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THE COEFFICIENT RING OF THE SKEW GROUP RING

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We let $R$ be an associative ring with an identity (unless explicitly stated otherwise). We let $G$ be a finite group of automorphisms of $R$. We consider two rings associated with $R$ and $G$. The first is the fixed ring of $R$ under $G$, $R^G = \{ r \in R \mid r^g = r \text{ for all } g \in G \}$. The second is the skew group ring or the crossed product, $R \ast G$, which as a left $R$ module is free with basis $\{ u_g \mid g \in G \}$ and $u_gr = r^gu_g$. Now $R$ can be viewed as a left $R \ast G$ module by defining $\sum_g x_g u_g r = \sum_g x_g r^g, x_g, r \in R$. We call a left $R \ast G$ submodule of $R$ a $G$-invariant left ideal of $R$. By the trace of $R$, $t(R)$, we mean the collection of all elements of $R^G$ of the form $\sum_g r^g, r \in R$. $t(R)$ is a two-sided ideal of $R^G$. Finally, the map that associates $\sum_g x_g u_g$ in $R \ast G$ to the right $R^G$ homomorphism $f(r) = \sum_g x_g r^g, r \in R$ is a ring homomorphism from $R \ast G$ to $\text{End}(R^G)$.

Now $f : R \ast G \rightarrow R$, $f(\sum_g x_g u_g) = \sum_g x_g$ is a left $R \ast G$, right $R$ map. Unlike the group ring $f$ is not a ring map, but $R$ is a left $R \ast G$ homomorphic image of $R \ast G$. Also the map from $R$ to $R \ast G$ that sends $r$ to $r(u_1 + u_g + \ldots + u_n)$ is a left $R \ast G$ map. So $R$ is a left $R \ast G$ submodule of $R \ast G$.

In [4, Theorem 2.8], J. FISHER and J. OSTERBURG showed that if $R^G$ has the ACC on semiprime ideals, then so does $R$, as long as $|G|$ is invertible in $R$.

**Theorem 1.** Assume that $G$ is a finite abelian group such that the order of $G$ is invertible in $R$. If $R \ast G$ satisfies the ACC on semiprime ideals, then $R$ satisfies the ACC on semiprime ideals.

**Proof.** Let $A_1 \subseteq A_2, \ldots \subseteq A_i$ be an ascending chain of $G$-invariant semiprime ideals of $R$. Then $(R \ast G)A_1 = A_1(R \ast G) \subseteq (R \ast G)A_2 = A_2(R \ast G) \cdots \subseteq (R \ast G)A_i = A_i(R \ast G)$ is an ascending chain of two-sided ideals of $R \ast G$. Now $(R \ast G)A_i, \text{ for } i = 1, 2, \ldots,$ is a semiprime ideal of $R \ast G$. Since $A_i$ is $G$-invariant for each $i$, $G$ acts on $R/A_i$. In fact, the map from $R \ast G$ to $(R/A_i) \ast G$ that associates $r_g u_g$ to $(r_g + A_i)u_g$ is an epimorphism with kernel $A_i(R \ast G)$. Now we have $(R/A_i) \ast G = R \ast G/(R \ast G) A_i$. Since $G$ is abelian and $R/A_i$ is semiprime with no order of $G$ torsion, we use [8, Proposition 3.3] to conclude that $(R \ast G)A_i$ is a semiprime ideal of $R \ast G$. 


By the hypothesis of the theorem, we conclude that the chain of ideals in $R \ast G$ terminates; hence, we have shown that every chain of $G$-invariant semiprime ideals of $R$ terminates. Using a result of Joe W. Fisher in [6], we conclude that this implies the ACC on semiprime ideals in $R$.

The next result is true even if there is order of $G$ torsion, i.e., there is $r \neq 0$ in $R$ such that $|G| r = 0$.

It is shown in [6] that if $R \ast G$ is Artinian or Noetherian, then $R$ is Artinian or Noetherian (respectively). The if part of the following theorem is due to D. Handelman, J. Lawrence, W. Schelter [8, Theorem 3.5c]. Our proof is slightly different.

**Theorem 2.** Assume that $R$ has no $|G|$-torsion. Then $R \ast G$ is a semiprime Goldie ring if and only if $R$ is a semiprime Goldie ring. Moreover, if the quotient ring of $R$ is $Q$, then the quotient ring of $R \ast G$ is $Q \ast G$, the skew group ring of $G$ with $Q$.

**Proof.** Assume $R$ is semiprime Goldie and $Q$ is the quotient ring. Since $|G|$ is regular in $R$, it is invertible in $Q$. The action of $G$ in $R$ can be extended to $Q$ by taking $(a^{-1}b)^g = (a^g)^{-1} b^g$. It is easy to see that $R \ast G$ is an order in $Q \ast G$.

Since $Q \ast G$ is f.g. over $Q$, it is Artinian. All we need to do is show that the Jacobson radical of $Q \ast G$ is 0. This follows from the fact that $|G|$ is invertible in $Q$, so $Q \ast G$ and $Q$ form a projective pair [5, Theorem 3, p. 99]. In this case, the Jacobson radical of $Q \ast G$ is zero by [12, Theorem 16.3, p. 65]. Thus $R \ast G$ is an order in a semisimple ring; hence, $R$ is semiprime Goldie.

Now to the converse. We show first that $R$ is semiprime, if $R \ast G$ is semiprime. Let $I$ be an ideal of $R$ such that $I^2 = 0$. Let $A = I + I^g + \ldots + I^h$, $G = \{1, g, \ldots, h\}$, then $A$ is $G$-invariant and $AR \ast G$ is an ideal of $R \ast G$. It is easy to see that a power of this ideal is 0. So $I = 0$.

