}(t) + \vartheta\sigma\, u^{(j)}(t))\|_{C(\bar{\Omega})} \leqq K_1^* K_2^* F^* |||v(t) + \vartheta\sigma\, u(t)||| \leqq \tag{15}$$

$$\leqq K_1^* K_2^* F^* (|||v(t)||| + R) \leqq C_6 .$$

If $0 \leqq k \leqq 2p/n - s^*$ then $2p/n - i - k \leqq 2p/n$ and so

$$\|D^k\, W(t)\|_{C(\bar{\Omega})} \leqq K_1^* K_2^* F^* |||u(t)||| , \tag{16}$$

$$\|D^{2p/n-i-k}\, V(t)\| \leqq \|V(t)\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* F^* |||u(t)||| .$$

If $2p/n - s^* + 1 \leqq k \leqq 2p/n$ then $2p/n - i - k \leqq 2p/n - s^*$ and thus

$$(17) \qquad \|D^{2p/n-i-k} V(t)\|_{C(\bar{\Omega})} \leqq K_1^* K_2^* F^* |||u(t)||| ,$$
$$\|D^k W(t)\| \leqq \|W(t)\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* F^* |||u(t)||| .$$

Now (12) follows from (13), (15), (16), (17) and from the boundedness of the functions $\mathfrak{m}_i(\tilde{r})$.

Secondly, let $2p/n - s^* + 1 \leqq i \leqq 2p/n$. Then with help of (14) we obtain that there exists at most one derivative D^v in (13) of an order $v > 2p/n - s^*$. Surely $v \leqq 2p/n$ and so this derivative can be estimated as follows:

$$(18) \qquad \|D^v f_{jk}(A) (v^{(j)}(t) + \vartheta\sigma\, u^{(j)}(t))\| \leqq K_1^* F^* |||v(t) + \vartheta\sigma\, u(t)||| \leqq C_7 .$$

The other derivatives are of an order $l \leqq 2p/n - s^*$ and so

$$(19) \qquad \|D^l f_{jk}(A) (v^{(j)}(t) + \vartheta\sigma\, u^{(j)}(t))\|_{C(\bar{\Omega})} \leqq K_1^* K_2^* F^* (|||v(t)||| + R) \leqq C_6 .$$

Because $k \leqq 2p/n - i \leqq s^* - 1 \leqq 2p/n - s^*$, $2p/n - i - k \leqq s^* - 1 \leqq 2p/n - s^*$, the following relations hold:

$$(20) \qquad \|D^k W(t)\|_{C(\bar{\Omega})} \leqq K_1^* K_2^* F^* |||u(t)||| ,$$
$$\|D^{2p/n-i-k} V(t)\|_{C(\bar{\Omega})} \leqq K_1^* K_2^* F^* |||u(t)||| .$$

The relations (13), (18), (19), (20) imply (12). So we have proved (ii) of (2.4.2).

Now, the validity of (2.2.3) and (i) of (2.4.2) is an easy consequence of (3.1.2), (ii) of (2.4.2) and of the linearity of F_L with respect to the variable u. It is evident that (2.2.2) and $F_L(t, O_{/I}) = 0$ hold. The relation (2.2.1) follows from (2.4.2):

$$\|A^{1/n}[F(t, v(t) + u(t)) - F(t, v(t))]\| \leqq \|A^{1/n} F_L(t, u(t))\| + $$
$$+ \|A^{1/n} F_N(t, u(t))\| \leqq (K + KR^v) |||u(t)||| .$$

The last statement of the theorem is a consequence of Theorems 3.1.1, 2.4.1, 2.5.1.

3.3 THE TIMOSHENKO OPERATOR AND THE OPERATOR OF THE SECOND ORDER

In the first part of this section, we shall deal with the operator defined by the relation

$$(3.3.1) \qquad \mathscr{L} u(t) = u''''(t) + a\, u'''(t) + (b_1 A^{1/2} + b_2)\, u''(t) +$$
$$+ (c_1 A^{1/2} + c_2)\, u'(t) + (d_1 A + d_2 A^{1/2} + d_3)\, u(t) ,$$

where $a, b_1, b_2, c_1, c_2, d_1, d_2, d_3$ are real constants.

