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Polarity compatible with a closure system

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A symmetric relation on a non empty set is called a polarity, in general. In [8] a C-polarity \( q_c(\Omega) \) on a closure space \((S, \Omega)\) is defined — \( S \) is a non empty set and \( \Omega \) is a closure system on \( S \) — in the following way: \( a q_c(\Omega) b \iff a \cap b \subseteq C \), where \( a, b \in S, C \subseteq S \) and \( M \) denotes the closure of \( M \subseteq S \) in \( \Omega \). This C-polarity is a generalization of some polarities from [1], [3], [4], [5] and [7] defined in l-groups, po-groups, lattices and semigroups (see [8], § 3). We denote \( p(A, C) = \{ x \in S : x \cap \bar{a} \subseteq \bar{C} \text{ for each } a \in A \} \), \( p^{n+1}(A, C) = p[p^n(A, C) \cup C] \) for every \( A, C \subseteq S \) and a positive integer \( n \); \( \Gamma_c(S, \Omega) = \{ p(A, C) : A \subseteq S \} \), \( \Gamma(S, \Omega) = \bigcap \{ \Gamma_c(S, \Omega) : C \subseteq S \} \). A set \( A \subseteq S \), \( A = p^2(A, C) \) is called a C-polar.

In § 1 of this paper we compare a C-polarity and a (general) polarity using the results of F. Šik (see [6]). It is shown that the set \( \Gamma_c(S, \Omega) \) of all C-polars on a closure space \((S, \Omega)\) is a complete Boolean algebra for each \( C \subseteq S \), ordered by the set-inclusion (Corollary 1.4). Further, a polarity on \((S, \Omega)\) compatible with \( \Omega \) is investigated. This polarity is characterized by the fact that all polars are closed. A C-polarity \( q_c(\Omega) \) is compatible with \( \Omega \) for each \( C \in \Omega \) if and only if \( \Omega \) is an algebraic closure system and a distributive lattice (Theorem 1.11).

In § 2 we show that a C-polarity on a closure space is a polarity defined in [7] on a suitable semigroup and some connections of these polarities are given.

A C-polarity on special closure spaces (topological spaces of Bourbaki, spaces with closed points) is investigated in § 3.

1. POLARITY COMPATIBLE WITH A CLOSURE SYSTEM

1.1. Definition. A symmetric binary relation \( \delta \) in a non empty set \( S \) is called a polarity in \( S \). For each \( A \subseteq S \) we define a set \( \delta(A) = \{ x \in S : x \delta a \text{ for each } a \in A \} \) and \( \delta^n(A) = \delta[\delta^{n-1}(A)] \) for each positive integer \( n \). If \( A = \delta^2(A) \), then \( A \) is called a \( \delta \)-polar. The set of all \( \delta \)-polars in \( S \) will be denoted by \( \Gamma_\delta(S) \) (or briefly \( \Gamma \)).

1.2. ([6], Theorem 3.) Let \( \delta \) be a polarity in a set \( S \). Then \( \Gamma_\delta(S) \) is a complete lattice, infima in \( \Gamma \) are set meets, \( S \) and \( \Lambda = \{ s \in S : s \delta x \text{ for each } x \in S \} \) are the
greatest and the least element of $\Gamma$, respectively, and the map $A \in \Gamma \mapsto \delta(A)$ is an involution, i.e. $\delta^2(A) = A$, $\delta(\bigvee A_a) = \bigwedge \delta(A_a)$, $\delta(\bigwedge A_a) = \bigvee \delta(A_a)$ for all $A, A_a \in \Gamma$.

B) Let $\delta$ be an antireflexive polarity in a set $S$. Then $\Gamma_\delta(S)$ is complemented and $\delta(A)$ is a complement of $A \in \Gamma_\delta(S)$.

