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Finite Abelian semigroups represented into the power set of finite groups


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Finite abelian groups have very well-defined structures and are direct sums of cyclic groups. If $2^G$ is the collection of nonempty subsets of a semigroup $G$, then $AB = \{ab \mid a \in A, b \in B\}$ defines a semigroup for $2^G$. Although finite abelian groups have been investigated, $2^G$ is a relatively new object for research. Byrd, Lloyd, Pederson, and Stepp studied the automorphisms of $2^G$ (see [2]) and have made contributions to the understanding of $2^G$.

If one allows $G$ to be any abelian group and not just finite then Trnková in [5] proved that every abelian semigroup is embeddable (one-to-one homomorphism) into $2^G$ for some abelian group $G$. But $2^G$ for an arbitrary abelian group is rather untractable. So further restriction was needed. In [1], Bilyeu and Lau studied the collection (hyperspace) of compact subsets of a compact group and certain topological embeddings were derived.

But underlying all the general studies, a very basic question has not been settled:

**Problem.** If $S$ is a finite abelian semigroup, then is $S$ embeddable in $2^G$ for some finite abelian group $G$?

A finite abelian semigroup is said to be *representable* (in this paper) if it is embeddable in $2^G$ for some finite abelian group $G$. A $z$-semigroup is a semigroup having a unique idempotent which is a zero for the semigroup (see Yamada [6] and [7]). If $S$ is a finite semigroup, then it has a minimal ideal denoted by $M(S)$ and $S/M(S)$ is the Rees quotient. If $S$ has an identity 1, then $H(1)$ is the group of units.

We were not able to solve the general problem but were able to prove that if finite abelian $z$-semigroups are representable, then finite abelian semigroups are representable. The following lemmas are helpful to establish this fact.
Lemma 1. If $G_1, \ldots, G_n$ are finite groups, then $\prod_{i=1}^{n} 2^{G_i}$ is embeddable in $2^{\prod G_i}$.

Proof. Use the function which sends $(A_1, \ldots, A_n)$ to $A_1 \times \ldots \times A_n$.

Lemma 2. If $S$ is a finite abelian semigroup and for each pair $x \neq y$ in $S$, there is a homomorphism $f$ from $S$ into $2^G$ for some finite abelian group $G$ so that $f(x) \neq f(y)$, then $S$ is representable.

Proof. Since there are finitely many homomorphisms from $S$ into $2^{G_1}, \ldots, 2^{G_n}$ to separate points, then $S$ is embeddable in $\prod 2^{G_i}$, hence in $2^{\prod G_i}$ by Lemma 1.

Lemma 3. If $S, T$ are semigroups and $i : S \to T$ is a one-to-one homomorphism, then $i^* : 2^S \to 2^T$ is a one-to-one homomorphism where $i^*(A) = i(A)$.

Lemma 4. If $S$ is a semigroup and $\sigma : 2^{2^S} \to 2^S$ is defined by $\sigma(\mathcal{A}) = \bigcup \{A \mid A \in \mathcal{A}\}$, then $\sigma$ is a homomorphism.

Theorem. If each finite abelian $z$-semigroup is representable, then every finite abelian semigroup is representable.

Proof. Induct on the order of $S$ where $S$ is a finite abelian semigroup. Suppose $M(S)$ has more than one element. Let $e = e^2 \in M(S)$. Note that $M(S)$ is a group since $S$ is abelian. Then $f : S \to M(S)$ by $f(x) = xe$ and $p : S \to S/M(S)$ would separate points. But $S/M(S)$ has an order less than that of $S$. By induction, $S/M(S)$ is representable.

We can now assume that $S$ has a zero. Choose $e = e^2 \neq 0$ so that it is minimal with respect to the idempotent ordering of all nonzero idempotents. Again $f : S \to Se$ by $f(x) = xe$ and $S \to S/Se$ separate points. Hence we can assume that $S = Se$, i.e., $S$ has an identity $1$ and has only two idempotents $0$ and $1$.