We introduce conditions ensuring the exponential stability or the stability of this operator. To this problem see also J. Barták [2].

With help of the Hurwitz theorem (see I. G. Malkin [15], p. 75) we can obtain the following lemma the proof of which is similar to the proof of Lemma 1 from J. Barták [2].

Lemma 3.3.1. *Let*

$$\begin{aligned} &a > 0\,, \\ &c_1 s^{1/2} + c_2 > 0\,, \\ &d_1 s + d_2 s^{1/2} + d_3 > 0\,, \\ &a(b_1 s^{1/2} + b_2)(c_1 s^{1/2} + c_2) - a^2(d_1 s + d_2 s^{1/2} + d_3) - \\ &\quad - (c_1 s^{1/2} + c_2)^2 > 0 \quad \text{for} \quad s \geqq \delta\,. \end{aligned} \tag{3.3.2}$$

Then to every $S_0 \geqq \delta$ *there exists a number* $\omega(S_0) < 0$ *such that if* $s \in [\delta, S_0]$, $\lambda_i(s)$ $(i = 1, 2, 3, 4)$ *are roots of the equation* (1.5.2), *then* $\operatorname{Re} \lambda_i(s) \leqq \omega(S_0)$, $(i = 1, 2, 3, 4)$. *If, moreover,*

$$ab_1c_1 - a^2 d_1 - c_1^2 > 0\,, \quad c_1 > 0\,, \quad d_1^2 + d_2^2 > 0\,, \tag{3.3.3}$$

then there exists a constant $\omega < 0$ *such that* $\operatorname{Re} \lambda_i(s) \leqq \omega$ *for* $s \geqq \delta$, $i = 1, 2, 3, 4$.

Theorem 3.3.1. *Let* (3.3.2) *be fulfilled and let*

$$d_1 \neq 0\,, \quad b_1^2 - 4d_1 \neq 0\,. \tag{3.3.4}$$

Then the operator $\mathscr{L}$ *defined by the relation* (3.3.1) *is stable. If, moreover,* (3.3.3) *is satisfied then the operator* $\mathscr{L}$ *is exponentially stable.*

Proof. The equation (1.5.3) has a form $\Lambda^4 + b_1\Lambda^2 + d_1 = 0$ in this case. Direct calculation shows that if the condition (3.3.4) is satisfied then all roots Λ_i are simple. This implies the existence of a number $S_0 \geqq \delta$ such that $\lambda_i(s) \neq \lambda_j(s)$ if $i \neq j$ $(i, j = 1, 2, 3, 4)$ for $s \geqq S_0$. Now, the statement of the theorem follows easily from Theorem 1.5.1 and Lemma 3.3.1.

Using Theorem 3.3.1 we can obtain criteria for stability of solution of the classical Timoshenko equation. This has been done in J. Barták [2], pp. 138–139.

In the second part of this section we shall investigate the operator defined by the relation

$$\mathscr{L}\, u(t) = u''(t) - 2\bar{b}\varepsilon\, u'(t) + [A - \varepsilon \sum_{i=1}^{q} \bar{a}_i f_i(A)]\, u(t)\,, \tag{3.3.5}$$

where $\bar{b}$, $\bar{a}_i$ $(i = 1, \ldots, q)$ are real constants and

(3.3.6) $f_i : \sigma(A) \to R_1$ are continuous functions such that $|f_i(s)| \leqq F_1^* s^{1/2}$ for $s \in \sigma(A)$, $i = 1, \ldots, q$ and for a suitable constant F_1^*.