C) Let $\delta$ be an antireflexive polarity in a set $S$ with a property $(D\beta)$: $x \not\sim y \Rightarrow$ there exists $z \in S$ such that $z \not\sim z$, $z \prec x$, $z \prec y$, where $\prec$ is a quasiorder in $S$ induced by $\delta(a \prec b \Leftrightarrow \{u \delta b \Rightarrow u \delta a\})$. Then $\Gamma_\delta(S)$ is a complete Boolean algebra.

1.3. ([6], Theorem 4.) A) Let $\mathcal{B}$ be a complete lattice of subsets of a set $S$, let infima in $\mathcal{B}$ be set meets and let $A \rightarrow A'$ be a map of $\mathcal{B}$ into $\mathcal{B}$, fulfilling $A'' = A$, $(\bigvee A_a)' = \bigwedge A_a'$ for all $A, A_a \in \mathcal{B}$. Denote by $X$ the greatest element of $\mathcal{B}$. Then there exists a unique polarity $\delta$ in $X$ such that $\Gamma_\delta(X) = \mathcal{B}$.

B) Let $\mathcal{B}$ be as in A) and in addition, let $A'$ be a complement of $A$ for any $A \in \mathcal{B}$. Then $\delta$ is antireflexive.

C) Let $\mathcal{B}$ be a complete Boolean algebra of subsets of a set $S$, let infima in $\mathcal{B}$ be set meets. Denote by $X$ the greatest element of $\mathcal{B}$. Then there exists a unique polarity $\delta$ in $X$ such that $\Gamma_\delta(X) = \mathcal{B}$. Furthermore, $\delta$ is antireflexive and $\delta$ has the property $(D\beta)$ from 1.2.

Remark. The polarity $\delta$ from 1.3 is defined in the following way: $x \not\sim y \Leftrightarrow y \in \bar{x}$, where $\bar{x} = \bigcap\{A \in \mathcal{B} : x \in A\}$.

1.4. Corollary. The set $\Gamma_c(S, \Omega)$ of all C-polars on a closure space $(S, \Omega)$ is a complete Boolean algebra for each $C \subseteq S$, ordered by set-inclusion. Further, $\bigwedge_{i \in I} p(A_i, C) = \bigcap_{i \in I} p(A_i, C)$, $\bigvee_{i \in I} p(A_i, C) = p^2(\bigcup_{i \in I} p(A_i, C), C)$ for every $A_i \subseteq S$, $i \in \in I \neq \emptyset$ and a complement of a C-polar $p(A, C)$ is $p^2(A, C)$ for each $A \subseteq S$. The greatest element of $\Gamma_c(S, \Omega)$ is $S = p(\emptyset, C)$ and the smallest element of $\Gamma_c(S, \Omega)$ is $C = p(S, C)$.

Proof. C-polarity $\gamma_c(\Omega)$ is a symmetric and antireflexive relation in $S$ and we shall prove the property $(D\beta)$ from 1.2, C): If $x \not\sim \gamma_c(\Omega) y$, then $\bar{x} \cap \bar{y} \subseteq \bar{C}$ and if we choose $z \in (\bar{x} \cap \bar{y}) \cap \bar{C}$, then $\bar{z} \cap \bar{z} = \bar{z}$ non $\subseteq \bar{C}$, i.e., $z \not\sim \gamma_c(\Omega) z$. Further, if $u \gamma_c(\Omega) x$, then $\bar{u} \cap \bar{x} \subseteq \bar{C}$ and $\bar{u} \cap \bar{z} \subseteq \bar{u} \cap (\bar{x} \cap \bar{y}) \subseteq \bar{u} \cap \bar{C}$, i.e., $u \gamma_c(\Omega) z$ and $z \prec x$ in the quasiorder $\prec$ induced by $\gamma_c(\Omega)$ in $S$. Similarly, we can prove $z \prec y$. The rest follows from 1.2.

