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ON TOLERANCES ON PERIODIC SEMIGROUPS

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A binary relation on a set S is said to be a *tolerance* on S if it is reflexive and symmetric. We say that a tolerance ϱ on a semigroup S is *compatible with* S if for any four elements x_1, x_2, y_1, y_2 of S for which $x_1 \varrho y_1$, $x_2 \varrho y_2$ we have $x_1x_2 \varrho y_1y_2$. Let $\mathcal{T}$ denote the class of all semigroups such that every tolerance compatible with S is a congruence on S (i.e. a transitive relation on S). It is known that every group belongs to $\mathcal{T}$. Any semigroup with at least three elements belonging to $\mathcal{T}$ is simple (see [1]). Hence it follows that every commutative semigroup with at least three elements belongs to $\mathcal{T}$ if and only if it is a group (see [2]). In this note we shall give a necessary and sufficient condition for a periodic semigroup to belong to $\mathcal{T}$.

Let I and J be non-empty sets and let G be a group. Let $P : I \times J \to G$. Put $p_{ij} = P(i, j)$ for $i \in I$ and $j \in J$. Denote by $M(G, I, J, P)$ the Rees matrix semigroup with the following multiplication: $(g, i, j)(h, r, s) = (gp_{jr}h, i, s)$, where $g, h \in G$, $i, r \in I$ and $j, s \in J$.

Lemma. *A semigroup* $M(G, I, J, P)$ *belongs to* $\mathcal{T}$ *if and only if* card $I \leqq 2$ *and* card $J \leqq 2$.

Proof. Let $S = M(G, I, J, P)$ belong to $\mathcal{T}$. By contradiction, we assume that card $I \geqq 3$. Then we can suppose that $I = I_1 \cup I_2$, where card $I_1 \geqq 2$, card $I_2 \geqq 2$ and card $I_1 \cap I_2 = 1$. Put $(g, i, j) \varrho (h, r, s)$ if and only if $g, h \in G$, $j, s \in J$ and either $i, r \in I_1$ or $i, r \in I_2$. It is easy to show that ϱ is a tolerance compatible with S. Now we have $(g, i, j) \varrho (g, k, j)$ and $(g, k, j) \varrho (g, r, j)$, where $g \in G$, $i \in I_1 \setminus I_2$, $k \in I_1 \cap I_2$, $r \in I_2 \setminus I_1$ and $j \in J$, but (g, i, j) non $\varrho(g, r, j)$. The tolerance ϱ is not a congruence. Thus we obtain that card $I \leqq 2$. Similarly we can prove that card $J \leqq 2$.

Let $S = M(G, I, J, P)$, where card $I \leqq 2$ and card $J \leqq 2$. We shall prove that S belongs to $\mathcal{T}$. Let $i \in I$, $j \in J$. Put $G_{ij} = \{(g, i, j);\ g \in G\}$. It is known that G_{ij} is a subgroup of S and $e_{ij} = (p_{ji}^{-1}, i, j)$ is the unit of G_{ij}. Let $x \in G_{is}$. Then $x = (g, i, s)$, where $g \in G$ and $s \in J$. We have $e_{ij}x = (p_{ji}^{-1}, i, j)(g, i, s) = (g, i, s) = x$ and so

$$(1) \qquad e_{ij}x = x \quad \text{for all} \quad x \in G_{is}\,.$$

Dually we obtain that

(2) $$xe_{ij} = x \quad \text{for all} \quad x \in G_{rj}\,.$$

Let ϱ be a tolerance compatible with S. We shall show that ϱ is a transitive relation on S. Suppose that $x \varrho y$, $y \varrho z$ and $y \in G_{ab}$. Since $y^{-1} \varrho y^{-1}$, we have $xe_{ab} = xy^{-1}y \varrho \varrho yy^{-1}z = e_{ab}z$ and so

(3) $$xe_{ab} \varrho e_{ab}z$$

and dually

(4) $$e_{ab}x \varrho ze_{ab}\,.$$

Let $x \in G_{tu}$ and $z \in G_{vw}$. Then we have the following possibilities:

Case 1. $t = a$ and $w = b$. Then according to (1), (2) and (4) we have $x \varrho z$.

