Antonio M. Lopez, Jr.; John K. Luedeman
Quasi-injective $S$-systems and their $S$-endomorphism semigroup

Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 1, 97–104

Persistent URL: http://dml.cz/dmlcz/101581
QUASI-INJECTIVE S-SYSTEMS AND THEIR S-ENDOMORPHISM SEMIGROUP

ANTONIO M. LOPEZ, JR., New Orleans, and JOHN K. LUEDEMAN, Clemson

(Received March 14, 1977)

Patterned after the theory of modules over a ring, P. BERTHIAUME [1] introduced the concepts of injective and weakly-injective S-systems. He exhibited examples of such S-systems and showed that properties holding true for a right ring module need not hold for a right S-system. For example, a weakly injective S-system need not be injective; in ring theory, this is part of Baer's Theorem. In this paper, we study another weak form of injectivity called quasi-injectivity. Quasi-injective modules have been studied by JOHNSON and WONG [6], FAITH and UTUMI [3], and B. OSOFSKY [8], among others. Recently M. SATYANARAYANA [9] investigated quasi- and weakly-injective S-systems. Our paper is a study of quasi-injective S-systems and their S-endomorphism semigroup. We characterize the smallest quasi-injective essential extension of an S-system $M_0$ contained in $I(M_0)$, its injective hull. Further we give conditions for $\text{Hom}_S(M, M)$ to be (von Neumann) regular and obtain as corollaries a result of M. BOTERO DE MEZA [2] dealing with the regularity of the maximal right quotient semigroup $Q(S)$ of a semigroup $S$, and a generalization for S-systems of Faith and Utumi's result on the regularity of the endomorphism ring of a quasi-injective module.

1. PRELIMINARIES

**Definition 1.1.** A right S-system $M$ with zero, denoted $M_0$, is a set $M$, a semigroup $S$ with zero, and a function $M \times S \rightarrow M$ such that $(m, s) \rightarrow ms$ and the following properties hold:

(i) $(ms)t = m(st)$ for $m \in M$ and $s, t \in S$.

(ii) $M$ contains an element $\mathcal{Z}$ (necessarily unique) such that $\mathcal{Z}s = \mathcal{Z}$ for all $s \in S$.

(iii) for all $m \in M$, $m0 = \mathcal{Z}$, where $0$ is the zero of $S$.

Dually we can define a left S-system with zero. In this paper all our S-system will be right S-systems with zero.
Definition 1.2. A subsystem $N$ of $M_S$, denoted $N_S \leq M_S$, is a subset of $M$ such that $ns \in N$ for all $n \in N$ and $s \in S$.

Definition 1.3. A (right) congruence $\alpha$ on $M_S$ is an equivalence relation defined on $M$ such that if $a \alpha b$ then $(as) \alpha (bs)$ for $a, b \in M$ and all $s \in S$.

Definition 1.4. An $S$-homomorphism $f : A_S \rightarrow B_S$ is a mapping from $A$ to $B$ such that for any $a \in A$ and $s \in S$, $f(as) = f(a)s$.

The set of all $S$-homomorphisms from $A_S$ to $B_S$ is denoted by $\text{Hom}_S(A, B)$. Under composition of functions $\text{Hom}_S(M, M)$ is a semigroup called the $S$-endomorphism semigroup of $M_S$. If the elements of $K = \text{Hom}_S(M, M)$ are regarded as left operators then $M$ is a $(K, S)$-bisystem; that is to say, $M$ is a right $S$-system and a left $K$-system such that $h(ms) = (hm)s$ for $h \in K$, $m \in M$, and $s \in S$.

Definition 1.5. An $S$-system $M_S$ is injective if for each one-to-one $S$-homomorphism $g : P_S \rightarrow R_S$ and each $S$-homomorphism $h : P_S \rightarrow M_S$, there exists an $S$-homomorphism $\tilde{h} : R_S \rightarrow M_S$ such that $\tilde{h}g = h$.