1.5. Proposition. Let $\delta$ be an antireflexive polarity in $S$, $S \neq \emptyset$. Then $\gamma_\delta(\Gamma_\delta(S)) \supseteq \equiv \delta$. Further, $\gamma_\delta(\Gamma_\delta(S)) = \delta$ if and only if $\delta^2(a) \cap \delta^2(b) = \delta^2(\emptyset)$ implies $a \delta b$ for $a, b \in S$. 

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1.6. Corollary. If \( \delta \) is an antireflexive polarity in \( S \), which fulfils \( \delta \neq \emptyset \) and has the property \((D\beta)\) from 1.2, then \( \delta = \Gamma_\emptyset(\Gamma_\emptyset(S)) \).

Proof. If \( x \) non \( \delta \ y, x, y \in S \), then \((D\beta)\) implies the existence of an element \( z \in S \) such that \( z \) non \( \delta \ z, z < x, z < y \). The relation \( z < x \) means: \( s \delta x \Rightarrow s \delta z, (s \in S) \), i.e., \( z \in \delta^2(x) \). Similarly \( z \in \delta^2(y) \). Further, \( \delta^2(\emptyset) = \delta(S) = \{s \in S: s \delta x \text{ for every } x \in S\} \). If \( z \in \delta^2(\emptyset) \), then \( z \) \( \delta \) \( z \), a contradiction. Then \( \delta^2(\emptyset) \neq \delta^2(x) \cap \delta^2(y) \) and the rest follows from 1.5.

Remark. C-polarity \( \varrho_C(\Omega) \) is antireflexive and has the property \((D\beta)\) (see the proof of 1.4) and thus \( \varrho_C(\Omega) = \varrho_\emptyset(\Gamma_C(S, \Omega)) \).

1.7. Definition. Let \( \delta \) be a relation on a closure space \((S, \Omega)\). We say that \( \delta \) is compatible with \( \Omega \), when \( s \delta A \Rightarrow s \delta \overline{A} \) for every \( s \in S \) and \( A \subseteq S \).

Remark. \( s \delta A \) means \( s \delta a \) for each \( a \in A \).

1.8. Proposition. Let \( \delta \) be a symmetric relation on a closure space \((S, \Omega)\). Then it holds:

1) \( \delta \) is compatible with \( \Omega \) if and only if \( \delta(\overline{A}) = \delta(A) \) for each \( A \subseteq S \).

2) If \( \delta \) is compatible with \( \Omega \), then \( \Gamma(\delta) \subseteq \Omega \).

Proof. 1) is clear. 2) If \( x \in \overline{\delta(A)} \), then \( x \in \delta^2(\{x\}) \subseteq \delta^2(\overline{\delta(A)}) = \delta^2(\delta(A)) = \delta(A) \) and \( \delta(A) \subseteq \delta(A) \).

1.9. Proposition. 1) C-polarity \( \varrho_C(\Omega) \) is a symmetric relation and \( \varrho_C(\Omega)(A) = p(A, C) \text{ for every } A, C \subseteq S, \Gamma_C(S, \Omega) = \Gamma(\varrho_C(\Omega)) \).

2) C-polarity \( \varrho_C(\Omega) \) is compatible with \( \Omega \) if and only if \( \Gamma_C(S, \Omega) \subseteq \Omega \).

3) C-polarity \( \varrho_C(\Omega) \) is compatible with \( \Omega \) for each \( C \subseteq S \) if and only if \( \Gamma(S, \Omega) = \Omega \).

Proof. 1) \( \varrho_C(\Omega)(A) = \{s \in S: s \varrho_C(\Omega) a \text{ for each } a \in A\} = p(A, C) \text{ for every } A, C \subseteq S \). \( \Gamma_C(S, \Omega) \subseteq \Omega \) implies \( p(A, C) \cap \overline{A} \subseteq p(A, C) \cap p^2(A, C) = p(A, C) \cap \overline{p^2(A, C)} = \overline{C} \) and similarly \( p(\overline{A}, C) \cap \overline{A} \subseteq \overline{C} \). From [8], 1.7 we have \( p(A, C) \subseteq p(\overline{A}, C) \subseteq p(A, C) \). \( \Rightarrow: \) see 1.8.2.