Case 2. $v = a$ and $u = b$. It follows from (1), (2) and (3) that $x \varrho z$.

Case 3a. $t \neq a$ and $v \neq a$. Since card $I \leqq 2$, we have $t = v$. By (1) we obtain that

(5) $$x = e_{tb}x \varrho e_{tb}y \quad \text{and} \quad e_{tb}y \varrho e_{tb}z = z\,.$$

It is clear that $e_{tb}y \in G_{tb}$. If $w = b$, then by (5) and Case 1 we have $x \varrho z$. If $u = b$, then it follows from (5) and Case 2 that $x \varrho z$. If $w \neq b \neq u$, then in virtue of card $J \leqq 2$ we have $w = u$ and so by (2) and (5) we obtain that $x = xe_{au} \varrho e_{tb}ye_{au}$ and $e_{tb}ye_{au} \varrho ze_{au} = z$. It is clear that $e_{tb}ye_{au} \in G_{tu}$ and so it follows from Case 1 that $x \varrho z$.

Case 3b. $w \neq b$ and $u \neq b$. This is dual to Case 3a.

Case 3c. $t \neq a$ and $u \neq b$. According to Cases 3a and 3b, we can suppose that $v = a$ and $w = b$. It follows from (2) that $x = xe_{au} \varrho ye_{au}$. It is clear that $ye_{au} \in G_{au}$. Since $y \varrho x$ and $x \varrho ye_{au}$, it follows from Case 3a that $y \varrho ye_{au}$. Now, since $ye_{au} \varrho y$ and $y \varrho z$, it follows from Case 1 that $ye_{au} \varrho z$. Finally, since $x \varrho ye_{au}$ and $ye_{au} \varrho z$, it follows from Case 2 that $x \varrho z$.

Case 3d. $w \neq b$ and $v \neq a$. This is analogous to Case 3c.

Consequently, ϱ is a congruence on S and so $S \in \mathscr{T}$.

Theorem. *Let S be a semigroup with at least three elements. Then the following conditions on S are equivalent*:

1. *S belons to $\mathscr{T}$ and some power of each element of S lies in a subgroup of S.*

2. *S is isomorphic to the Rees matrix semigroup $M(G, I, J, P)$, where* card $I \leqq 2$ *and* card $J \leqq 2$.

Proof. $1 \Rightarrow 2$. If a semigroup S with at least three elements belongs to $\mathcal{T}$, then according to Theorem 4 of [1], S is simple. If some power of each element of a simple semigroup S lies in a subgroup of S, then it follows from Theorem 2.55 (MUNN W. D.) of [3] that S is completely simple. Then, by Theorem 3.5 (Rees D.) of [3], S is isomorphic to $M(G, I, J, P)$ and so according to Lemma card $I \leqq 2$ and card $J \leqq 2$.

$2 \Rightarrow 1$. This follows from Lemma.

Corollary 1. *Let S be a semigroup with at least three elements. Then the following conditions on S are equivalent:*

1. *S is a periodic semigroup belonging to $\mathcal{T}$.*

2. *S is isomorphic to the Rees matrix semigroup $M(G, I, J, P)$, where G is a periodic group,* card $I \leqq 2$ *and* card $J \leqq 2$.

Corollary 2. *Let S be a semigroup with at least three elements. Then the following conditions on S are equivalent:*

1. *S is a finite semigroup belonging to $\mathcal{T}$.*

2. *S is isomorphic to the Rees matrix semigroup $M(G, I, J, P)$, where G is a finite group,* card $I \leqq 2$ *and* card $J \leqq 2$.

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