Definition 1.6. An $S$-system $M_S$ is weakly-injective if for any right ideal $R$ of $S$ and $f \in \text{Hom}_S(R, M)$ there exists an element $m \in M$ such that $f(r) = mr$ for all $r \in R$.

Definition 1.7. An $S$-system $M_S$ is quasi-injective if for $N_S \leq M_S$ and $f \in \text{Hom}_S(N, M)$ there exists an $S$-homomorphism $\tilde{f} : M_S \rightarrow M_S$ such that $\tilde{f}|_N = f$.

In [1], Berthiaume showed that a weakly-injective $S$-system need not be injective. However, the converse is true. Also, it is clear that $M_S$ being injective implies that $M_S$ is quasi-injective, but the converse here is false. In fact, quasi-injective does not imply weakly-injective, as shown by the following example adapted from [9].

Example 1.8. Let $S$ be the semigroup \{0, $a, b$\} with $ab = a^2 = a$ and $ba = b^2 = b$. Now $S$ considered as an $S$-system over itself is quasi-injective but it is not weakly injective since the identity map is not determined by left multiplication by an element of $S$. Consequently, it is not injective.

Definition 1.9. A subsystem $N$ is large (or essential) in $M_S$ if for any $P_S$ and any $S$-homomorphism $f : M_S \rightarrow P_S$ whose restriction to $N$ is one-to-one, then $f$ is itself one-to-one. In such a case, we say that $M_S$ is an essential extension of $N_S$.

The main result of Berthiaume's work in [1] is that every $S$-system has a maximal essential extension which is injective and unique up to $S$-isomorphism over $M_S$. This maximal essential extension which is injective is called the injective hull of $M_S$ and is denoted by $I(M_S)$.

Definition 1.10. A nonzero subsystem $N$ of $M_S$ is intersection large ($\cap$-large) if for all nonzero subsystems $X$ of $M, X \cap N \neq \emptyset$. This will be denoted by $N_S \leq' M_S$. 

98
Equivalently, a nonzero subsystem \( N_s \subseteq M_s \) if and only if for all \( \emptyset \neq m \in M \) there exists \( s \in S^1 \) (an identity adjoined) such that \( \emptyset \neq ms \in N \). Feller and Gantos in [4] proved that every large subsystem of \( M_S \) is \( \cap \)-large. The converse is false.

**Definition 1.11.** The singular congruence \( \psi_M \) on \( M_S \) is a right congruence defined by \( a \psi_M b \) if and only if \( ax = bx \) for all \( x \) in some \( \cap \)-large right ideal of \( S \).

In [5], Hinkle showed that when \( \psi_M = i \), the identity congruence on \( M \), the concepts of large and \( \cap \)-large are the same. He also showed that \( M_S \) being weakly-injective and \( \psi_M = i \) imply that \( M_S \) is injective. Example 1.8 shows that \( M_S \) being quasi-injective and \( \psi_M = i \) does not imply that \( M_S \) is itself injective.

### 2. The Injective Hull

Let \( M_S \) be an \( (H, S) \)-system with zero, let \( I = I(M_S) \), its injective hull, and let \( H = \text{Hom}_S(I, I) \) the \( S \)-endomorphism semigroup of \( I \). We know that \( I \) is the minimal injective essential extension containing \( M_S \). Is there a minimal quasi-injective essential extension of \( M_S \) contained in \( I \) as in ring theory?

**Lemma 2.1.** If \( M \) is an \((H, S)\)-bisubsystem of \( I \), then \( M \) is quasi-injective.

**Proof.** Let \( N_S \subseteq M_S \) and \( f : N_S \rightarrow M_S \) an \( S \)-homomorphism. Since \( M_S \subseteq I \), \( f \) can be extended to an \( S \)-homomorphism \( \bar{f} \in H \). But \( \bar{f}(M) \subseteq M \) so \( f \) can be extended to an \( S \)-homomorphism of \( M \) into \( M \), namely \( \bar{f}|_M \).