Theorem 3.3.2. *Let the operator $\mathscr{L}$ be defined by the relation (3.3.5) and let (3.3.6) hold. Then to every $\eta > 0$ there exists a number $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0]$ then the operator $\mathscr{L}$ is of the type $\varepsilon\bar{b}$ and we can put $C(\mathscr{L}) = 1 + \eta$ in Definition 1.2.1. If $\bar{b} = 0$, $\bar{a}_i = 0$ $(i = 1, \ldots, q)$ then the operator $\mathscr{L}$ is of the type 0 and we can put $C(\mathscr{L}) = 1$.*

Proof. Denote $\Delta = \Delta(s) = s - \varepsilon \sum_{i=1}^{q} \bar{a}_i f_i(s) - \varepsilon^2 \bar{b}^2$. Suppose that the number $\eta > 0$ is given. Let us find a number $\varepsilon_0 > 0$ in such a way that $\Delta > 0$, $\sqrt{(s/\Delta)} \leqq 1 + \eta$, $|\varepsilon| \, |\bar{b}|/\sqrt{\Delta} \leqq \eta$ for every $\varepsilon \in (0, \varepsilon_0]$. The statement of the theorem follows now easily from the relation

$$m(t; s) = m_1(t; s) = e^{\varepsilon \bar{b} t} \sin (t \sqrt{\Delta})/\sqrt{\Delta} .$$

3.4 THE CASE OF THE BOUNDED OPERATOR

In this section, we shall suppose that A is a bounded operator.

Lemma 3.4.1. *If the operator A is bounded, then there exists a positive number Δ such that $\sigma(A) \subseteq [\delta, \Delta]$.*

Lemma 3.4.2. *Let the operator A be bounded and $\alpha \geqq 0$. Then $\delta^\alpha \|x\| \leqq \|A^\alpha x\| \leqq \Delta^\alpha \|x\|$ for $x \in H$ and $\delta^\alpha \leqq \|A^\alpha\| \leqq \Delta^\alpha$.*

Proof. By Lemma 3.4.1 we have $\delta \leqq s \leqq \Delta$ for $s \in \sigma(A)$ and thus

$$(\delta^\alpha \|x\|)^2 = \delta^{2\alpha} \int_{\sigma(A)} \mathrm{d}\|E(s)\, x\|^2 \leqq \int_{\sigma(A)} s^{2\alpha}\, \mathrm{d}\|E(s)\, x\|^2 = \|A^\alpha x\|^2 \leqq$$

$$\leqq \int_{\sigma(A)} \Delta^{2\alpha}\, \mathrm{d}\|E(s)\, x\|^2 = (\Delta^\alpha \|x\|)^2 .$$

This proves the lemma.

Theorem 3.4.1. *If the operator A is bounded and $\alpha \geqq 0$ then*

(i) $F \in \mathscr{C}([t_0, +\infty), \mathscr{D}(A^\alpha))$ *if and only if* $F \in \mathscr{C}([t_0, +\infty), \mathscr{D}(A^0))$,

(ii) $F \in \mathscr{C}(\mathscr{D}(u_{/I}), \mathscr{D}(A^\alpha))$ *if and only if* $F \in \mathscr{C}(\mathscr{D}(u_{/I}), \mathscr{D}(A^0))$,

(iii) $\mathscr{D}(\mathscr{L}) = \bigcup_{\substack{I=[a,b)\subseteq R^+ \\ I=[a,b]\subseteq R^+}} \bigcap_{i=0}^{n} \mathscr{C}^i(I, \mathscr{D}(A^0))$,

(iv) $\mathscr{U} = \bigcup_{\substack{I=[a,b)\subseteq R^+ \\ I=[a,b]\subseteq R^+}} \bigcap_{i=0}^{n-1} \mathscr{C}^i(I, \mathscr{D}(A^0))$, *where obviously* $\mathscr{D}(A^0) = H$.

Proof. This theorem is an easy consequence of Lemma 3.4.2.

Let us introduce the following notation:

$$|||u(t)|||_p = [\sum_{i=0}^{n-1} \|u^{(i)}(t)\|^p]^{1/p} \quad \text{for an integer} \quad p \geqq 1 \quad \text{and} \quad u \in \mathcal{U} .$$

Lemma 3.4.3. *If* $p \geqq 1$, $q \geqq 1$ *then* $|||u(t)|||_p \leqq n|||u(t)|||_q$.

