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THE COEFFICIENT RING OF THE SKEW GROUP RING

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We let R be an associative ring with an identity (unless explicitly stated otherwise). We let G be a finite group of automorphisms of R. We consider two rings associated with R and G. The first is the fixed ring of R under G, $R^G = \{r \text{ in } R \mid r^g = r \text{ for all } g \text{ in } G\}$. The second is the skew group ring or the crossed product, $R * G$, which as a left R module is free with basis $\{u_g \mid g \in G\}$ and $u_g r = r^g u_g$. Now R can be viewed as a left $R * G$ module by defining $\sum_G x_g u_g r = \sum_G x_g r^g$, x_g, r in R. We call a left $R * G$ submodule of R a G-invariant left ideal of R. By the trace of R, $t(R)$, we mean the collection of all elements of R^G of the form $\sum_G r^g$, r in R. $t(R)$ is a two-sided ideal of R^G. Finally, the map that associates $\sum_G x_g u_g$ in $R * G$ to the right R^G homomorphism $f(r) = \sum_G x_g r^g$, r in R is a ring homomorphism from $R * G$ to End (R_{R^G}).

Now $f : R * G \to R$, $f(\sum x_g u_g) = \sum x_g$ is a left $R * G$, right R map. Unlike the group ring f is not a ring map, but R is a left $R * G$ homomorphic image of $R * G$. Also the map from R to $R * G$ that sends r to $r(u_1 + u_g + \ldots + u_h)$ is a left $R * G$ map. So R is a left $R * G$ submodule of $R * G$.

In [4, Theorem 2.8], J. FISHER and J. OSTERBURG showed that if R^G has the ACC on semiprime ideals, then so does R, as long as $|G|$ is invertible in R.

Theorem 1. *Assume that G is a finite abelian group such that the order of G is invertible in R. If $R * G$ satisfies the ACC on semiprime ideals, then R satisfies the ACC on semiprime ideals.*

Proof. Let $A_1 \subseteq A_2, \ldots \subseteq A_i$ be an ascending chain of G-invariant semiprime ideals of R. Then $(R * G) A_1 = A_1(R * G) \subseteq (R * G) A_2 = A_2(R * G) \ldots \ldots \subseteq (R * G) A_i = A_i(R * G)$ is an ascending chain of two-sided ideals of $R * G$. Now $(R * G) A_i$, for $i = 1, 2, \ldots$, is a semiprime ideal of $R * G$. Since A_i is G-invariant for each i, G acts on R/A_i. In fact, the map from $R * G$ to $(R/A_i) * G$ that associates $r_g u_g$ to $(r_g + A_i) u_g$ is an epimorphism with kernel $A_i(R * G)$. Now we have $(R/A_i) * * G = R * G/(R * G) A_i$. Since G is abelian and R/A_i is semiprime with no order of G torsion, we use [8, Proposition 3.3] to conclude that $(R * G) A_i$ is a semiprime ideal of $R * G$.

By the hypothesis of the theorem, we conclude that the chain of ideals in $R * G$ terminates; hence, we have shown that every chain of G-invariant semiprime ideals of R terminates. Using a result of Joe W. Fisher in [6], we conclude that this implies the ACC on semiprime ideals in R.

The next result is true even if there is order of G torsion, i.e., there is $r \neq 0$ in R such that $|G|\, r = 0$.

It is shown in [6] that if $R * G$ is Artinian or Noetherian, then R is Artinian or Noetherian (respectively). The if part of the following theorem is due to D. HANDELMAN, J. LAWRENCE, W. SCHELTER [8, Theorem 3.5c]. Our proof is slightly different.

Theorem 2. *Assume that R has no $|G|$-torsion. Then $R * G$ is a semiprime Goldie ring if and only if R is a semiprime Goldie ring. Moreover, if the quotient ring of R is Q, then the quotient ring of $R * G$ is $Q * G$, the skew group ring of G with Q.*

Proof. Assume R is semiprime Goldie and Q is the quotient ring. Since $|G|$ is regular in R, it is invertible in Q. The action of G in R can be extended to Q by taking $(a^{-1}b)^g = (a^g)^{-1}\, b^g$. It is easy to see that $R * G$ is an order in $Q * G$.

Since $Q * G$ is f.g. over Q, it is Artinian. All we need to do is show that the Jacobson radical of $Q * G$ is 0. This follows from the fact that $|G|$ is invertible in Q, so $Q * G$ and Q form a projective pair [5, Theorem 3, p. 99]. In this case, the Jacobson radical of $Q * G$ is zero by [12, Theorem 16.3, p. 65]. Thus $R * G$ is an order in a semisimple ring; hence, R is semiprime Goldie.

Now to the converse. We show first that R is semiprime, if $R * G$ is semiprime. Let I be an ideal of R such that $I^2 = 0$. Let $A = I + I^g + \ldots + I^h$, $G = \{1, g, \ldots, h\}$, then A is G-invariant and $AR * G$ is an ideal of $R * G$. It is easy to see that a power of this ideal is 0. So $I = 0$.