**Lemma 2.2.** If \( \psi_M = i_M \) then \( \psi_I = i_I \).

**Proof.** This follows immediately from the fact that \( M_S \) is large in \( I \) and Theorem 7 in [1].

**Lemma 2.3.** Let \( f, g \in \text{Hom}_S(M, M) \) and suppose \( f \) and \( g \) agree on an \( \cap \)-large subsystem \( N_S \) of \( M_S \). If \( \psi_M = i \), then \( f = g \).

**Proof.** Let \( x \in M \), then for \( c \in x^{-1}N = \{ s \in S : xs \in N \} \), an \( \cap \)-large right ideal of \( S \), we have \( f(x) c = g(x) c \). Since \( \psi_M = i \) then \( f(x) = g(x) \).

**Theorem 2.4.** If \( M_S \) is quasi-injective and \( \psi_M = i \), then \( M \) is an \((H, S)\)-bisubsystem of \( I \).

**Proof.** Let \( h \in H \). Since \( M_S \subseteq I \) then \( h^{-1}(M) \subseteq I \) and so \( \emptyset \neq h^{-1}(M) \cap M \subseteq I \). Let \( N = h^{-1}(M) \cap M \) and define an \( S \)-homomorphism \( a : N_S \rightarrow M_S \) by \( x \rightarrow h(x) \). Since \( M_S \) is quasi-injective there exists \( b \in \text{Hom}_S(M, M) \) such that \( b(x) = a(x) \) for all \( x \in N \). Since \( I \) is injective, there exists \( c \in H \) such that \( c(x) = b(x) \) for all \( x \in M \). Hence \( c(n) = b(n) = a(n) = h(n) \) for all \( n \in N \). Since \( \psi_M = i \) then \( \psi_I = i \) by Lemma
2.2, and so \( c = h \) by Lemma 2.3. But \( c(M) \subseteq M \) so \( h(M) \subseteq M \). Hence \( M \) is an \((H, S)\)-bisubsystem of \( I \).

**Corollary 2.5.** Let \( M_S \) be an \( S \)-system for which \( \psi_M = i \). Then \( M_S \) is quasi-injective if and only if \( M = HM \) where \( HM = \{ f(m) \in I \mid f \in H \text{ and } m \in M \} \).

**Proof.** We note that \( HM \) is the smallest fully invariant \((H, S)\)-bisubsystem of \( I \) containing \( M \) and it is quasi-injective.

Note that if \( M_S \) is quasi-injective and \( K = \text{Hom}_S(M, M) \), then any \( K \)-invariant subsystem of \( M_S \) is also quasi-injective.

**Theorem 2.6.** Let \( M_S \) be an \( S \)-system for which \( \psi_M = i \). Then \( M_S \) is quasi-injective if and only if \( \text{Hom}_S(M, M) \cong \text{Hom}_S(I, I) \).

**Proof.** Let \( K = \text{Hom}_S(M, M) \). If \( H \cong K \) then \( M \) is an \((H, S)\)-bisubsystem of \( I \) and so by Lemma 2.1 must be quasi-injective. Conversely, consider \( \phi : K \rightarrow H \) defined by \( a \rightarrow \bar{a} \) where \( \bar{a} : I \rightarrow I \) is the quasi-injective extension of \( a : M \rightarrow M \subseteq I \).

Since \( \psi_M = i \) this mapping is well defined, one-to-one and a semigroup homomorphism. Furthermore, \( M_S \) being quasi-injective implies by Theorem 2.4 that \( M \) is an \((H, S)\)-bisubsystem of \( I \).

We now show that \( HM \) is the smallest quasi-injective essential extension of \( M \) contained in \( I \).

**Theorem 2.7.** Let \( M_S \) be an \( S \)-system with \( \psi_M = i \). Then \( HM \) is the intersection of all quasi-injective \( S \)-subsystems of \( I \) containing \( M \).