3) \( \Rightarrow: \Gamma(S, \Omega) = \Omega \) implies \( \Gamma_C(S, \Omega) \subseteq \Omega \) for each \( C \subseteq S \) and the rest follows from 2. \( \Rightarrow: \) see 1.8.2.

1.10. Lemma. Let \((S, \Omega)\) be a closure system and \( \Omega \) a distributive lattice with operations \( \overline{A} \land \overline{B} = \overline{A \land B}, \overline{A} \lor \overline{B} = \overline{A \lor B} \) for every \( A, B \subseteq S \). Then \( \overline{x} \land \overline{\bigcup \{N: N \subseteq A \text{ finite}\}} \subseteq \overline{x} \land \bigcup \{a: a \in A\} \) for every \( x \in S, A \subseteq S \).
Proof. $\bar{x} \cap \bigcup \{N : N \subseteq A \text{ finite}\} = \bigcup \{\bar{x} \cap N : N \subseteq A \text{ finite}\} = \bigcup \{\bar{x} \cap \{a_{1N}, \ldots, a_{kN}\} : N = \{a_{1N}, \ldots, a_{kN}\} \subseteq A \text{ finite}\}$.

1.11. Theorem. The following assertions are equivalent:

1) $C$-polarity $q_C(\Omega)$ is compatible with $\Omega$ for each $C \subseteq \Omega$.

2) $x \cap A = x \cap \bigcup \{a : a \in A\}$ for every $x \in S$, $A \subseteq S$.

3) $\Omega$ is an algebraic closure system and a distributive lattice with operations
   \[ A \land B = A \cap B, \quad A \lor B = A \lor B \text{ for every } A, B \subseteq S. \]

Remark. 1. An algebraic closure system $\Omega$ on $S$ is a closure system with the property:
   \[ \bar{A} = \bigcup \{N : N \subseteq A \text{ finite}\} \text{ for each } \bar{N} \in S \text{ (see [2]).} \]

2. The assertion 2 is the same kind of distributivity in $\Omega : \bar{x} \cap \bigcup \{a : a \in A\} = x \cap A = x \cap \bigcup \{a : a \in A\}$.

Proof. 1 $\Rightarrow$ 2: The fact $\bigcup \{\bar{x} \cap \bar{a} : a \in A\} = \bigcup \{\bar{x} \cap a : a \in A\}$ implies $x \in \bigcup \{\bar{x} \cap a : a \in A\}$. Thus $\bar{x} \cap A = \bar{x} \cap \bigcup \{a : a \in A\}$.

3 $\Rightarrow$ 1: Let $X, Y, Z \subseteq S$. Then $(X \cup Z) \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \subseteq (X \cap Y) \cup (X \cap Z)$.

Finally, $X \cup Z \cap Y \cup Z \subseteq K = (X \cap Y) \cup Z$ and $\Omega$ is a distributive lattice. If $x \in \bar{A}$, then $\bar{x} \subseteq \bar{A}$ and $\bar{x} \cap \bigcup \{N : N \subseteq A \text{ finite}\} \subseteq \bar{x} \cap \bigcup \{a : a \in A\} = \bar{x} \cap A = \bar{x}$ (see Lemma 1.10). This fact implies $x \in \bar{x} \subseteq \bigcup \{N : N \subseteq A \text{ finite}\}$ and $\bar{A} \subseteq \bigcup \{N : N \subseteq A \text{ finite}\}$ is clear and $\Omega$ is an algebraic closure system.

