Kandiah Dayanithy

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A NOTE ON A RESULT OF KENDALL

K. DAYANITHY, Kuala Lumpur

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In this note we shall be concerned with continuous-time Markov processes which are homogeneous in time and have a countable number of states. Such systems may, be described by a collection $\{P(t) : 0 \leqq t < +\infty\}$ of matrices, where $P(t) = \{p_{jk}(t) : j, k = 1, 2, \ldots\}$ $(0 \leqq t < +\infty)$ satisfies the following conditions:

$$p_{jk}(t) \geqq 0 \; ; \quad \sum_{\alpha=1}^{+\infty} p_{j\alpha}(t) = 1 \; ; \quad \sum_{\alpha=1}^{+\infty} p_{j\alpha}(s)\, p_{\alpha k}(t) = p_{jk}(s + t) \; ; \quad \text{and}$$

$$\lim_{t=0+} p_{jk}(t) = \delta_{jk} = p_{jk}(0)$$

(the above relations are to hold for all positive integers j and k and for all real non-negative s and t). We shall restrict ourselves to irreducible processes.

It is shown in [2] that every irreducible Markov process has at least one positive sub-invariant measure $\{m_j : j = 1, 2, \ldots\}$; thus

$$\sum_{\alpha=1}^{+\infty} m_\alpha\, p_{\alpha k}(t) \leqq m_k \,,$$

for each positive integer k and each real non-negative t. This sub-invariant measure allows us to define, for each $t \geqq 0$, a bounded linear transformation $T(t)$ on l^2 in the following manner:

$$[T(t)\, x]_k = \sum_{\alpha=1}^{+\infty} x_\alpha (m_\alpha / m_k)^{1/2}\, p_{\alpha k}(t) \,,$$

for each $x = \{x_\alpha : \alpha = 1, 2, \ldots\} \in l^2$, where $[T(t)\, x]_k$ denotes the k-th component of $T(t)\, x$ $(k = 1, 2, \ldots)$.

Then $\{T(t) : 0 \leqq t < +\infty\}$ is a weakly continuous one-parameter semi-group of contractions and hence is strongly continuous on $[0, +\infty[$. KENDALL [2] uses this fact and Sz.-Nagy's theorem on unitary dilations to obtain a unitary representations of the transition probabilities of irreducible Markov processes [2, Theorem II].

A further representation is obtained by Kendall [2, Theorem IV] for a narrower class of Markov processes which may be stated as follows:

If the operator $T(t)$ is self-adjoint for *each* $t \geqq 0$, then the transition probabilities may be uniquely represented in the form

$$p_{jk}(t) = (m_k/m_j)^{1/2} \int_0^{+\infty} e^{-t\tau}\, G_{jk}(d\tau) \quad (t \geqq 0),$$

where $\{G_{jk} : j, k = 1, 2, \ldots\}$ is a symmetric system of real-valued functions of bounded variation on $[0, +\infty[$.

Theorem VII thereof gives a set of necessary and sufficient conditions for this to be so. This condition is stated in terms of the Doob-Kolmogorov limits

$$q_{jk} = p'_{jk}(0+) \quad (j, k = 1, 2, \ldots);$$

and is that they should satisfy the "reversibility" condition:

$$m_j q_{jk} = m_k q_{kj} \quad (j, k = 1, 2, \ldots).$$

In this case each of the matrices $P(t)$, where $t \geqq 0$, satisfies the reversibility condition with respect to the same sub-invariant measure, which now becomes an invariant measure. In general, the Doob-Kolmogorov limits do not determine the process uniquely and a set of conditions, called conservation conditions, sufficient to ensure unicity of the generated process is as follows:

$$0 \leqq q_{jk} < +\infty \qquad (j, k = 1, 2, \ldots; j \neq k);$$

$$\sum_{\substack{\alpha=1 \\ \alpha \neq j}}^{+\infty} q_{j\alpha} = -q_{jj} (\equiv q_j) < +\infty \quad (j = 1, 2, \ldots);$$

and the set of equations

$$\sum_{\alpha=1}^{+\infty} q_{j\alpha} y_\alpha = \lambda y_j \quad (j = 1, 2, \ldots)$$

possesses no non-zero bounded solution $y = \{y_j : j = 1, 2, \ldots\}$ for some, and hence for all, positive λ.