**Proof.** Let \( P \) be a quasi-injective subsystem of \( I \) containing \( M \). We must show that \( HM \subseteq P \), but it is sufficient to show that \( aP \subseteq P \) for all \( a \in H \). To this end then let \( a \in H \). Since \( M \subseteq I \) and \( M \subseteq P \subseteq I \) then both \( P \) and \( a^{-1}(P) \) are \( \cap \)-large \( S \)-subsystems of \( I \) and so \( \emptyset = a^{-1}(P) \cap P \) is an \( \cap \)-large \( S \)-subsystem of \( P \). Consider the mapping \( a^{-1}(P) \cap P \rightarrow P \) defined by \( x \rightarrow a(x) \). Since \( P \) is quasi-injective then there exists an \( \bar{a} \in \text{Hom}_S(P, P) \) such that \( \bar{a}(x) = a(x) \) for all \( x \in a^{-1}(P) \cap P \). Since \( I \) is injective there exists \( \bar{a} \in H \) such that \( \bar{a}(y) = a(y) \) for all \( y \in P \). Thus \( \bar{a}P \subseteq P \).

But by Lemma 2.2 and 2.3, \( \bar{a}(x) = a(x) \) for all \( x \in a^{-1}(P) \cap P \subseteq I \) implies that \( \bar{a} = a \), and so \( aP \subseteq P \).

Since there are \( S \)-systems which are quasi-injective but not injective (Example 1.8) we can have \( HM \subseteq I, HM \neq I \). The condition that \( \psi_M = i \) cannot be omitted in the previous theorem as the following example demonstrates.

**Example 2.8.** Let \( Q^* \) represent the noncomplete chain of rationals with largest element \( +\infty \) and \( q \cdot q' = q \) if and only if \( q \leq q' \). Thus \( Q^*_0 \), has for its injective hull the chain of extended reals \( \mathbb{R}^* \). Berthiaume [1] showed that every noncomplete chain is weakly injective. Satyanarayana [9] showed that since \( Q^*_0 \) has an identity it must
also be quasi-injective. Here \( \psi_{Q^*} \neq i \) because if \( \psi_{Q^*} = i \) then the maximal right quotient semigroup \( Q(Q^*) \approx B(Q^*) \), the bicommutator of the injective hull of \( Q^* \), [7; Corollary 3.1] which is a contradiction since \( Q^* = Q(Q^*) \) and \( R^* = B(Q^*) \). Hence \( Q_{Q^*} \) is quasi-injective and \( \psi_{Q^*} \neq i \). In this case, \( H = \text{Hom}_{Q^*}(R^*, R^*) \) and considering the mapping \( f : R^* \to R^* \) defined by \( r \to (\sqrt{2}) \cdot r \), we say that \( HQ^* \neq Q^* \). Hence \( HQ^* \) is not the smallest quasi-injective essential extension contained in \( R^* \).

3. THE S-ENDOMORPHISM SEMIGROUP OF A QUASI-INJECTIVE S-SYSTEM

In addition to the notation of the previous section we let \( K = \text{Hom}_S(M, M) \) and define for \( m \in M \) the mapping \( \lambda_m : S_S \to M_S \) by \( s \to ms \). Let

\[
J(M_S) = \{ m \in M : \lambda_m \text{ is one-to-one only on one element right ideals of } S \}.
\]

**Lemma 3.1.** \( J(M_S) \) is an \( S \)-subsystem of \( M_S \).

**Proof.** It is clear that \( J(M_S) \) is not empty since \( \emptyset \in J(M_S) \). Let \( m \in J(M_S) \) and \( s \in S \), we must show that \( ms \in J(M_S) \). Let \( A \) be a right ideal of \( S \) with more than one element, denoted \( |A| \geq 2 \). Consider the right ideal \( sA \) of \( S \). Either \( sA = 0 \) or \( |sA| \geq 2 \).