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1.12. Proposition. Let \((S, \Omega)\) be a closure system. Then it holds: 1. Let \(A, C, D \subseteq S, A \supseteq C\). Then \(p(A, C) \cap p(C, D) = p(A, D)\) if and only if \(\varrho_d(\Omega)\) is compatible with \(\Omega\) and \(\overline{D} \subseteq \overline{C}\).

2. If \(\varrho_d(\Omega)\) is compatible with \(\Omega\) and \(\overline{D} \subseteq \overline{C}\), then \(\overline{D} = \overline{C} \cap p(C, D)\).

Proof. 1. \(\Rightarrow: p(\overline{A}, D) = p(\overline{A}, A) \cap p(A, D) = S \cap p(A, D) = p(A, D)\) and 1.8.1 implies the compatibility of \(\varrho_d(\Omega)\) with \(\Omega\). \(\overline{D} = p(S, D) = p(S, C) \cap p(C, D) \subseteq p(S, C) = \overline{C}\).

\(\Leftarrow:\) If \(x \in p(A, D)\), then \(\overline{x} \cap \overline{a} \subseteq \overline{D} \subseteq \overline{C}\) for each \(a \in A\) and \(x \in p(A, C)\). Further, \(\overline{x} \cap \overline{c} \subseteq \overline{x} \cap \bigcup \{\overline{a} : a \in A\} \subseteq \overline{D}\) for each \(c \in C\). It means that \(x \in p(C, D)\), i.e., \(p(A, D) \subseteq p(A, C) \cap p(C, D)\). If \(x \in p(A, C) \cap p(C, D)\), then \(\overline{x} \cap \overline{c} \subseteq \overline{C}, \overline{x} \cap \overline{c} \subseteq \overline{D}\) for every \(a \in A, c \in C\). Compatibility of \(\varrho_d(\Omega)\) with \(\Omega\) implies \(x \in p(C, D) = (p(C, D))\), i.e., \(\overline{x} \cap \overline{C} = \overline{x} \cap \bigcup \{\overline{y} : y \in C\} = \bigcup \{\overline{x} \cap \overline{y} : y \in C\} \subseteq \overline{D}\). Finally, \(\overline{x} \cap \overline{a} \subseteq \overline{x} \cap \bigcap \overline{C} \subseteq \overline{D}\) and \(x \in p(A, D), p(A, C) \cap p(C, D) \subseteq p(A, D)\).

2. From 1 it follows that \(D = p(S, D) = p(S, C) \cap p(C, D) = \overline{C} \cap p(C, D)\).

2. POLARITY ON SEMIGROUPS

2.1. Definition. (See [7].) Let \((S, \cdot)\) be a semigroup. A mapping \(x : \exp S \rightarrow \exp S\) fulfilling the following conditions:

I. \(A \subseteq S \Rightarrow A \subseteq A_x\),

II. \(A, B \subseteq S, A \subseteq B_x \Rightarrow A_x \subseteq B_x\),

III. \(A \subseteq S \Rightarrow S \cdot A_x \subseteq A_x\),

IV. \(A, B \subseteq S \Rightarrow A \cdot B_x \subseteq (A \cdot B)_x\)

is called an ideal mapping and a set \(A \subseteq S\) with the property \(A_x = A\) is called an \(x\)-ideal in \(S\). A system of all \(x\)-ideals in \(S\) for a given ideal mapping is called an \(x\)-system.

2.2. Proposition. If \(\Omega\) is a closure system on a set \(S\) and \(x : \exp (\exp S) \rightarrow \exp (\exp S)\) such that \(\mathcal{A} = \{X \in \exp S : X \subseteq \overline{A}, X \neq 0\}\) for a suitable \(A \in \mathcal{A}\) for each \(\mathcal{A} \subseteq \exp S\), then \(x\) is an ideal mapping in the commutative semigroup \((\exp S, \cdot)\), where \(A \cdot B = \overline{A} \cap \overline{B}\) for every \(A, B \subseteq S\).