If $R \ast G$ is semiprime Goldie, then $R$ when viewed as a subring of $R \ast G$ inherits the ACC on left annihilators. By considering $R$ as a left $R \ast G$ submodule of $R \ast G$, we see that $R$ has finite Goldie dimension as an $R \ast G$ module. By [4, Corollary 1.3], we conclude $R$ is Goldie.

For each $g$ in $G$, we let $C_g = \{r \in R \mid rx = x^g r \text{ for all } x \text{ in } R\}$. Now $C_1$ is the center of $R$ and each $C_g$ is a module over $C_1$. We say $g$ is inner, if $C_g$ contains a regular element, $r$. Note $rx = x^g r$ is the left common multiple property. Thus if $C_g$ contains a regular element we can form a classical quotient ring that contains $r^{-1}$. In this quotient ring $x^g = r x r^{-1}$. We call an automorphism outer, if it is not inner. $G$ is called outer, if every automorphism, except the identity, is outer. In our next result, we allow $G$ torsion.

**Theorem 3.** Let $R$ be a prime Goldie ring and $G$ and outer group of automorphisms of $R$. Then $R \ast G$ is a prime Goldie ring.

**Proof.** Put $Q$ equal to the quotient ring of $R$. As usual, we extend the action of $G$ to $Q$. Since regular elements of $Q$ are invertible in $Q$, $G$ remains outer as a group of
automorphisms of $Q$. By [8, Proposition 1.1], the skew group ring of $Q$ with $G$, $Q \ast G$ is simple. Thus $R \ast G$ is an order of $Q \ast G$, a simple Artinian ring; hence, $R \ast G$ is prime Goldie.

The following example shows that the converse is not quite true. Let $R = Z \times Z$, $Z$ the integers, a semiprime Goldie ring with quotient ring $T = Q \times Q$, $Q$ the rationals. Let $g(a, b) = (b, a)$ and $G = \langle g \rangle$. Now $T \ast G$ is simple Artinian, hence $R \ast G$ is prime Goldie, but $R$ is not prime.

In [9, p. 350], V. K. Kharchenko defined the notion of $G$-prime, if $A, B$ are $G$-invariant ideals of $R$ such that $AB = 0$, then $A = 0$ or $B = 0$. Furthermore, $R$ is $G$-prime if and only if $\bigcap_{g \in G} P^g = 0$, $P$ a prime ideal of $R$. We note that $R$ is $G$-prime means $R$ is a subdirect sum of $G$ isomorphic prime rings. See [9, Lemma 1, p. 450].

**Theorem 4.** Let $R \ast G$ be a prime Goldie ring, then $R$ is a $G$-prime Goldie ring. So $R$ is semiprime Goldie.

**Proof.** Just as the proof of Theorem 3.

The left Krull dimension of $R$ we denote by $K \dim R$. The reader should consult [7] for all of the relevant facts concerning Krull dimension.

**Theorem 5.** Assume $|G|$ is invertible in $R$. Then $R$ is semiprime with Krull dimension if and only if $R \ast G$ is semiprime with Krull dimension.

**Proof.** (only if) By [7, Corollary 3.4, p. 20] $R$ is semiprime Goldie. Thus by Theorem 2 $R \ast G$ is semiprime. Since $R \ast G$ has Krull dimension as a left $R$ module, it has Krull dimension as a left $R \ast G$ module.

(if) Clearly as a left $R \ast G$ module $R$ has Krull dimension. Since $|G|$ is invertible in $R$, we conclude that the fixed ring has Krull dimension by [5, Theorem 2.2, p. 104].

Now, D. Farkas and R. Snider show in [3] that $R$ is a submodule of a f.g. $R^G$ module. Hence, if $R^G$ has Krull dimension so does $R$.

We now consider left perfect rings. These are rings such that modulo the Jacobson radical, $J(R)$, they are Artinian. Also $J(R)$ is left $T$-nilpotent. We will use the following characterization of an ideal $A$, being left $T$-nilpotent, for any left $R$ module $M \neq 0$, $AM$ is a proper submodule of $M$. See [1, Lemma 28.3, p. 314].

**Theorem 6.** Assume $R$ has no $|G|$-torsion. Then $R$ is left perfect if and only if $R \ast G$ is left perfect.

**Proof.** It is well-known that left perfect rings have the DCC on principal right ideals. Thus $|G|$ is invertible in $R$. Each automorphism of $R$, $g$, induces an automorphism on $\bar{R} = R/J(R)$ as follows, $g(r + J(R)) = r^g + J(R)$. We denote this map by $\bar{g}$. The association $g$ to $\bar{g}$ is a group homomorphism from $G$ to the group automorphism of $\bar{R}$. Let $H$ be the kernel of this map and $\bar{G} = G/H$. We form $\bar{R} \ast \bar{G}$, which is a homomorphic image of $R \ast G$. Namely, apply the map $\bar{\ast} : R \to \bar{R}$ to the
coefficients of $R \ast G$. The kernel of this homomorphism is $J(R) R \ast G$, but by [11, Theorem 16.3, p. 65], $J(R \ast G)$ is the kernel. Thus we have $R \ast G/J(R \ast G)$ is Artinian.

We now consider $T$-nilpotence. To this end let $M$ be an arbitrary left $R \ast G$ module. Now $J(R \ast G) M$ is $J(R) M$ from the above, and $J(R) M$ is a proper submodule, since $J(R)$ is $T$-nilpotent. Hence $J(R \ast G)$ is left $T$-nilpotent and we have shown $R \ast G$ is left perfect, if $R$ is left perfect. The converse follows from $J(R)$ is contained in $J(R \ast G)$.

References


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