Case 1. Suppose \( sA = 0 \) then for all \( a_1 \neq a_2 \in A \), \( sa_1 = sa_2 = 0 \) and so \( m(sa_1) = m(sa_2) = \emptyset \). Consequently \( \lambda_{ms} \) is not one-to-one on \( A \) and thus \( ms \in J(M_S) \).

Case 2. Suppose \( |sA| \geq 2 \) then there exists \( sa_1 \neq sa_2 \in sA \) such that \( m(sa_1) = m(sa_2) \) because \( m \in J(M_S) \). Hence \( \lambda_{ms} \) is not one-to-one on \( A \) and \( ms \in J(M_S) \).

**Lemma 3.2.** \( J(M_S) \) is \( K \)-invariant.

**Proof.** Let \( f \in K \) and \( m \in J(M_S) \). Since \( f \) is an \( S \)-homomorphism then \( f(ms) = f(m) \cdot s = f(\lambda_m(s)) = f \circ \lambda_m(s) \). Suppose \( f(m) \notin J(M_S) \) then \( \lambda_{f(m)} \) is one to one on a right ideal \( R \) of \( S \) with \( |R| \geq 2 \). Since \( m \in J(M_S) \) then there exists \( r_1 \neq r_2 \in R \) such that \( \lambda_m(r_1) = \lambda_m(r_2) \). But then \( f(\lambda_m(r_1)) = f(\lambda_m(r_2)) \) and so \( f \circ \lambda_m(r_1) = f \circ \lambda_m(r_2) \). Thus \( \lambda_{f(m)} \) is not one-to-one on \( R \); a contradiction.

Thus \( J(M_S) \) is a \( (K, S) \)-subsystem of \( M_S \) and when \( M_S \) is quasi-injective, \( J(M_S) \) is also. Furthermore, when \( \psi_M = i \) and \( M_S \) is quasi-injective, \( J(M_S) \) is an \( (H, S) \)-subsystem of \( M_S \). We now define the set

\[
T(M_S) = \{ f \in K : f^{-1}(J(M_S)) \subseteq M_S \}.
\]

Clearly the zero mapping \( \theta \in K \) is in \( T(M_S) \) and \( \{ f \in K : f^{-1}(\emptyset) \subseteq M_S \} \subseteq T(M_S) \).

**Lemma 3.3.** If \( J(M_S) = \{ \emptyset \} \), then

\[
T(M_S) = \{ f \in K : f^{-1}(\emptyset) \subseteq M_S \} = \{ \emptyset \}.
\]

101
Proof. Let $0 \neq f \in T(M_S)$, then there exists $\emptyset \neq m \in M_S$ such that $f(m) \neq \emptyset$. Since $J(M_S) = \{\emptyset\}$ then $f(m) \notin J(M_S)$ so there exists a right ideal $R$ of $S$ with $|R| \geq 2$ such that $\lambda_{f(m)}$ is one-to-one on $R$. Consider now the $S$-subsystem $mR$ and note that $|mR| \geq 2$. Now $f$ is one-to-one on $mR$ and since $f^{-1}(\emptyset) \subseteq M_S$ then $f^{-1}(\emptyset) \cap mR \neq \emptyset$. This is a contradiction since if $x \in f^{-1}(\emptyset) \cap mR, f(x) = \emptyset$ and since $x \in mR$, then $f(x) = f(\emptyset)$ which implies that $x = \emptyset$ since $f$ is one-to-one on $mR$. Hence $T(M_S) = \{\emptyset\}$.

Theorem 3.4. Let $M_S$ be a quasi-injective $S$-system. If $\psi_M = i$ and $J(M_S) = \{\emptyset\}$, then $K = \text{Hom}_S(M, M)$ is regular.