Proof. I. \(\mathcal{A} \subseteq \mathcal{A}_x\) is clear. II. If \(\mathcal{A} \subseteq \mathcal{B}_x, Y \in \mathcal{A}_x\), then there exists \(A \in \mathcal{A}\) such that \(Y \subseteq \overline{A}\). It means that \(A \in \mathcal{B}_x\), i.e., there exists \(B \in \mathcal{B}\) such that \(A \subseteq \overline{B}\) and \(Y \subseteq \overline{A} \subseteq \overline{B} = \overline{B}, Y \in \mathcal{B}_x\). Finally, \(\mathcal{A}_x \subseteq \mathcal{B}_x\). III. If \(\mathcal{A} \subseteq \exp S, X \in \mathcal{A}_x, Y \in \exp S\), then there exists \(A \in \mathcal{A}\) such that \(X \subseteq \overline{A}\), i.e., \(X \cdot Y = \overline{X} \cap \overline{Y} \subseteq \overline{A} \cap \overline{Y} \subseteq \overline{A}\). It implies \(X \cdot Y \in \mathcal{A}_x\) and \(\mathcal{A}_x \cdot \exp S \subseteq \mathcal{A}_x\). IV. If \(\mathcal{A}, \mathcal{B} \subseteq \exp S, X \in \mathcal{A}, Y \in \mathcal{B}_x\), then there exists \(B \in \mathcal{B}\) such that \(Y \subseteq \overline{B}\) and thus \(X \cdot Y = \overline{X} \cap \overline{Y} \subseteq \overline{X} \cap \overline{Y} \in \mathcal{A}_x \cdot \mathcal{B}_x\).
2.3. Definition. (See [7].) Let \((S, \cdot, e)\) be a commutative semigroup with a zero \(e\) (i.e., \(s \cdot e = e \cdot s = e\) for each \(s \in S\)). We define a symmetric relation \(\delta'\) in \(S\), called \(\delta'\)-polarity, in the following way:

\[
\begin{align*}
&x \delta' y \iff x \cdot y = e \quad \text{for } x, y \in S, \ x \neq y, \\
&x \delta' x \iff x = e \quad \text{for } x \in S.
\end{align*}
\]

2.4. Proposition. If \((S, \cdot, e)\) is a commutative semigroup with a zero \(e\), \(\Omega\) is an \(x\)-system on \(S\) and \(s \cdot s = e\) implies \(s = e\), for \(s \in S\), then it holds: \(\delta' = \varrho_{(e)}(\Omega) \iff \{e\}_x = \{e\}\).

Proof. \(\Rightarrow\): \(p \in \{e\}_x \Rightarrow \{p\}_x \cap \{p\}_x = \{e\}_x \subseteq \{e\}_x \Rightarrow \varrho_{(e)}(\Omega) p \Rightarrow p \delta' p \Rightarrow p = e \Rightarrow \{e\}_x = \{e\}\).

\(\Leftarrow\): If \(a \delta' b\), \(a \neq b\), \(a, b \in S\), then \(a \cdot b = e\), \(b \in \delta'(a)\), \(\{b\}_x \subseteq \delta'(a)\), \(\{a\}_x \subseteq \delta'(a)\) — see [7], 1.7. Now, if \(p \in \{a\}_x \cap \{b\}_x\), then \(p \in \delta'(a) \cap \delta'(a) = \{e\}_x = \{e\}\) and \(\{a\}_x \cap \{b\}_x \subseteq \{e\}_x\), i.e., \(a \varrho_{(e)}(\Omega) b\). If \(a \delta' b\), \(a = b\), then \(a = b = e\) and \(\{a\}_x \cap \{b\}_x \subseteq \{e\}_x\), i.e., \(a \varrho_{(e)}(\Omega) b\).

