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ON THE DENSITY OF CERTAIN SETS IN ARITHMETICAL SEMIGROUPS

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Dedicated to the memory of Professor Pál Turán

In [4] RÉNYI investigated the number-theoretical function

$$\Delta(n) = \Omega(n) - \omega(n),$$

where $\omega(n)$ denotes the number of distinct prime factors and $\Omega(n)$ the total number of prime factors of n. He showed that the generating function of the sequence of densities d_k of those integers n for which $\Delta(n) = k$ is given by the following identity:

$$\sum_{k=0}^{\infty} d_k z^k = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p - z}\right), \quad |z| < 2$$

(in the product, p runs over the primes). Rényi's method does not give an estimate for the number of integers n for which $\Delta(n) = k$ with a remainder term. Later COHEN [1] refined Rényi's result for d_1 $(=(6/\pi^2)\sum_p 1/p(p+1))$ proving that

$$\sum_{\substack{n \leq x \\ \Delta(n)=1}} 1 = d_1 x + O(x^{1/2} \operatorname{loglog} x),$$

while DELANGE [2] proved that in general we have

$$\sum_{\substack{n \leq x \\ \Delta(n)=k}} 1 = d_k x + o(x^{1/2}(\operatorname{loglog} x)^k).$$

Delange's proof is based on the fact that the Riemann zeta function has no zero with the real part equal to 1 and on theorems of Tauberian type.

The sequence of integers n for which $\Delta(n) = 1$ is a special case of sequences of integers of the form

$$p_1^k p_2^k \dots p_r^k \cdot e \tag{1}$$

where e runs over all s-free integers and $\{p_1, \dots, p_r\}$ over all the r-tuples of distinct rational primes with the g.c.d. $(p_1 \dots p_r, e) = 1$ provided $k \geqq s \geqq 2$, $r \geqq 1$ are fixed throughout.

In the present note we are going to extend Cohen's result to sequences of the form (1), however, with the variation that $e, p_1, \dots, p_r$ need not be necessarily rational integers but elements of a given abstract arithmetical semigroup. Thus, in particular, our result yields the analogue of Cohen's result for the generalized Rényi's formula as stated in Proposition 5.3.6 of [3]. To our best knowledge similar extensions of Delange's result have remained an open problem for the present.

As to our proof technique, we replace Cohen's counting argument based on the properties of Möbius function by another one which, perhaps, will simplify the scheme of our calculations. Unless otherwise stated, the terminology and notation of [3] will be used. Nevertheless, for the convenience of the reader we repeat here two basic definitions.

Let G be a free commutative semigroup (written multiplicatively) with identity element 1. Suppose that G has a countable set P of generators – called the *primes*. Such a semigroup G will be called an *arithmetical semigroup* if in addition there exists a real – valued norm mapping $|\ |$ on G such that

(i) $|ab| = |a| \cdot |b|$ for all a, b in G,

(ii) the total number $N_G(x)$ of elements $a \in G$ with norm $|a| \leqq x$ is finite for each real $x > 0$.

Moreover, in what follows we shall assume tacitly that the arithmetical semigroups G under consideration satisfy Knopfmacher's Axiom A.

Axiom A. *There exist positive constants A and δ, and a constant η with $0 \leqq \eta < \delta$ such that*

$$N_G(x) = Ax^\delta + O(x^\eta) \quad as \quad x \to \infty .$$

Let G be an arithmetical semigroup and $k \geqq s \geqq 2$, $r \geqq 1$ integers. Define $K_{k,r,s}$ as the set of elements of the form (1), where e is s-free in G, $(p_1 \dots p_r, e) = 1$ and $\{p_1, \dots, p_r\}$ runs over all r-tuples of distinct primes $p_i \in P$ in G. Let, as usual,

$$K_{k,r,s}(x) = \sum_{\substack{|n| \leqq x \\ n \in K_{k,r,s}}} 1 .$$

Finally, for the purpose of our main result define

$$\alpha(k, r, s) = \sum_{\{p_1, \dots, p_r\}} \prod_{i=1}^{r} \frac{|p_i|^\delta - 1}{(|p_i|^{s\delta} - 1) \cdot |p_i|^{\delta(k-s+1)}} . \tag{2}$$