Proof. Let us denote $u_i = \|u^{(i)}\|$. Then

$$|||u(t)|||_p^q = [\sum_{i=0}^{n-1} u_i^p]^{q/p} \leqq [\sum_{i=0}^{n-1} u_i]^q \leqq n^q \max{}^q (u_0, \ldots, u_{n-1}) \leqq$$

$$\leqq n^q \sum_{i=0}^{n-1} u_i^q = (n|||u(t)|||_q)^q .$$

This proves the lemma.

Lemma 3.4.4. *Let the operator A be bounded and* $p \geqq 1$. *Then there exist positive constants* d_1^*, d_2^* *such that* $d_1^*|||u(t)|||_p \leqq |||u(t)||| \leqq d_2^*|||u(t)|||_p$ *for* $u \in \mathcal{U}$, $t \in \mathcal{D}(u)$.

Proof. By Lemmas 3.4.2 and 3.4.3 it holds

$$(1) \qquad |||u(t)||| = [\sum_{i=0}^{n-1} \|A^{(n-i)/n} u^{(i)}(t)\|^2]^{1/2} \leqq [\sum_{i=0}^{n-1} \Delta^{2(n-i)/n} \|u^{(i)}(t)\|^2]^{1/2} \leqq$$

$$\leqq K_1|||u(t)|||_2 \leqq n\, K_1|||u(t)|||_p ,$$

$$(2) \qquad |||u(t)|||_p \leqq n[\sum_{i=0}^{n-1} \|u^{(i)}(t)\|^2]^{1/2} \leqq n[\sum_{i=0}^{n-1} \delta^{-2(n-i)/n} \|A^{(n-i)/n} u^{(i)}(t)\|^2]^{1/2} \leqq$$

$$\leqq K_2|||u(t)||| ,$$

for suitable positive constants K_1, K_2. The relations (1), (2) prove the lemma.

As an easy consequence of Lemmas 3.4.2, 3.4.4 we get

Theorem 3.4.2. *If the operator A is bounded and* $p \geqq 1$ *then*

(i) *All types of stability and instability introduced in Definitions* 1.1.1, 1.1.2, 1.1.4 *with respect to the norm* $|||\cdot|||$ *are equivalent to the corresponding types of stability and instability, respectively, with respect to the norm* $|||\cdot|||_p$.

(ii) *The stability at constantly acting disturbances with respect to the norms* $|||\cdot|||$, $\|\cdot\|_{\mathcal{D}(A^{1/n})}$ *is equivalent to the stability at constantly acting disturbances with respect to the norms* $|||\cdot|||$, $\|\cdot\|$, *or* $|||\cdot|||_p$, $\|\cdot\|_{\mathcal{D}(A^{1/n})}$ *or* $|||\cdot|||_p$, $\|\cdot\|$.

Theorem 3.4.3. *Let the operator A be bounded. Then*

(i) *Theorems* 2.2.1, 2.3.1 *remain valid if we replace the condition* (2.2.1) *by the following one*:

There exist numbers K, $R > 0$, $p \geqq 1$, $q \geqq 1$ such that $\|F(t, v(t) + u(t)) - F(t, v(t))\| \leqq \Delta^{-1/n} d_1^* K |||u(t)|||_p$ *for* $u \in \mathscr{U}$ *satisfying* $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, *and for* $t \in \mathscr{D}(u)$ *such that* $|||u(t)|||_q \leqq R$.

(ii) *Theorems* 2.2.2, 2.3.2 *remain valid if we replace the condition* (2.2.3) *by the following one*:

There exist numbers K_1, K_2, $R_1 > 0$, $\nu > 0$, $p_1 \geqq 1$, $p_2 \geqq 1$, $p_3 \geqq 1$ *such that if* $u \in \mathscr{U}$ *fulfils* $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ *and* $t \in \mathscr{D}(u)$ *is such that* $|||u(t)|||_{p_1} \leqq R_1$ *then* $\|F_L(t, u(t))\| \leqq \Delta^{-1/n} d_1^* K_1 |||u(t)|||_{p_2}$, $\|F_N(t, u(t))\| \leqq K_2 |||u(t)|||_{p_3}^{1+\nu}$.