Conversely, if \(a \varrho_{(e)}(\Omega) b\), \(a \neq b\), then \(\{a\}_x \cap \{b\}_x \subseteq \{e\}_x = \{e\}\) and \(a \cdot b \in \{a\}_x \cap \{b\}_x = \{e\}\), i.e., \(a \cdot b = e\), \(a \delta' b\). If \(a \varrho_{(e)}(\Omega) b\), \(a = b\), then \(\{a\}_x = \{e\}\) and \(a = e\), i.e., \(a \delta' b\).

Notation. \((\exp S)_C = \{X \in \exp S : X \supseteq C\}\), \(\Omega(\exp S) = \{A : A \subseteq \exp S\}\), \(\Omega_C(\exp S) = \{A : A \subseteq \exp S\}_C\).

2.5. Corollary. If \((S, \Omega)\) is a non-empty set with a closure system \(\Omega\), \(C \in \Omega\), then \((\exp S)_C, \cdot, C)\) is a commutative semigroup with a zero \(C\), where \(A \cdot B = A \cap B\) for every \(A, B \in (\exp S)_C\), and the restriction \(x_C\) of the mapping \(x\) from Proposition 2.2 on \(\exp S\) is an ideal mapping in \((\exp S), \cdot)\) and also in \((\exp S)_C, \cdot)\). Further, \(\delta'\)-polarity in \((\exp S)_C, \cdot)\) is a \(C\)-polarity \(\varrho_{(C)}(\Omega_C(\exp S))\) in \((\exp S)_C, \Omega_C(\exp S))\).

Proof. The first part of Corollary can be proved similarly as Proposition 2.2. The second part follows from Proposition 2.4 and the fact \(\{C\}_x = \{X \in \exp S_C : X \supseteq C\} = \{C\}\).

2.6. Proposition. For a \(C\)-polarity \(\varrho_C(\Omega)\) on a closure system \((S, \Omega)\) and a \(\delta'\)-polarity on a commutative semigroup \((\exp S)_C, \cdot)\), where \(C \in \Omega\), it holds:

1. \(X \in \delta'(A) \iff \overline{X} \subseteq p(\overline{A}, C)\) for \(A, X \in (\exp S)_C\).

2. Moreover, if \(\varrho_C(\Omega)\) is compatible with \(\Omega\), then \(\delta'(A) = (p(A, C))_C, \overline{\delta'(A)} = (p(A, C))_C, \overline{\delta'(A)} = \overline{\delta'(p(A, C))}\), where \(A \in (\exp S)_C\) and \(x_C\) is the ideal mapping on \((\exp S)_C\) from 2.5.

Proof. 1. \(X \in \delta'(A) \iff X \cdot A = C \iff \overline{X} \cap \overline{A} = C \iff \overline{X} \subseteq p(\overline{A}, C)\).
2. The fact \( \delta'(A) = (p(A, C))_{xc} \) follows from 1 and 1.9, 2): \( X \in \delta'(A) \iff X \subseteq p(A, C) \iff X \in \{p(A, C)\}_{xc} \iff X \in \{p(A, C)\}_{xc} \). Further, for each \( X \in \delta'(p(A, C)) \) and each \( Z \in \{p(A, C)\}_{xc} \) we have \( Z \subseteq p(A, C) \) and \( X \cdot Z = X \cap Z \subseteq X \cap p(A, C) = = X \cap p(A, C) = X \cdot p(A, C) = C \), i.e., \( X \in \delta'[(p(A, C))_{xc}] \). It means \( \delta'(p(A, C)) \subseteq \subseteq \delta'[(p(A, C))_{xc}] \) and from \( p(A, C) \in (p(A, C))_{xc} \) we have \( \delta'(p(A, C)) \subseteq \subseteq \delta'[(p(A, C))_{xc}] \). Finally, \( \delta'(A) = \delta'(\delta'(A)) = \delta'[(p(A, C))_{xc}] = \delta'(p(A, C)) = = (p^2(A, C))_{xc} \).