Proof. Let $0 \neq f \in K$, then there exists $\emptyset \neq x \in M_S$ such that $f(x) \neq \emptyset$ and so $f(x) \notin J(M_S)$. Hence there exists a right ideal $R$ of $S$ with $|R| \geq 2$ such that $\lambda_{f(x)}$ is one-to-one on $R$. Hence considering the $S$-subsystem $xR$ we can say that $f$ is one-to-one on $xR$ and $|xR| \geq 2$. By Zorn's Lemma, there is a maximal $S$-subsystem on which $f$ is one-to-one, call it $D_f$. Define the $S$-homomorphism $g : f(D_f) \to D_f$ by $y = f(z) \to z$. Since $M_S$ is quasi-injective then we can extend $g$ to $\bar{g} \in K$ such that $\bar{g} \big|_{f(D_f)} = g$. Let $D_f = f^{-1}(f(D_f))$, then for $t \in D_f, f(t) = f(r)$ for some $r \in D_f$. Hence for $t \in D_f$ we have

$$f \bar{g}(f(t)) = f(\bar{g}(f(t))) = f(\bar{g}(f(r))) = f(r) = f(t).$$

Thus if $D_f \subseteq M_S$ we have by Lemma 2.3 that $f \bar{g}f = f$ on $M_S$. Hence suppose $D_f$ is not an $\bigcap$-large subsystem of $M_S$, then there exists $A_S \subseteq M_S$ such that $|A_S| \geq 2$ and $A_S \cap D_f = \{\emptyset\}$. Let $\emptyset \neq a \in A_S$ such that $f(a) \neq \emptyset$. Then $f(a) \notin J(M_S)$ so there exists a right ideal $Y$ of $S$ such that $|Y| \geq 2$ and $f(a) y_1 \neq f(a) y_2$ for all $y_1 \neq y_2 \in Y$. Hence $f$ is one-to-one on $aY \subseteq M_S$. But $D_f \subseteq D_f$ so $A_S \cap D_f = \{\emptyset\}$ implies $D_f \cap aY = \{\emptyset\}$. Now $D_f \cup aY \supseteq D_f$ so $f$ is not one-to-one on $D_f \cup aY$ by the maximality of $D_f$. Hence there exists $d \in D_f$ and $a y \in aY$ such that $d \neq a y$ but $f(d) = f(ay)$. Thus $a y \in f^{-1}(f(D_f)) = D_f$. But $D_f \cap aY = \emptyset$ since $D_f \cap A_S = \emptyset$ and so $a y = \emptyset$. Thus $f(d) = \emptyset = f(\emptyset)$ and $d = \emptyset$: a contradiction since $a y \neq d$. Thus $D_f \subseteq M_S$ and $K$ is regular.

Corollary 3.5. Let $M_S$ be an $S$-system with $H = \text{Hom}_S(I, I)$ where $I$ is the injective hull of $M_S$. If $\psi_M = i$ and $J(M_S) = \{\emptyset\}$, then $H$ is regular.

Proof. It suffices to show that $J(M_S) = \{\emptyset\}$ implies $J(I) = \{\emptyset\}$. Let $0 \neq t \in J(I)$. Since $M_S \subseteq I$ then $t^{-1}M$ is an $\bigcap$-large right ideal of $S$ and $|t^{-1}M| \geq 2$. Hence there exists $0 \neq s \in S$ such that $0 \neq ts \in M$. We now show that $ts \in J(M_S)$ which gives a contradiction. Let $R$ be any right ideal of $S$ with $|R| \geq 2$. Then either $sR = 0$ or $|sR| \geq 2$.

Case 1. If $sR = 0$ then for $r_1 \neq r_2 \in R, t(s r_1) = t(s r_2)$ so $\lambda_{ts}$ is not one-to-one on $R$. 

102
Case 2. If $|sR| \geq 2$ then there exists $sr_1 \neq sr_2 \in sR$ such that $t(sr_1) = t(sr_2)$ because $t \in J(I)$. Hence once again $\lambda_{ts}$ is not one-to-one on $R$.