Theorem. *Let G be an arithmetical semigroup satisfying Axiom* A. *If $k = s$, then*

$$K_{k,r,k}(x) = \frac{A \,.\, \alpha(k,r,k)}{\xi_G(k\delta)} x^\delta + \begin{cases} O(x^{\delta/k}(\operatorname{loglog} x)^r) & \text{if } \eta < \delta/k\,, \\ O(x^{\delta/k} \log x \,.\, (\operatorname{loglog} x)^r) & \text{if } \eta = \delta/k\,, \\ O(x^\eta) & \text{if } \eta > \delta/k\,. \end{cases}$$

If $k > s$, then

$$K_{k,r,s}(x) = \frac{A \,.\, \alpha(k,r,s)}{\xi_G(k\delta)} x^\delta + \begin{cases} O(x^{\delta/s}) & \text{if } \eta < \delta/s\,, \\ O(x^{\delta/s} \log x) & \text{if } \eta = \delta/s\,, \\ O(x^\eta) & \text{if } \eta > \delta/s\,. \end{cases}$$

Proof. Given an r-tuple $\{p_1, \ldots, p_r\}$ of distinct primes, denote by $K_{k,r,s}^{\{p_1,\ldots,p_r\}}$ the set of those elements of $K_{k,r,s}$ which are divisible by $(p_1 \ldots p_r)^k$. Since every element of $K_{k,r,s}$ is uniquely representable in the form (1), we have

$$(3) \qquad K_{k,r,s}(x) = \sum_{\substack{\{p_1,\ldots,p_r\} \\ |p_1 \ldots p_r| \leqq x^{1/k}}} K_{k,r,s}^{\{p_1,\ldots,p_r\}}(x)\,.$$

However, $K_{k,r,s}^{\{p_1,\ldots,p_r\}}(x)$ equals the number of s-free elements with norms $\leqq x/|p_1 \ldots p_r|^k$ in the semigroup $G\langle p_1 \ldots p_r\rangle$ of those elements of G which are coprime to $p_1 \ldots p_r$. It follows from the proof of Proposition 4.1.3 of [3] that $G\langle p_1 \ldots p_r\rangle$ satisfies Axiom A with

$$(4) \qquad N_{G\langle p_1 \ldots p_r\rangle}(x) = A \prod_{i=1}^{r} (1 - |p_i|^{-\delta})\, x^\delta + O\Big(\prod_{i=1}^{r} (1 - |p_i|^{-\delta})\, x^\eta\Big)$$

where the O-constant depends only on G and r but not on the p_i's.

Since the summation in (3) is over r-tuples $\{p_1, \ldots, p_r\}$, the estimation of the number of s-free elements in $G\langle p_1 \ldots p_r\rangle$ from Proposition 4.5.5 of [3] cannot be used here for its O-constants depend on the p_i's by means of the coefficient $A \prod_{i=1}^{r} (1 - |p_i|^{-\delta})$ of the main term in (4). To extract this dependence on the p_i's we shall use the following more exact estimations which can be verified in turn

$$(5) \qquad \sum_{\substack{|n| \leqq x \\ n \in G\langle p_1 \ldots p_r\rangle}} |n|^{-z} = \begin{cases} \xi_{G\langle p_1 \ldots p_r\rangle}(z) + O\Big(\prod_{i=1}^{r} (1 - |p_i|^{-\delta})\, x^{\delta - z}\Big) & \text{if } z > \delta\,, \\ O\Big(\prod_{i=1}^{r} (1 - |p_i|^{-\delta}) \log x\Big) & \text{if } z = \delta\,, \\ O\Big(\prod_{i=1}^{r} (1 - |p_i|^{-\delta})\, x^{\delta - z}\Big) & \text{if } z < \delta\,, \end{cases}$$