3. POLARS ON SPECIAL CLOSURE SPACES

3.1. Proposition. Let \( q_c(\Omega) \) be compatible with \( \Omega \) for each \( C \subseteq S \). Then \( \Omega \) defines a topological space of Bourbaki on \( S \) if and only if \( I(S, \Omega) \) is a sublattice of the lattice \( (\exp S, \cup, \cap) \).

Proof. \( \Rightarrow: \) For every \( P, Q \in I(S, \Omega) \) it holds \( P \cup Q = \bar{P} \cup Q = P \cup \bar{Q} \in \Omega = = I(S, \Omega) \), see 1.9.3.

\( \Leftarrow: \) If \( A, B \subseteq S \), then \( \bar{A} \cup \bar{B} = p(S \setminus A, A) \cup p(S \setminus B, B) \in I(S, \Omega) = \Omega \) (see 1.9.3 and [8], 1.5,a)), i.e., \( \bar{A} \cup \bar{B} = \bar{A} \cup \bar{B} = A \cup B \).

3.2. Proposition. If \( \emptyset \in \Omega \), then \( p(A, \emptyset) = \{s \in S : s \subseteq S \setminus \bar{A} \} \) for each \( A \subseteq S \).

Proof. If \( s \in p(A, \emptyset) \), then \( \bar{s} \cap \bar{a} \subseteq \emptyset = \emptyset \) for each \( a \in \bar{A} \), i.e., \( \bar{s} \cap \bar{A} = 0 \) and \( \bar{s} \subseteq S \setminus \bar{A} \). Further, if \( \bar{s} \subseteq S \setminus \bar{A} \), then \( \bar{s} \cap \bar{a} \subseteq (S \setminus \bar{A}) \cap \bar{A} = 0 \) for each \( a \in \bar{A} \) and \( s \in p(A, \emptyset) \).

3.3. Theorem. If \( \{s\} \cup \emptyset \in \Omega \) for each \( s \in S \), then \( p(A, C) = (S \setminus A) \cup C \) for every \( A, C \subseteq S \). If \( q_c(\Omega) \) is compatible with \( \Omega \) and \( p(A, C) = (S \setminus A) \cup C \) for every \( A, C \subseteq S \), then \( \{s\} \cup \emptyset \in \Omega \) for each \( s \in S \).

Proof. If \( x \in p(A, C) \setminus \bar{C} \), then \( \bar{x} \cap \bar{a} \subseteq \bar{C} \) for every \( a \in A \) and \( x \neq a \), i.e., \( x \in S \setminus A, p(A, C) \subseteq \bar{C} \cup (S \setminus A) \). Further, \( \bar{C} \subseteq p(A, C) \) (see [8], 1.1,a)) and \( \bar{s} \cap \bar{a} = = (\{s\} \cup \emptyset) \cap (\{a\} \cup \emptyset) = (\{s\} \cap \{a\}) \cup \emptyset = \emptyset \subseteq C \) for every \( s \in S \setminus A \) and \( a \in A \), i.e., \( \bar{C} \cup (S \setminus A) \subseteq p(A, C) \).

If \( q_c(\Omega) \) is compatible with \( \Omega \), then we have \( p(S \setminus \{s\}, \emptyset) = (S \setminus (S \setminus \{s\})) \cup \emptyset = = \{s\} \cup \emptyset \) for each \( s \in S \) and thus \( \{s\} \cup \emptyset \in \Omega \), see 1.8.2.

3.4. Corollary. Let \( \{s\} \in \Omega \) for each \( s \in S \). Then \( q_c(\Omega) \) is compatible with \( \Omega \) if and only if \( \Omega = \exp S \).

Proof. \( \Rightarrow: \) It is \( \emptyset \in \Omega \) and \( S \setminus A = p(A, \emptyset) = p(\bar{A}, \emptyset) = S \setminus \bar{A} \) (see 3.3), i.e., \( A \in \Omega \) for each \( A \subseteq S \). The second implication is clear.
References


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