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Varieties of $l$-groups are torsion classes

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In [3], Martinez introduced the notion of a torsion class of lattice ordered groups. A class \( \mathcal{F} \) is a torsion class provided

1) \( G \in \mathcal{F} \) and \( N \) an \( l \)-ideal of \( G \) imply \( G/N \in \mathcal{F} \),

2) \( G \in \mathcal{F} \) and \( H \) a convex \( l \)-subgroup of \( G \) imply \( H \in \mathcal{F} \), and

3) if \( \mathcal{A} \) is a collection of convex \( l \)-subgroups of \( G \) and for each \( A \in \mathcal{A} \), \( A \in \mathcal{F} \), then \( \bigvee \mathcal{A} \in \mathcal{F} \), where \( \bigvee \mathcal{A} \) denotes the convex \( l \)-subgroup of \( G \) generated by \( \mathcal{A} \).

The idea of torsion class was intended to generalize, among other things, varieties (equationally defined classes). Indeed, in [3], Martinez notes that every representable variety is a torsion class, and also the variety of normal valued \( l \)-groups is a torsion class. The main (and only) result of the present paper is to close the gap by showing that every variety of \( l \)-groups is a torsion class.

The proof depends on two important properties of normal valued \( l \)-groups (Theorems 1 and 2 below). If \( G \) is an \( l \)-group and \( g \in G \), a value of \( g \) is any convex \( l \)-subgroup of \( G \) maximal with respect to missing \( g \). Every value \( K \) has a unique cover \( K^* \) which is the intersection of all convex \( l \)-subgroups of \( G \) properly containing \( K \). If each value \( K \) is a normal subgroup of its cover \( K^* \), then \( G \) is said to be normal valued. The normal valued \( l \)-groups form a variety; in fact, it is the largest proper variety of \( l \)-groups:

**Theorem 1** [2]. If an \( l \)-group \( N \) satisfies an equation which is not satisfied by every \( l \)-group, then \( N \) is normal valued.

**Theorem 2** [3]. The normal valued \( l \)-groups form a torsion class.

If \( g \) is an element of the \( l \)-group \( G \), \( G(g) \) denotes the convex \( l \)-subgroup of \( G \) generated by \( g \). As a final bit of terminology, \( G \) is a **lex extension** of a prime convex \( l \)-subgroup \( K \) if \( b \neq e \) and \( a \land b = e \) imply \( a \in K \). In this case, if \( e < g \notin K \) then \( K < g \) [1, pp. 2.23, 2.24].

**Lemma.** Let \( G \) be a subdirectly irreducible normal valued \( l \)-group generated by \( g_1, \ldots, g_n \). Then \( G = G(g_k) \) for some \( 1 \leq k \leq n \).
Proof. Let \( C \) be a value of some element of the minimal \( l \)-ideal of \( G \). Then 
\[ \{g_1, \ldots, g_n\} \not\subseteq C. \]
Let \( K \) be the largest member of the non-empty finite chain \( \{M \mid C \subseteq M, M \text{ a value of some } g_i\} \). Then \( K \) is a value of \( g_k \) for some \( 1 \leq k \leq n \). Also, \( K \) is normal in its cover \( K^* \); in fact, \( K^* = G \) and \( G/K \) is \( l \)-isomorphic to a subgroup of the archimedean ordered group of real numbers. Moreover, \( G \) is a lex extension of \( K \).

For suppose that \( b \neq e \) and \( a \wedge b = e \). Since \( \bigcap_{g \in G} g^{-1}Cg \) is an \( l \)-ideal of \( G \) which clearly does not contain the minimal \( l \)-ideal, it must be that \( \bigcap_{g \in G} g^{-1}Cg = \{e\} \). Hence, there exists \( g \in G \) such that \( b \notin g^{-1}Cg \). But any (conjugate of a) value must be prime, and so \( a \in g^{-1}Cg \subseteq g^{-1}Kg = K \). Therefore, \( G \) is a lex extension of \( K \). Since \( g_k \notin K \) and \( G/K \) is an archimedean \( o \)-group, it follows that \( G = G(g_k) \).

**Theorem 3.** Every variety of \( l \)-groups is a torsion class.

Proof. The first two properties in the definition of torsion class obviously hold for any variety. To verify the third property, we assume that \( H \) is an \( l \)-group, \( \mathcal{A} \) is a collection of convex \( l \)-subgroups of \( H \), and each member of \( \mathcal{A} \) satisfies the equation 
\[ p(x_1, \ldots, x_m) = e. \]
If every \( l \)-group satisfies the equation \( p(x_1, \ldots, x_m) = e \), then certainly so does \( \bigvee \mathcal{A} \), the convex \( l \)-subgroup of \( H \) generated by \( \mathcal{A} \). If not every \( l \)-group satisfies \( p(x_1, \ldots, x_m) = e \), then by Theorem 1, every member of \( \mathcal{A} \) is normal valued. By Theorem 2, \( \bigvee \mathcal{A} \) is also normal valued. Let \( h_1, h_2, \ldots, h_m \in \bigvee \mathcal{A} \). We wish to show that \( p(h_1, \ldots, h_m) = e \). Since \( \bigvee \mathcal{A} \) is just the subgroup of \( H \) generated by \( \mathcal{A} \) [1, Theorem 1.4], \( h_i = \prod_{j} g_{ij} \) for some \( g_{ij} \in \bigcup \mathcal{A} \). Let \( G \) be the \( l \)-subgroup of \( \bigvee \mathcal{A} \) generated by \( \{g_{ij}\} \). As an \( l \)-subgroup of a normal valued \( l \)-group, \( G \) is also normal valued. Let \( \tilde{G} \) be any subdirectly irreducible factor of \( G \) and denote the natural map \( g \mapsto \tilde{g} \). Then \( \tilde{G} \) is normal valued and generated by \( \{\tilde{g}_{ij}\} \). Therefore, by the lemma, \( \tilde{G} = G(\tilde{g}_{ki}) \) for some \( k, l \). Since \( g_{et} \in A \) for some \( A \in \mathcal{A} \), and since the image \( A \cap \tilde{G} \) is a convex \( l \)-subgroup of \( \tilde{G} \), \( \tilde{G} = A \cap \tilde{G} \). Because \( A \) satisfies \( p(x_1, \ldots, x_m) = e \), so does \( A \cap \tilde{G} = \tilde{G} \). Finally, \( G \) is a subdirect product of subdirectly irreducible factors, each of which satisfies \( p(x_1, \ldots, x_m) = e \), and therefore, so does \( G \). In particular, \( p(h_1, \ldots, h_m) = e \).

References


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