The present note endeavours to give a similar but weaker representation under the milder hypothesis that just *one* of the matrices $P(t)$, where $t > 0$, satisfies the reversibility condition with respect to a sub-invariant measure of the process. Without any loss of generality we may assume that $P(1)$ satisfies this condition, thus making $T(1)$ a self-adjoint operator on l^2. We observe that, in this case, the discrete-time Markov chain $\{P(n) : n = 0, 1, 2, \ldots\}$ thus defined is reversible. Hence what we are concerned with are non-reversible Markov processes which have a reversible skeleton.

Theorem. *Let* $\{m_j : j = 1, 2, \ldots\}$ *be a positive sub-invariant measure associated with an irreducible Markov process, and suppose that*

$$m_j p_{jk}(1) = m_k p_{kj}(1) \quad (j, k = 1, 2, \ldots) .$$

Then there exists a system $\{G_{jk}^{(n)} : j, k = 1, 2, \ldots; n = 0, \pm 1, \pm 2, \ldots\}$ *of complex-valued Borel measures on the positive half-real-line* $[0, +\infty[$ *such that*

$$p_{jk}(t) = \sum_{n=-\infty}^{+\infty} e^{\pi i n t} \int_0^{+\infty} e^{-\lambda t}\, G_{jk}^{(n)}(\mathrm{d}\lambda) \quad (t \geqq 1) ,$$

for each pair j, k of positive integers, where the summation of the above Fourier-type series is to be performed using a suitable kernel.

Moreover, for each pair (j, k) *of positive integers we have*

$$G_{jk}^{(-n)}(.) = \overline{G_{jk}^{(n)}(.)} \quad (n = 0, \pm 1, \pm 2, \ldots) ;$$

hence $G_{jk}^{(0)}(.)$ *is real-valued.*

Proof. Suppose that the conditions of the theorem are satisfied. Then $\{T(t) : 0 \leqq \leqq t < +\infty\}$ is a one-parameter semi-group of contractions on l^2, strongly continous on $[0, +\infty[$, and such that $T(1)$ is self-adjoint on l^2. Let us write P for $T(1)$ and let the unique resolution of the identity for P be F_0. Further let $H_+ = F_0(]0, +\infty[)\, l^2$, $H_- = F_0(]-\infty, 0[)\, l^2$ and $H_0 = F_0(\{0\})\, l^2$. Then $l^2 = H_- \oplus H_0 \oplus H_+$ is an orthogonal decomposition of l^2; and this decomposition reduces the semi-group, since P commutes with the semi-group. Let the corresponding decomposition of the semi-group be

$$T(t) = T_-(t) \oplus T_0(t) \oplus T_+(t) \quad (t \geqq 0) .$$

Then $T_0(t) = 0$ $(t \geqq 1)$; $T_\pm(1) = P_\pm$, where $P_\pm$ are the components of P in $H_\pm$ respectively. Further $\{T_\pm(t) : 0 \leqq t < +\infty\}$ are one-parameter semi-groups of contractions on $H_\pm$ respectively, and are strongly continuous on $[0, +\infty[$.

Now consider the semi-group $\{T_+(t) : 0 \leqq t < +\infty\}$. We have

$$T_+(t) = P_+^{1-t} P_+^{t-1}\, T_+(t) \quad (t \geqq 1) ,$$

where P_+^{t-1} is a bounded linear operator on H_+ and P_+^{1-t} is a closed linear operator in H_+, both are defined using the familiar operational calculus for self-adjoint operators. Since P commutes with the semi-group, we have

$$T_+(t) = \{P_+^{1-t}\, T_+(t)\} \,.\, P^{t-1} \quad (t \geqq 1) ,$$

where $P_+^{1-t}\, T_+(t)$ is a closed linear operator in H_+, for each $t \geqq 0$. We next show that this operator is indeed bounded.

