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VARIETIES OF l-GROUPS ARE TORSION CLASSES

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In [3], MARTINEZ introduced the notion of a torsion class of lattice ordered groups. A class $\mathscr{I}$ is a torsion class provided

1) $G \in \mathscr{I}$ and N an l-ideal of G imply $G/N \in \mathscr{I}$,

2) $G \in \mathscr{I}$ and H a convex l-subgroup of G imply $H \in \mathscr{I}$, and

3) if $\mathscr{A}$ is a collection of convex l-subgroups of G and for each $A \in \mathscr{A}$, $A \in \mathscr{I}$, then $\bigvee \mathscr{A} \in \mathscr{I}$, where $\bigvee \mathscr{A}$ denotes the convex l-subgroup of G generated by $\mathscr{A}$.

The idea of torsion class was intended to generalize, among other things, varieties (equationally defined classes). Indeed, in [3], Martinez notes that every representable variety is a torsion class, and also the variety of normal valued l-groups is a torsion class. The main (and only) result of the present paper is to close the gap by showing that *every* variety of l-groups is a torsion class.

The proof depends on two important properties of normal valued l-groups (Theorems 1 and 2 below). If G is an l-group and $g \in G$, a *value* of g is any convex l-subgroup of G maximal with respect to missing g. Every value K has a unique cover K^* which is the intersection of all convex l-subgroups of G properly containing K. If each value K is a normal subgroup of its cover K^*, then G is said to be *normal valued*. The normal valued l-groups form a variety; in fact, it is the largest proper variety of l-groups:

Theorem 1 [2]. *If an l-group N satisfies an equation which is not satisfied by every l-group, then N is normal valued.*

Theorem 2 [3]. *The normal valued l-groups form a torsion class.*

If g is an element of the l-group G, $G(g)$ denotes the convex l-subgroup of G generated by g. As a final bit of terminology, G is a *lex extension* of a prime convex l-subgroup K if $b \neq e$ and $a \wedge b = e$ imply $a \in K$. In this case, if $e < g \notin K$ then $K < g$ [1, pp. 2.23, 2.24].

Lemma. *Let G be a subdirectly irreducible normal valued l-group generated by $g_1, \ldots, g_n$. Then $G = G(g_k)$ for some $1 \leqq k \leqq n$.*

Proof. Let C be a value of some element of the minimal l-ideal of G. Then $\{g_1, \ldots, g_n\} \nsubseteq C$. Let K be the largest member of the non-empty finite chain $\{M \mid C \subseteq \subseteq M, M$ a value of some $g_i\}$. Then K is a value of g_k for some $1 \leqq k \leqq n$. Also, K is normal in its cover K^*; in fact, $K^* = G$ and G/K is l-isomorphic to a subgroup of the archimedean ordered group of real numbers. Moreover, G is a lex extension of K. For suppose that $b \neq e$ and $a \wedge b = e$. Since $\bigcap_{g \in G} g^{-1}Cg$ is an l-ideal of G which clearly does not contain the minimal l-ideal, it must be that $\bigcap_{g \in G} g^{-1}Cg = \{e\}$. Hence, there exists $g \in G$ such that $b \notin g^{-1}Cg$. But any (conjugate of a) value must be prime, and so $a \in g^{-1}Cg \subseteq g^{-1}Kg = K$. Therefore, G is a lex extension of K. Since $g_k \notin K$ and G/K is an archimedean o-group, it follows that $G = G(g_k)$.

Theorem 3. *Every variety of l-groups is a torsion class.*

Proof. The first two properties in the definition of torsion class obviously hold for any variety. To verify the third property, we assume that H is an l-group, $\mathscr{A}$ is a collection of convex l-subgroups of H, and each member of $\mathscr{A}$ satisfies the equation $p(x_1, \ldots, x_m) = e$. If every l-group satisfies the equation $p(x_1, \ldots, x_m) = e$, then certainly so does $\bigvee\mathscr{A}$, the convex l-subgroup of H generated by $\mathscr{A}$. If not every l-group satisfies $p(x_1, \ldots, x_m) = e$, then by Theorem 1, every member of $\mathscr{A}$ is normal valued. By Theorem 2, $\bigvee\mathscr{A}$ is also normal valued. Let $h_1, h_2, \ldots, h_m \in \bigvee\mathscr{A}$. We wish to show that $p(h_1, \ldots, h_m) = e$. Since $\bigvee\mathscr{A}$ is just the subgroup of H generated by $\mathscr{A}$ [1, Theorem 1.4], $h_i = \prod_j g_{ij}$ for some $g_{ij} \in \bigcup\mathscr{A}$. Let G be the l-subgroup of $\bigvee\mathscr{A}$ generated by $\{g_{ij}\}$. As an l-subgroup of a normal valued l-group, G is also normal valued. Let $\bar{G}$ be any subdirectly irreducible factor of G and denote the natural map $g \mapsto \bar{g}$. Then $\bar{G}$ is normal valued and generated by $\{\bar{g}_{ij}\}$. Therefore, by the lemma, $\bar{G} = \bar{G}(\bar{g}_{kl})$ for some k, l. Since $g_{kl} \in A$ for some $A \in \mathscr{A}$, and since the image $\overline{A \cap G}$ is a convex l-subgroup of $\bar{G}$, $\bar{G} = \overline{A \cap G}$. Because A satisfies $p(x_1, \ldots, x_m) = e$, so does $\overline{A \cap G} = \bar{G}$. Finally, G is a subdirect product of subdirectly irreducible factors, each of which satisfies $p(x_1, \ldots, x_m) = e$, and therefore, so does G. In particular, $p(h_1, \ldots, h_m) = e$.

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