(iii) *Theorems* 2.4.1, 2.5.1 *remain valid if we replace the condition* (2.4.2) *by the following one*:

There exist constants K, $R > 0$, $\nu > 0$, $p_i \geqq 1$ $(i = 1, 2, 3, 4)$ *such that*

(a) *if* $\mathscr{D}(u_i) \subseteq \mathscr{D}(v)$ *for* $u_i \in \mathscr{U}$ *and* $t \in \mathscr{D}(u_1) \cap \mathscr{D}(u_2)$ *is such that* $|||u_i(t)|||_{p_1} \leqq R$ $(i = 1, 2)$ *then* $\|F_L(t, u_1(t)) - F_L(t, u_2(t))\| \leqq K |||u_1(t) - u_2(t)|||_{p_2}$,

(b) *if* $u \in \mathscr{U}$ *fulfils* $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ *and* $t \in \mathscr{D}(u)$ *is such that* $|||u(t)|||_{p_3} \leqq R$ *then* $\|F_N(t, u(t))\| \leqq K |||u(t)|||_{p_4}^{1+\nu}$.

(iv) *Theorem* 2.6.1 *remains valid if we replace the condition* (2.6.1) *by the following one*:

There exist numbers $K > 0$, $R > 0$, $\nu > 0$, $p \geqq 1$, $q \geqq 1$ *so that* $\|F(t, v(t) + u(t)) - F(t, v(t))\| \leqq K |||u(t)|||_p^{1+\nu}$ *for* $u \in \mathscr{U}$ *such that* $\mathscr{D}(u) = [t_0, +\infty)$ *and* $t \in \mathscr{D}(u)$ *such that* $|||u(t)|||_q \leqq R$.

(v) *Theorem* 3.1.1 *remains valid if we replace the condition* (3.1.2) *by the following one*:

There exist real constants $\bar{a}_{ij}$, K_{ij}, $(i = 0, \ldots, n - 1, j = 1, \ldots, q_i)$ $p \geqq 1$ *such that* $\|(a_{ij}(t) - \bar{a}_{ij}) f_{ij}(A) u^{(i)}(t)\| \leqq \Delta^{-1/n} d_1^* K_{ij} |||u(t)|||_p$ *for* $u \in \mathscr{U}$, $t \in \mathscr{D}(u)$, $i = 0, \ldots, n - 1$, $j = 1, \ldots, q_i$.

Proof. This theorem follows immediately from Lemmas 3.4.2, 3.4.3, 3.4.4.

3.5 THE BOUNDED OPERATOR IN E_N

In this section we shall suppose that A is a bounded operator. Moreover, $H = E_N$ will be the N-dimensional Euclidean space. (So $\|\cdot\|$ will denote the norm in E_N.) Let $v = (v_1, \ldots, v_N) : \mathscr{D}(v) \to H$ be a maximal solution of the equation

$$\mathscr{L} u(t) = F(t, u(t)), \quad F \in \mathscr{C}(\mathscr{D}(u_{/\mathscr{D}(v)}), \mathscr{D}(A^0)). \tag{3.5.1}$$

We shall suppose

$$\begin{aligned} F(t, u(t)) &= f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) = (f_1(t, u(t), \ldots, u^{(n-1)}(t)), \\ &\quad f_2(t, u(t), \ldots, u^{(n-1)}(t)), \ldots, f_N(t, u(t), \ldots, u^{(n-1)}(t))), \quad \mathscr{D}(f) = \mathscr{D}(v) \times (E_N)^n. \end{aligned} \tag{3.5.2}$$

(3.5.3) There exist continuous **G**-derivatives of the function f with respect to the variables $u, u', \ldots, u^{(n-1)}$ up to the second order.

