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A CLASS OF SETS WHOSE DISTANCE SET FILLS AN INTERVAL

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1. INTRODUCTION

Let Λ be a class of linear sets. For sets A and B in Λ, let $\varrho(A, B) = \inf\{|a - b| : a \in A, b \in B\}$. We will denote the *diameter of* Λ by

$$\delta(\Lambda) = \sup\{\varrho(A, B) : A, B \in \Lambda\},$$

and the *distance set of* Λ will be denoted by $D(\Lambda) = \{\varrho(A, B) : A, B \in \Lambda\}$.

The following question was raised in [1]:

If $2^{\aleph_0} = \aleph_1$, *does there exist a class* Λ *of pairwise disjoint linear sets such that* $D(\Lambda)$ *fills an interval with left endpoint zero and length* $\delta(\Lambda)$?

We note that the answer is trivially affirmative by considering the class $\Lambda = \{\{x\} : x \in [0. 1]\}$. The author of [1] attempted to verify that such a class did indeed exist with the additional requirement that each member of the class have cardinality $\mathfrak{c}$. Here we present a result which demonstrates that the class Λ constructed in [1] does not meet the specified requirements, and then present a class Λ of linear sets which asserts the validity of her theorem.

2. THE NUMBER OF DISJOINT TRANSLATES OF THE CANTOR TERNARY SET

The construction of the class Λ in [1] is based upon the existence of a continuum number of pairwise disjoint translates of the Cantor ternary set C, but the number of such translates is at most countable as is seen in the following theorem.

Theorem. *There are at most a countable number of disjoint linear translates of* C.

Proof. If there did indeed exist $\mathfrak{c}$ disjoint translates of C, then, since the diameter of C is 1, there must be two of these disjoint translates $C + a$ and $C + b$ such that $a < b < a + 1$. Since the distance set of C, $D(C) = \{|x - y| : x, y \in C\}$, is the interval $[0, 1]$, it follows that $D(C + a) = [0, 1]$. Thus $(C + a) \cap (C + z) \neq \emptyset$ for any z in the interval $[a, a + 1]$, and the theorem is established by contradiction.

3. THE CONSTRUCTION OF Λ

Here we construct a class of linear sets which satisfies the properties stated in § 1. We begin by letting F denote the set of all decimal fractions in the interval $[0, 1]$ which may be expressed without using the digit 5.

Note 1. It is well known that F is a perfect nowhere dense set of measure zero.

Note 2. The distance set of F is the interval $[0, 1]$; furthermore, it is easily seen that for each distance $0 < d < 1$, there exist $\mathfrak{c}$ pairs of elements a and b in F such that $a - b = d$.

Given $x \in F$, $0 < x < 1$, we will denote a set constructed in the following manner by A_x: choose from the complement of F a sequence of intervals $I_n \downarrow x$, then choose a perfect set P_n of measure zero in each I_n; we will denote $\bigcup P_n \cup \{x\}$ by A_x. If instead we select a sequence of intervals $I_n \uparrow x$ contiguous to F and proceed as above, the resulting set will be denoted by B_x. Note that A_x and B_x are perfect sets of measure zero.

Assuming that $2^{\aleph_0} = \aleph_1$ (or more generally that the union of less than $\mathfrak{c}$ sets of measure zero is of measure zero), we describe the sets comprising the class Λ by transfinite induction. Let Ω denote the least ordinal of cardinality $\mathfrak{c}$, and let $d_0, d_1, \ldots$ $\ldots, d_\alpha, \ldots$ $(\alpha < \Omega)$ be a well ordering of the interval $(0, 1)$. Given d_0, choose $a_0, b_0 \in F$, $0 < b_0 < a_0 < 1$, such that $a_0 - b_0 = d_0$, and also choose sets A_{a_0} and B_{b_0} as described above.

Suppose that the points $a_\gamma, b_\gamma \in F$ and that the sets A_{a_γ} and B_{b_γ} have been chosen for each $\gamma < \alpha$ such that $a_\gamma - b_\gamma = d_\gamma$ and such that the collection of points $K = \{a_\gamma\} \cup \{b_\gamma\}$ consists of distinct points, and the collection of sets $L = \{A_{a_\gamma}\} \cup \{B_{b_\gamma}\}$ consists of mutually disjoint sets. Given d_α, by Note 2 there exist $a_\alpha, b_\alpha \in F$, $0 < b_\alpha < a_\alpha < 1$, such that $a_\alpha - b_\alpha = d_\alpha$, and since K contains less than $\mathfrak{c}$ points of F, we may choose $a_\alpha, b_\alpha \notin K$. In each interval contiguous to F less than $\mathfrak{c}$ perfect sets of measure zero have been selected in constructing the collection L, therefore we may choose sets A_{a_α} and B_{b_α} as described above to be mutually disjoint from the members of L. Consequently the class $\Lambda = \{A_{a_\alpha}\} \cup \{B_{b_\alpha}\}$ is a collection of $\mathfrak{c}$ mutually disjoint linear sets of cardinality $\mathfrak{c}$.

Note that $\delta(\Lambda) = 1$, and that for any $0 < d_\alpha < 1$, $\varrho(A_{a_\alpha}, B_{b_\alpha}) = d_\alpha$. Hence the class Λ satisfies the required properties.

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