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ON TOLERANCES ON PERIODIC SEMIGROUPS

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A binary relation on a set $S$ is said to be a tolerance on $S$ if it is reflexive and symmetric. We say that a tolerance $\mathfrak{g}$ on a semigroup $S$ is compatible with $S$ if for any four elements $x_1, x_2, y_1, y_2$ of $S$ for which $x_1 \mathfrak{g} y_1, x_2 \mathfrak{g} y_2$ we have $x_1 x_2 \mathfrak{g} y_1 y_2$. Let $\mathcal{T}$ denote the class of all semigroups such that every tolerance compatible with $S$ is a congruence on $S$ (i.e. a transitive relation on $S$). It is known that every group belongs to $\mathcal{T}$. Any semigroup with at least three elements belonging to $\mathcal{T}$ is simple (see [1]). Hence it follows that every commutative semigroup with at least three elements belongs to $\mathcal{T}$ if and only if it is a group (see [2]). In this note we shall give a necessary and sufficient condition for a periodic semigroup to belong to $\mathcal{T}$.

Let $I$ and $J$ be non-empty sets and let $G$ be a group. Let $P : I \times J \to G$. Put $p_{ij} = P(i, j)$ for $i \in I$ and $j \in J$. Denote by $M(G, I, J, P)$ the Rees matrix semigroup with the following multiplication: $(g, i, j) (h, r, s) = (g p_{jr} h, i, s)$, where $g, h \in G$, $i, r \in I$ and $j, s \in J$.

**Lemma.** A semigroup $M(G, I, J, P)$ belongs to $\mathcal{T}$ if and only if $\mathrm{card} \ I \leq 2$ and $\mathrm{card} \ J \leq 2$.

**Proof.** Let $S = M(G, I, J, P)$ belong to $\mathcal{T}$. By contradiction, we assume that $\mathrm{card} \ I \geq 3$. Then we can suppose that $I = I_1 \cup I_2$, where $\mathrm{card} \ I_1 \geq 2$, $\mathrm{card} \ I_2 \geq 2$ and $I_1 \cap I_2 = \emptyset$. Put $(g, i, j) \mathfrak{g} (h, r, s)$ if and only if $g, h \in G$, $i, j \in J$ and either $i \in I_1$ or $i \in I_2$. It is easy to show that $\mathfrak{g}$ is a tolerance compatible with $S$. Now we have $(g, i, j) \mathfrak{g} (g, k, j)$ and $(g, k, j) \mathfrak{g} (g, r, j)$, where $g \in G$, $i \in I_1 \setminus I_2$, $k \in I_2 \setminus I_1$, $r \in I_2 \setminus I_1$ and $j \in J$, but $(g, i, j) \mathfrak{g} (g, r, j)$. The tolerance $\mathfrak{g}$ is not a congruence. Thus we obtain that $I \leq 2$. Similarly we can prove that $J \leq 2$.

Let $S = M(G, I, J, P)$, where $\mathrm{card} \ I \leq 2$ and $\mathrm{card} \ J \leq 2$. We shall prove that $S$ belongs to $\mathcal{T}$. Let $i \in I$, $j \in J$. Put $G_{ij} = \{(g, i, j) ; g \in G\}$. It is known that $G_{ij}$ is a subgroup of $S$ and $e_{ij} = (p_{ji}^{-1}, i, j)$ is the unit of $G_{ij}$. Let $x \in G_{is}$. Then $x = (g, i, s)$, where $g \in G$ and $s \in J$. We have $e_{ij} x = (p_{ji}^{-1}, i, j) (g, i, s) = (g, i, s) = x$ and so

$$e_{ij} x = x \quad \text{for all} \quad x \in G_{is}.$$
Dually we obtain that

\[ xe_{ij} = x \quad \text{for all} \quad x \in G_{ef}. \]

Let \( \varrho \) be a tolerance compatible with \( S \). We shall show that \( \varrho \) is a transitive relation on \( S \). Suppose that \( x \varrho y \), \( y \varrho z \) and \( y \in G_{ab} \). Since \( y^{-1} \varrho y^{-1} \), we have \( xe_{ab} = xy^{-1}y \varrho \varrho y y^{-1}z = e_{ab}z \) and so

\[ xe_{ab} \varrho e_{ab}z \]

and dually

\[ e_{ab}x \varrho ze_{ab}. \]

Let \( x \in G_{tu} \) and \( z \in G_{eu} \). Then we have the following possibilities:

Case 1. \( t = a \) and \( w = b \). Then according to (1), (2) and (4) we have \( x \varrho z \).