Then

(3.5.4) $F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t))$, where $F_L(t, u(t)) =$

$$= \sum_{i=0}^{n-1} \boldsymbol{a}_i(t)\, u^{(i)}(t),$$

$$F_N(t, u(t)) = \sum_{i,j=0}^{n-1} \boldsymbol{r}_{ij}(t, u(t))\, u^{(i)}(t)\, u^{(j)}(t),$$

$$\boldsymbol{a}_i(t) = \frac{\partial \boldsymbol{f}}{\partial u^{(i)}}(t, v(t), v'(t), \ldots, v^{(n-1)}(t)),$$

$$\boldsymbol{r}_{ij}(t, u(t)) = \int_0^1 \int_0^1 \frac{\partial^2 \boldsymbol{f}}{\partial u^{(i)}\, \partial u^{(j)}}(t, v(t) + \vartheta\sigma\, u(t), \ldots$$

$$\ldots, v^{(n-1)}(t) + \vartheta\sigma\, u^{(n-1)}(t))\, \sigma\, \mathrm{d}\vartheta\, \mathrm{d}\sigma, \quad (i, j = 0, \ldots, n-1).$$

Theorem 3.5.1. *Let A be a bounded operator in $H = E_N$, let $F \in \mathscr{C}(\mathscr{D}(u_{/\mathscr{D}(v)}), \mathscr{D}(A^0))$ fulfil the conditions* (3.5.2), (3.5.3). *Let $v = (v_1, \ldots, v_N) : \mathscr{D}(v) \to H$ be a maximal solution of the equation* (3.5.1) *such that $\varrho_1 = \sup_{t \in \mathscr{D}(v)} |||v(t)|||_p < +\infty$ for some $p \geqq 1$. Further, let there exist a positive number $\varrho > \varrho_1$ such that all derivatives $\partial f_m / \partial u_k^{(i)}$, $\partial^2 f_m / \partial u_k^{(i)}\, \partial u_l^{(j)}$ $(m, k, l = 1, \ldots, N,\ i, j = 0, \ldots, n-1)$ exist and are continuous and bounded on $\mathscr{D}(v) \times K_\varrho$, where $K_\varrho = \{(x_0, x_1, \ldots, x_{n-1}) \mid x_i \in E_N, [\sum_{i=0}^{n-1} \|x_i\|^p]^{1/p} \leqq \leqq \varrho\}$.*

Then the condition (2.2.2) *and the conditions* (a), (b) *from* (iii) *of Theorem* 3.4.3 *are satisfied.*

Proof. The condition (2.2.2) is obviously fulfilled according to (3.5.4) and Theorem 3.4.1.

If $u \in \mathscr{U}$ is such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and $t \in \mathscr{D}(u)$ then it holds

$$\|\boldsymbol{a}_i(t)\, u^{(i)}(t)\| = \left\|\left(\sum_{k=1}^{N} \frac{\partial f_1}{\partial u_k^{(i)}}(t, v(t), \ldots, v^{(n-1)}(t))\, u_k^{(i)}(t), \ldots\right.\right.$$

$$\left.\left. \ldots, \sum_{k=1}^{N} \frac{\partial f_N}{\partial u_k^{(i)}}(t, v(t), \ldots, v^{(n-1)}(t))\, u_k^{(i)}(t)\right)\right\|, \quad (i = 0, \ldots, n-1).$$

This yields

(1) $$\|\boldsymbol{a}_i(t)\, u^{(i)}(t)\| \leqq C_i \|(\sum_{k=1}^{N} |u_k^{(i)}(t)|, \ldots, \sum_{k=1}^{N} u|_k^{(i)}(t)|)\| =$$

$$= C_i N^{1/2} \sum_{k=1}^{N} |u_k^{(i)}(t)| \leqq C_i N [\sum_{k=1}^{N} (u_k^{(i)}(t))^2]^{1/2} = C_i N \|u^{(i)}(t)\|,$$

where

$$C_i = \max_{m,k=1,\dots,N} \sup_{t\in\mathscr{D}(v)} \left| \frac{\partial f_m}{\partial u_k^{(i)}} (t, v(t), v'(t), \dots, v^{(n-1)}(t)) \right|, \quad (i = 0, \dots, n-1).$$

(By the assumption C_i are finite numbers.)