If $0 \leqq t \leqq 1$, clearly $P_+^{1-t}\, T_+(t)$ is a bounded linear operator on H_+. If $t > 1$, let $t = n + s$, where n is the integral part of t and $0 \leqq s < 1$. Then

$$P_+^{1-t}\, T_+(t) = P_+^{1-t}\, T_+(n)\, T_+(s) = P_+^{1-t} P_+^n\, T_+(s) = P_+^{1-s}\, T_+(s)\,.$$

Thus, for each $t \geqq 0$, $P_+^{1-t}\, T_+(t)$ is a bounded linear operator on H_+. Moreover, it is a periodic function of t, with period 1 (or a franction of it); and is strongly continuous on $[0, +\infty[$, since it is strongly continuous on $[0, 1]$.

For each integer n, let the corresponding Fourier coefficient be $\tilde{G}^{(2n)}$:

$$\tilde{G}^{(2n)} = \int_0^1 e^{-2\pi i n t} P_+^{1-t}\, T_+(t)\, \mathrm{d}t\,.$$

Each $\tilde{G}^{(2n)}$ is a bounded linear operator on H_+ and we have

$$P_+^{1-t}\, T_+(t) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n t} \tilde{G}^{(2n)} \quad (t \geqq 0)\,,$$

where the summation is to be performed using a suitable summability kernel, such as Fejér's or Poisson's, and in the strong operator topology of $B(H_+)$.

We next observe that P_+ is a positive operator on H_+ and hence, for each $t \geqq 1$,

$$P_+^{t-1} = \int_0^{+\infty} e^{-\lambda(t-1)}\, E_+(\mathrm{d}\lambda)\,,$$

where E_+ is a self-adjoint resolution of the identity on H_+. Thus, for each $t \geqq 1$, we have

$$T_+(t) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n t} \tilde{G}^{(2n)} \int_0^{+\infty} e^{-\lambda(t-1)}\, E_+(\mathrm{d}\lambda)\,,$$

where the summation is to be performed as before. But, since $\tilde{G}^{(2n)}$ is a bounded linear operator on H_+, we have

$$T_+(t) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n t} \int_0^{+\infty} e^{-\lambda(t-1)} \tilde{G}^{(2n)}\, E_+(\mathrm{d}\lambda) \quad (t \geqq 1)\,.$$

Similarly, by considering the semi-group $\{e^{-\pi i t}\, T_-(t) : 0 \leqq t < +\infty\}$, we have

$$T_-(t) = \sum_{n=-\infty}^{+\infty} e^{\pi i(2n+1)t} \int_0^{+\infty} e^{-\lambda(t-1)} \tilde{G}^{(2n+1)}\, E_-(\mathrm{d}\lambda) \quad (t \geqq 1)\,.$$

If we now observe that $T_0(t) = 0$ $(t \geqq 1)$, we have

$$T(t)\, e_i = \sum_{n=-\infty}^{+\infty} e^{\pi i n t} \int_0^{+\infty} e^{-\lambda(t-1)}\, \tilde{G}_i^{(n)}(\mathrm{d}\lambda) \quad (t \geqq 1)\,,$$

where, for each positive integer i, we denote by e_i the element of l^2 whose i-th component is 1 and whose all other components are zero; and

$$\left.\begin{aligned}\tilde{G}_i^{(2n)}(.) &= \tilde{G}^{(2n)}\, E_+(.)\, F_+ e_i \\ \tilde{G}_i^{(2n+1)}(.) &= \tilde{G}^{(2n+1)}\, E_-(.)\, F_- e_i\end{aligned}\right\} (n = 0, \pm 1, \pm 2, \ldots)\,,$$

$$F_+ = F_0(]\,0,\ +\infty[)\,, \quad F_- = F_0(]-\infty,\ 0[)\,.$$

Thus

$$(*) \qquad p_{jk}(t) = \sum_{n=-\infty}^{+\infty} e^{\pi i n t} \int_0^{+\infty} e^{-\lambda t}\, \hat{G}_{jk}^{(n)}(\mathrm{d}\lambda) \quad (t \geqq 1)\,,$$

where $\hat{G}_{jk}^{(n)}(\mathrm{d}\lambda) = (m_k/m_j)^{1/2}\, e^{\lambda} \langle e_j, \tilde{G}_k^{(n)}(\mathrm{d}\lambda)\rangle$ $(j, k = 1, 2, \ldots)$.