Thus in both cases $ts \in J(M_S)$.

The next corollary is similar to a result of M. Botero de Meza [2].

**Corollary 3.6.** Let $S$ be a monoid considered as a right $S$-system with zero over itself, and let $Q(S)$ be the maximal right quotient semigroup of $S$. If $\psi_S = i$ and $J(S) = 0$, then $Q(S)$ is regular.

**Proof.** Corollary 3.5 and [7, Corollary 3.2].

We now link this work with a result of Faith and Utumi [3] by considering the following set:

$$X(K) = \{f \in K : f \text{ is one-to-one only on one element $S$-subsystems of } M_S\}$$

**Lemma 3.7.** $T(M_S) \subseteq X(K)$.

**Proof.** Let $f \in T(M_S)$ then $f^{-1}(J(M_S)) \subseteq M_S$. Let $\emptyset \neq N_S \subseteq M_S$, then $f^{-1}(J(M_S)) \cap N_S \neq \{\emptyset\}$. Let $\emptyset \neq n \in f^{-1}(J(M_S)) \cap N_S$ then $f(n) \in J(M_S)$. Consequently, $\lambda_{f(n)}$ is one-to-one on only one element right ideals of $S$. So there exists $s_1 \neq s_2 \in S$ such that $f(n)s_1 = f(n)s_2$. But then $f$ is not one-to-one on $nS \subseteq N_S$ so $f \in X(K)$.

**Lemma 3.8.** If $J(M_S) = \{\emptyset\}$ then

$$X(K) = T(M_S) = \{f \in K : f^{-1}(\emptyset) \subseteq M_S\} = \{\emptyset\}.$$  

**Proof.** Let $f \in X(K)$ and suppose $f^{-1}(\emptyset)$ is not an $\bigcap$-large subsystem of $M_S$. Then there exists $\emptyset \neq T_S \subseteq M_S$ such that $f^{-1}(\emptyset) \cap T = \{\emptyset\}$; that is, $\{m \in M : f(m) = \emptyset\} \cap T = \{\emptyset\}$. So there exists $\emptyset \neq t \in T$ such that $f(t) \neq \emptyset$ and so $f(t) \notin f^{-1}(J(M_S))$. Furthermore, there exists a right ideal $R$ of $S$ with $|R| \geq 2$ such that $r_1 \neq r_2 \in R$ implies $f(t)r_1 \neq f(t)r_2$. Hence $f$ is one-to-one on $tR \subseteq T \subseteq M$. But this is a contradiction since $|tR| \geq 2$ and $f \in X(K)$. Thus $f^{-1}(\emptyset) \subseteq M_S$ and so $X(K) = \{f \in K : f^{-1}(\emptyset) \subseteq M_S\}$.

**Theorem 3.9.** If $S$ is a ring and $M_S$ is a quasi-injective right $S$-module then

$$X(K) = \{f \in K : \ker f \subseteq M_S\}.$$  

**Proof.** If $f \in X(K)$ but $\ker f = \{m \in M : f(m) = 0\}$ is not $\bigcap$-large in $M_S$ then there exists $\emptyset \neq T_S \subseteq M_S$ such that $\ker f \cap T_S = \{0\}$ so $f$ is one-to-one on $T_S$. This is a contradiction since $f \in X(K)$ so $\ker f \subseteq M_S$.

Faith and Utumi [3] showed that $K \setminus X(K)$ is a regular ring and when $X(K) = \{\emptyset\}$, $K$ is a regular ring. Thus Theorem 3.4 generalizes the second half of Faith and Utumi's result to quasi-injective $S$-system whose singular congruence is the identity congruence.
References


Authors' addresses: Antonio M. Lopez, Jr. Department of Mathematical Sciences, Loyola University, New Orleans, Louisiana 70118, U.S.A.; John K. Luedeman, Department of Mathematical Sciences, Clemson University, Clemson, South Carolina 29631, U.S.A.