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ON SUMMABILITY IN CONVERGENCE GROUPS

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I. In Novák's paper "On some problems concerning convergence space and groups" (see [1]) the following problem is given:

"Is there a sequence of points of a convergence commutative group such that in each subsequence of it there is a subsequence the limit sum of which exists and another subsequence the infinite sum of which does not exist?"

An elegant example of a space containing a sequence whose subsequences have both summable and unsummable subsequences was given by C. RYLL-NARDZEWSKI. The Continuum Hypothesis was essentially used in C. Ryll-Nardzewski's example.

In the last section we give an example of a normed space without using the Continuum Hypothesis.

II. In this section we consider vector measures $m : 2^N \to L$. By an orthogonal measure (=o.m.) we mean a measure $m : 2^N \to L$ which transforms every family of disjoint, nonvoid subsets of N into a system of linearly independent vectors in L.

For each family $\mathscr{A}$ of subsets of N we denote:

$$m(\mathscr{A}) =^{\mathrm{df}} \{m(A) : A \in \mathscr{A}\} ,$$

and

$$L(\mathscr{A}) =^{\mathrm{df}} \mathrm{Lin}\, \{m(A) : A \in \mathscr{A}\} .$$

By $I(A)$ we denote the family $\{B \subset A : B \text{ is an infinite subset}\}$. Let us observe that each o.m. $m : 2^N \to L$ is a monomorphism.

Lemma 1. *Let m be any o.m. and let $\mathscr{A}$ be a finite family of disjoint nonvoid subsets of N. Then for each set $A \subset N$, the set*

$$m(I(A)) \cap L(\mathscr{A})$$

is finite.

Proof. Let $\mathscr{A} = \{A_1, \ldots, A_n\}$. Moreover, we assume that $B \subset A$, and

$$m(B) = a_1\, m(A_1) + \ldots + a_n\, m(A_n) . \tag{1}$$

By the transformation (1) we obtain

$$(2) \qquad \sum_i (a_i - 1)\, m(A_i \cap B) + \sum_i a_i\, m(A_i \setminus B) - m(B \setminus \bigcup_i A_i) = 0$$

where all sets are disjoint.

The orthogonality of the measure m together with (2) implies $\bigcup_i A_i \supset B$. Further, for each $i = 1, \ldots, n$ at least one of the sets $A_i \cap B$ or $A_i \setminus B$ is nonvoide. Hence the coefficient a_i is 0 or 1, and the number of elements of the set $m(I(A)) \cap L(\mathscr{A})$ is equal to the number of all n-element sequences of 0, 1. This proves Lemma 1.

Lemma 2. *Let m be any o.m. and let E be a subspace of L of infinite algebraic dimension. Then for each infinite set $A \subset N$ we have* $\operatorname{card} m(I(A)) \cap E \leqq \dim E$.

Proof. First we consider the case when $F \subset L$ and F is a linear space of a finite dimension. Let $u_1, \ldots, u_n$ be one of the largest collection of linearly independent vectors in $F \cap m(I(A))$.

By $\mathscr{I}$ we denote the family of all atoms of a ring generated by $m^{-1}(u_1), \ldots, m^{-1}(u_n)$. It is clear that

$$F \cap m(I(A)) \subset L(\mathscr{I}).$$

Hence Lemma 1 implies that the set $F \cap m(I(A))$ is finite. By the representation $E = \bigcup_{\mathscr{F}} F$, where $\mathscr{F}$ is the class of all finite dimensional spaces generated by a fixed basis of E, we have

$$(3) \qquad m(I(A)) \cap E = \bigcup_{\mathscr{F}} m(I(A)) \cap F$$

i.e. the set $m(I(A)) \cap E$ is a union of finite sets. The equality

$$\operatorname{card} \mathscr{F} = \dim E$$

together with (3) yields the assertion of Lemma 2.

Let $\mathscr{A}$ be a family of subsets of N. We say that a linear space $E \subset L$ is an *m-dissection* of $\mathscr{A}$ iff for each $A \in \mathscr{A}$ there are vectors $u \in E$, $v \notin E$ such that

$$m^{-1}(u), m^{-1}(v) \subset' A.$$

Theorem. *Let $m : 2^N \to L$ be an orthogonal measure, and let $\mathscr{N}$ be the collection of all infinite subsets of N. Then there exists a subspace $E \subset L$ which is an m-dissection of $\mathscr{N}$.*

Proof. Let ω denote the smallest ordinal number of a power of continuum, and let $\{A_\alpha\}_{\alpha < \omega}$ be a transfinite sequence of all members of $\mathscr{N}$. We use transfinite induction to define two increasing sequences $\{E^i_\alpha\}_{\alpha>\omega}$ $(i = 1, 2)$ of linear subspaces of L such that:

(i) $E^1_\alpha \cap E^2_\alpha = \{0\}$,

(ii) $\dim (E^1_\alpha \oplus E^2_\alpha) \leqq \aleph_0 + \operatorname{card} \alpha$,

(iii) for $i = 1, 2$ and for $\alpha > 0$ there exists $u^i_\alpha \in E^i_\alpha$ such that $m^{-1}(u^i_\alpha) \subset A_\alpha$.

Define E^1_0 as the space Lin $\{m(A) : A$ is a finite set$\}$, and $E^2_0 = \{0\}$.

Suppose that $0 < \alpha < \omega$ and that E^i_β, $i = 1, 2$, have been defined for each β, $0 \leqq \beta < \alpha$.

Since m is a monomorphism, Lemma 2 implies that we can choose two linearly independent vectors u^i_α $(i = 1, 2)$ in $m(I(A_\alpha))$, such that Lin $(u^1_\alpha, u^2_\alpha) \cap \bigcup_{\beta<\alpha} E^1_\beta \oplus E^2_\beta = 0$. Let us define E^i_α as $\bigcup_{\beta<\alpha} E^i_\beta \oplus \operatorname{Lin}(u^i_\alpha)$ $(i = 1, 2)$ and assume that $E = \bigcup_{\alpha<\omega} E^i_\alpha$. It is clear that E satisfies the assertion of the theorem.

III. By μ we denote the orthogonal measure from 2^N into the Hilbert space $(l^2, \| \|_2)$, such that

$$\mu(A) = \sum_{n\in A} \frac{1}{n} e_n$$

where $(e_n)_{n=1,2,\ldots}$ is an orthonormal basis of l^2. Moreover, let $E \subset l^2$ be a μ-dissection of $\mathcal{N}$. Then, by virtue of Theorem, it is easy to see that each subsequence of a sequence $a_n = (1/n)\, e_n$ contains a subsequence which is summable in $(E, \| \|_2)$, and a subsequence which is unsummable in $(E, \| \|_2)$.

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References

[1] General Topology and Its Relations to Modern Analysis and Algebra. Proceedings of the Kanpur Topological Conference, 1968. Academia, Publishing House of the Czechoslovak Academy of Sciences. Prague 1971.

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