If $R * G$ is semiprime Goldie, then R when viewed as a subring of $R * G$ inherits the ACC on left annihilators. By considering R as a left $R * G$ submodule of $R * G$, we see that R has finite Goldie dimension as an $R * G$ module. By [4, Corollary 1.3], we conclude R is Goldie.

For each g in G, we let $C_g = \{r \in R \mid rx = x^g r \text{ for all } x \text{ in } R\}$. Now C_1 is the center of R and each C_g is a module over C_1. We say g is *inner*, if C_g contains a regular element, r. Note $rx = x^g r$ is the left common multiple property. Thus if C_g contains a regular element we can form a classical quotient ring that contains r^{-1}. In this quotient ring $x^g = rxr^{-1}$. We call an automorphism *outer*, if it is not inner. G is called outer, if every automorphism, except the identity, is outer. In our next result, we allow G torsion.

Theorem 3. *Let R be a prime Goldie ring and G and outer group of automorphisms of R. Then $R * G$ is a prime Goldie ring.*

Proof. Put Q equal to the quotient ring of R. As usual, we extend the action of G to Q. Since regular elements of Q are invertible in Q, G remains outer as a group of automorphisms of Q. By [8, Proposition 1.1], the skew group ring of Q with G, $Q * G$ is simple. Thus $R * G$ is an order of $Q * G$, a simple Artinian ring; hence, $R * G$ is prime Goldie.

The following example shows that the converse is not quite true. Let $R = Z \times Z$, Z the integers, a semiprime Goldie ring with quotient ring $T = Q \times Q$, Q the rationals. Let $g(a, b) = (b, a)$ and $G = \langle g \rangle$. Now $T * G$ is simple Artinian, hence $R * G$ is prime Goldie, but R is not prime.

In [9, p. 350], V. K. KHARCHENKO defined the notion of G-prime, if A, B are G-invariant ideals of R such that $AB = 0$, then $A = 0$ or $B = 0$. Furthermore, R is G-prime if and only if $\bigcap_G P^g = 0$, P a prime ideal of R. We note that R is G-prime means R is a subdirect sum of G isomorphic prime rings. See [9, Lemma 1, p. 450].

Theorem 4. *Let $R * G$ be a prime Goldie ring, then R is a G-prime Goldie ring. So R is semiprime Goldie.*

Proof. Just as the proof of Theorem 3.

The *left Krull dimension of* R we denote by $K \dim R$. The reader should consult [7] for all of the relevant facts concerning Krull dimension.

Theorem 5. *Assume $|G|$ is invertible in R. Then R is semiprime with Krull dimension if and only if $R * G$ is semiprime with Krull dimension.*

Proof. (only if) By [7, Corollary 3.4, p. 20] R is semiprime Goldie. Thus by Theorem 2 $R * G$ is semiprime. Since $R * G$ has Krull dimension as a left R module, it has Krull dimension as a left $R * G$ module.

(if) Clearly as a left $R * G$ module R has Krull dimension. Since $|G|$ is invertible in R, we conclude that the fixed ring has Krull dimension by [5, Theorem 2.2, p. 104]. Now, D. FARKAS and R. SNIDER show in [3] that R is a submodule of a f.g. R^G module. Hence, if R^G has Krull dimension so does R.

We now consider left perfect rings. These are rings such that modulo the Jacobson radical, $J(R)$, they are Artinian. Also $J(R)$ is left T-nilpotent. We will use the following characterization of an ideal A, being left T-nilpotent, for any left R module $M \neq 0$, AM is a proper submodule of M. See [1, Lemma 28.3, p. 314].

Theorem 6. *Assume R has no $|G|$-torsion. Then R is left perfect if and only if $R * G$ is left perfect.*

Proof. It is well-known that left perfect rings have the DCC on principal right ideals. Thus $|G|$ is invertible in R. Each automorphism of R, g, induces an automorphism on $\bar{R} = R/J(R)$ as follows, $g(r + J(R)) = r^g + J(R)$. We denote this map by $\bar{g}$. The association g to $\bar{g}$ is a group homomorphism from G to the group automorphism of $\bar{R}$. Let H be the kernel of this map and $\bar{G} = G/H$. We form $\bar{R} * \bar{G}$, which is a homomorphic image of $R * G$. Namely, apply the map $^{-}: R \to \bar{R}$ to the coefficients of $R * G$. The kernel of this homomorphism is $J(R)\, R * G$, but by [11, Theorem 16.3, p. 65], $J(R * G)$ is the kernel. Thus we have $R * G/J(R * G)$ is Artinian.

We now consider T-nilpotence. To this end let M be an arbitrary left $R * G$ module. Now $J(R * G)\, M$ is $J(R)\, M$ from the above, and $J(R)\, M$ is a proper submodule, since $J(R)$ is T-nilpotent. Hence $J(R * G)$ is left T-nilpotent and we have shown $R * G$ is left perfect, if R is left perfect. The converse follows from $J(R)$ is contained in $J(R * G)$.

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