Case 2. \( v = a \) and \( u = b \). It follows from (1), (2) and (3) that \( x \varrho z \).

Case 3a. \( t \neq a \) and \( v \neq a \). Since card \( I \leq 2 \), we have \( t = v \). By (1) we obtain that

\[ x = e_{tb}x \varrho e_{tb}y \quad \text{and} \quad e_{tb}y \varrho e_{tb}z = z. \]

It is clear that \( e_{tb}y \in G_{tu} \). If \( w = b \), then by (5) and Case 1 we have \( x \varrho z \). If \( u = b \), then it follows from (5) and Case 2 that \( x \varrho z \). If \( w \neq b \neq u \), then in virtue of card \( J \leq 2 \) we have \( w = u \) and so by (2) and (5) we obtain that \( x = xe_{au} \varrho e_{tb}ye_{au} \) and \( e_{tb}ye_{au} \varrho ze_{au} = z \). It is clear that \( e_{tb}ye_{au} \in G_{tu} \) and so it follows from Case 1 that \( x \varrho z \).

Case 3b. \( w \neq b \) and \( u \neq b \). This is dual to Case 3a.

Case 3c. \( t \neq a \) and \( u \neq b \). According to Cases 3a and 3b, we can suppose that \( v = a \) and \( w = b \). It follows from (2) that \( x = xe_{au} \varrho ye_{au} \). It is clear that \( ye_{au} \in G_{au} \).

Since \( y \varrho x \) and \( x \varrho ye_{au} \), it follows from Case 3a that \( y \varrho ye_{au} \). Now, since \( ye_{au} \varrho y \) and \( y \varrho z \), it follows from Case 1 that \( ye_{au} \varrho z \). Finally, since \( x \varrho ye_{au} \) and \( ye_{au} \varrho z \), it follows from Case 2 that \( x \varrho z \).

Case 3d. \( w \neq b \) and \( v \neq a \). This is analogous to Case 3c.

Consequently, \( \varrho \) is a congruence on \( S \) and so \( S \in \mathcal{T} \).

**Theorem.** Let \( S \) be a semigroup with at least three elements. Then the following conditions on \( S \) are equivalent:

1. \( S \) belongs to \( \mathcal{T} \) and some power of each element of \( S \) lies in a subgroup of \( S \).
2. \( S \) is isomorphic to the Rees matrix semigroup \( M(G, I, J, P) \), where card \( I \leq 2 \) and card \( J \leq 2 \).

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Proof. 1 ⇒ 2. If a semigroup S with at least three elements belongs to \( \mathcal{F} \), then according to Theorem 4 of [1], S is simple. If some power of each element of a simple semigroup S lies in a subgroup of S, then it follows from Theorem 2.55 (Mun W. D.) of [3] that S is completely simple. Then, by Theorem 3.5 (Rees D.) of [3], S is isomorphic to \( M(G, I, J, P) \) and so according to Lemma \( \text{card } I \leq 2 \) and \( \text{card } J \leq 2 \).

2 ⇒ 1. This follows from Lemma.

**Corollary 1.** Let S be a semigroup with at least three elements. Then the following conditions on S are equivalent:

1. S is a periodic semigroup belonging to \( \mathcal{F} \).
2. S is isomorphic to the Rees matrix semigroup \( M(G, I, J, P) \), where G is a periodic group, \( \text{card } I \leq 2 \) and \( \text{card } J \leq 2 \).

**Corollary 2.** Let S be a semigroup with at least three elements. Then the following conditions on S are equivalent:

1. S is a finite semigroup belonging to \( \mathcal{F} \).
2. S is isomorphic to the Rees matrix semigroup \( M(G, I, J, P) \), where G is a finite group, \( \text{card } I \leq 2 \) and \( \text{card } J \leq 2 \).

**References**


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