Adding $(*)$ to its conjugate and dividing by 2, we have

$$p_{jk}(t) = \sum_{n=-\infty}^{+\infty} e^{\pi i n t} \int_0^{+\infty} e^{-\lambda t}\, G_{jk}^{(n)}(\mathrm{d}\lambda) \quad (t \geqq 1)\,,$$

where

$$G_{jk}^{(n)}(\mathrm{d}\lambda) = \tfrac{1}{2}\{\hat{G}_{jk}^{(n)}(\mathrm{d}\lambda) + \overline{\hat{G}_{jk}^{(-n)}(\mathrm{d}\lambda)}\}\,,$$

for each pair (j, k) of positive integers. This completes the proof of the theorem.

If we are using Fejér's kernel in summing the series for $T_+(t)$ and $T_-(t)$, we get the following combined kernel given by:

$$p_{jk}(t) = \lim_{n=\infty} \sum_{r=-2n}^{2n+1} \left(1 - \frac{|[r/2]|}{r+1}\right) e^{\pi i r t} \int_0^{+\infty} e^{-\lambda t}\, G_{jk}^{(r)}(\mathrm{d}\lambda)\,,$$

where $[r/2]$ denotes the integral part of $r/2$.

We observe that $G_{jk}^{(n)} = 0$ for each non-zero value of n if, and only if, each matrix $P(t)$, where $t \geqq 0$, is reversible with respect to the sub-invariant measure $\{m_j : j = 1, 2, \ldots\}$.

An important question remains to be answered, that is, whether there exist any non-reversible Markov processes which possess a reversible skeleton. One such process was given by Speakman [3]. Here Speakman constructs two three-state Markov chains, one of which is reversible while the other is of the above description. The non-reversible process has infinitesimal matrix (that is the matrix of Doob-Kolmogorov limits) Q given by:

$$Q = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix};$$

the individual transition probabilities are given by:

$$p_{11}(t) = p_{22}(t) = p_{33}(t) = \tfrac{1}{3} + \tfrac{2}{3}e^{-3t/2}\cos\left(\sqrt{(3t)}/2\right);$$

$$p_{12}(t) = p_{23}(t) = p_{31}(t) = \tfrac{1}{3} + \tfrac{2}{3}e^{-3t/2}\cos\left(\sqrt{(3t)}/2 - 2\pi/3\right);$$

and

$$p_{13}(t) = p_{21}(t) = p_{32}(t) = \tfrac{1}{3} + \tfrac{2}{3}e^{-3t/2}\cos\left(\sqrt{(3t)}/2 - 4\pi/3\right).$$

Q satisfies the conservation conditions stated earlier and hence the only Markov chain it generates is the above (which is thus the Feller minimal process associated with Q). Further [1, 1, 1] is an invariant measure for the process and the matrix $P(t)$ is reversible with respect to this invariant measure whenever t is an integral multiple of $4\pi/\sqrt{3}$. But the process is reversible for no positive sub-invariant measure since, Q obviously is non-reversible with respect to any such measure. Finally we observe that the above expressions for the transition probabilities is already recognisable as a particular instance of our expansion theorem, with suitable Dirac measures for $G_{jk}^{(n)}$.

I am very grateful to Professor David Kendall of the University of Cambridge for suggesting the above Speakman process as an illustration of a non-reversible Markov process with a reversible skeleton.

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Author's address: Department of Mathematics, University of Malaya, Kuala Lumpur, Malaysia.