Using (1) and the linearity of the function F_L with respect to the variables $u, u', \dots$ $\dots, u^{(n-1)}$ we obtain

$$\|F_L(t, u_1(t)) - F_L(t, u_2(t))\| \leqq \sum_{i=0}^{n-1} \|a_i(t)\,(u_1^{(i)}(t) - u_2^{(i)}(t))\| \leqq$$

$$\leqq \sum_{i=0}^{n-1} C_i N \|u_1^{(i)}(t) - u_2^{(i)}(t)\| \leqq \max_{i=0,\dots,n-1} C_i N |||u_1(t) - u_2(t)|||_1 .$$

So the condition (a) is proved. Now let us find a number $R > 0$ so small that

(2) $$(v(t) + \vartheta\sigma\, u(t), v'(t) + \vartheta\sigma\, u'(t), \dots, v^{(n-1)}(t) + \vartheta\sigma\, u^{(n-1)}(t)) \in K_\varrho$$

for all $\vartheta, \sigma \in [0, 1]$, $u \in \mathscr{U}$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and for $t \in \mathscr{D}(u)$ satisfying $|||u(t)|||_p \leqq R$.

From (2) in virtue of the boundedness of the derivatives $\partial^2 f_m / \partial u_k^{(i)} \partial u_l^{(j)}$ we obtain that

$$C_{ij} = \max_{m,k,l=1,\dots,N} \sup_{t\in\{t\in\mathscr{D}(v) \mid |||u(t)|||_p \leqq R\}} \left| \int_0^1 \int_0^1 \frac{\partial^2 f_m}{\partial u_k^{(i)} \partial u_l^{(j)}} (t, v(t) + \vartheta\sigma\, u(t), \right.$$

$$\left. v'(t) + \vartheta\sigma\, u'(t), \dots, v^{(n-1)}(t) + \vartheta\sigma\, u^{(n-1)}(t))\, \sigma \,\mathrm{d}\vartheta \,\mathrm{d}\sigma \right|$$

are finite numbers. Thus

$$\|r_{ij}(t, u(t))\, u^{(i)}(t)\, u^{(j)}(t)\| = \left\| \left(\sum_{k,l=1}^{N} \int_0^1 \int_0^1 \frac{\partial^2 f_1}{\partial u_k^{(i)} \partial u_l^{(j)}} (t, v(t) + \vartheta\sigma\, u(t), \dots \right. \right.$$

$$\dots v^{(n-1)}(t) + \vartheta\sigma\, u^{(n-1)}(t))\, \sigma \,\mathrm{d}\vartheta \,\mathrm{d}\sigma\, u_k^{(i)}(t)\, u_l^{(j)}(t), \dots$$

$$\dots, \sum_{k,l=1}^{N} \int_0^1 \int_0^1 \frac{\partial^2 f_N}{\partial u_k^{(i)} \partial u_l^{(j)}} (t, v(t) + \vartheta\sigma\, u(t), \dots, v^{(n-1)}(t) +$$

$$\left. \left. + \vartheta\sigma\, u^{(n-1)}(t))\, \sigma \,\mathrm{d}\vartheta \,\mathrm{d}\sigma\, u_k^{(i)}(t)\, u_l^{(j)}(t) \right) \right\| \leqq$$

$$\leqq C_{ij} \|(\sum_{k,l=1}^{N} |u_k^{(i)}(t)\, u_l^{(j)}(t)|, \dots, \sum_{k,l=1}^{N} |u_k^{(i)}(t)\, u_l^{(j)}(t)|)\| =$$

$$= C_{ij} N^{1/2} \sum_{k,l=1}^{N} |u_k^{(i)}(t)\, u_l^{(j)}(t)| \leqq K_{ij}(\|u^{(i)}(t)\|^2 + \|u^{(j)}(t)\|^2),$$

where K_{ij} are suitable constants. So

$$\|F_N(t, u(t))\| \leqq K \sum_{i=0}^{n-1} \|u^{(i)}(t)\|^2 = K \||u(t)\||_2^2$$

for $u \in \mathscr{U}$ for which $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and for $t \in \mathscr{D}(u)$ satisfying $\||u(t)\||_p \leqq R$. This proves the condition (b). The theorem is proved.

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