$$\sum_{\substack{|n| \leqq x \\ n \in G\langle p_1 \ldots p_r\rangle}} \frac{\mu_G(n)}{|n|^z} = \xi_{G\langle p_1 \ldots p_r\rangle}^{-1}(z) + O\Big(\prod_{i=1}^{r} (1 - |p_i|^{-\delta})\, x^{\delta - z}\Big) \quad \text{if } z > \delta\,,$$

where the constants involved can be assumed to be uniform in r-tuples $\{p_1, \ldots, p_r\}$ of distinct primes p_i in G.

If $G_s\langle p_1 \ldots p_r\rangle(x)$ denotes the total number of s-free element with norms at most x in the semigroup $G\langle p_1 \ldots p_r\rangle$, then (see [5])

$$G_s\langle p_1 \ldots p_r\rangle(x) = \sum_{\substack{|n| \leq x^{1/s} \\ n \in G\langle p_1 \ldots p_r\rangle}} \mu_G(n) \, . \, N_{G\langle p_1 \ldots p_r\rangle}\left(\frac{x}{|n|^s}\right).$$

Using (5) we immediately obtain the required result that

$$G_s\langle p_1 \ldots p_r\rangle(x) = \frac{A \prod_{i=1}^{r} (1 - |p_i|^{-\delta})}{\xi_G(s\delta) \prod_{i=1}^{r} (1 - |p_i|^{-s\delta})} +$$

$$+ \begin{cases} O(\prod_{i=1}^{r} (1 - |p_i|^{-\delta})\, x^{\delta/s}) & \text{if } \eta < \delta/s\,, \\ O(\prod_{i=1}^{r} (1 - |p_i|^{-\delta})\, x^{\eta} \log x) & \text{if } \eta = \delta/s\,, \\ O(\prod_{i=1}^{r} (1 - |p_i|^{-\delta})\, x^{\eta}) & \text{if } \eta > \delta/s \end{cases}$$

with O-constants not depending on the p_i's.

This estimation yields after a short calculation

$$K_{k,r,s}^{\{p_1 \ldots p_r\}}(x) = \frac{A \, . \, x^{\delta}}{\xi_G(s\delta)} \prod_{i=1}^{r} \frac{|p_i|^{\delta} - 1}{(|p_i|^{s\delta} - 1) \, . \, |p_i|^{\delta(k-s+1)}} +$$

$$+ \begin{cases} O(|p_1 \ldots p_r|^{-k\delta/s} \, . \, x^{\delta/s}) & \text{if } \eta < \delta/s\,, \\ O(|p_1 \ldots p_r|^{-k\delta/s} \, . \, x^{\eta} \, . \log x) & \text{if } \eta = \delta/s\,, \\ O(|p_1 \ldots p_r|^{-k\eta} \, . \, x^{\eta}) & \text{if } \eta > \delta/s \end{cases}$$

again with O-constants not depending on the p_i's. The proof can be now completed by using [3, p. 170]

$$\sum_{\substack{\{p_1 \ldots p_r\} \\ |p_1 \ldots p_r| \leq x^{1/k}}} |p_1 \ldots p_r|^{-\delta} = O((\log\log x)^r)$$

and

$$\sum_{\substack{\{p_1, \ldots, p_r\} \\ |p_1 \ldots p_r| \leq x^{1/k}}} \prod_{i=1}^{r} \frac{|p_i|^{\delta} - 1}{(|p_i|^{s\delta} - 1) \, . \, |p_i|^{\delta(k-s+1)}} = \alpha(k, r, s) + O(x^{-\delta + \delta/k})\,.$$

Corollary. *If $k \geqq s \geqq 2$, $r \geqq 1$, then the set $K_{k,r,s}$ has the asymptotic density*

$$\alpha(k, r, s) \,.\, \xi_G^{-1}(k\delta)$$

with $\alpha(k, r, s)$ being defined in (2).

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