Suppose $H(1) = \{1\}$. Then $I = S \setminus H(1)$ is a finite abelian $z$-semigroup. Let $j$ be an embedding of $I$ into $2^S$ for some finite abelian group $G$. Let $H$ be a finite abelian group having more than one element. Then $J : S \to 2^{G \times H}$ defined by:

$$J(x) = \begin{cases} j(x) \times H & \text{if } x \neq 1, \\ \{(1, 1)\} & \text{if } x = 1, \end{cases}$$

is an embedding.

Assume that the set of idempotents of $S$ is $\{0, 1\}$ and $H(1) \neq \{1\}$.

Let $H = H(1)$. Since $|I \cup \{1\}| < |S|$, then by induction, we have $j : I \cup \{1\} \to 2^G$ an embedding for some finite abelian group $G$. Let

1. $J : H \times (I \cup \{1\}) \to H \times 2^G$ be defined by $J(h, x) = (h, j(x))$,
2. $K : H \times 2^G \to 2^{H \times G}$ be defined by $K(h, A) = \{h\} \times A$,
3. $m : H \times (I \cup \{1\}) \to S$ be defined by $m(h, x) = hx$.
Then
\[ m^{-1}(x) = \begin{cases} \{(h, h^{-1}x) \mid h \in H\} & \text{if } x \in I, \\ \{(x, 1)\} & \text{if } x \in H(1). \end{cases} \]

Claim. \( M : S \to 2^{H \times (I \cup \{1\})} \) is a homomorphism where \( M(x) = m^{-1}(x) \).

Let \( x, y \in S \). Then \( M(x) M(y) \subseteq M(xy) \) since \( m \) is a homomorphism.

Case A. Suppose \( x \in H \) and \( y \in I \). Then \( xy \in I \). Let \( (h, z) \in M(xy) \). Then \( hz = xy \),
\[ m^{-1}(x) = (x, 1) \text{ and } (h, z) = (x, 1)(hx^{-1}, z) \in M(x) M(y). \]

Case B. Suppose \( x \in H \) and \( y \in H \). Then \( M(xy) = (xy, 1) = (x, 1)(y, 1) = M(x)M(y). \)

Case C. Suppose \( x, y \in I \). Let \( (h, z) \in M(xy) \). Then \( hz = xy \). Hence \( (h, z) = (h, h^{-1}x)(1, y) \in M(x)M(y). \)

Consider \( i : S \to 2^{H \times G} \) defined by composing these four functions:
\[ S \xrightarrow{M} 2^{H \times (I \cup \{1\})} \xrightarrow{J^*} 2^{H \times 2^G} \xrightarrow{K^*} 2^{2^{H \times G}} \xrightarrow{\sigma} 2^{H \times G}. \]

We shall prove that \( i = \sigma K^* J^* M \) is an embedding. It is clear that it is a homomorphism.

Case 1. Let \( x, y \in I \).
\[ i(x) = \sigma K^* J^* M(x) = \sigma K^* \{(h, h^{-1}x) \mid h \in H\} = \sigma \{(h) \times j(h^{-1}x) \mid h \in H\} = \bigcup_{h \in H} \{h\} \times j(h^{-1}x). \]
\[ i(y) = \bigcup_{h \in H} \{h\} \times j(h^{-1}y). \]

Suppose \( i(x) = i(y) \). Then \( \{1\} \times j(x) \subseteq \bigcup_{h \in H} \{h\} \times j(h^{-1}y) \). Hence \( \{1\} \times j(x) \subseteq \{1\} \times j(y) \). Conversely, \( \{1\} \times j(y) \subseteq \{1\} \times j(x) \). But \( j(x) = j(y) \) implies \( x = y \).

Case 2. Let \( x, y \in H \).
\[ i(x) = \sigma K^* J^* M(x) = \sigma K^* \{(x, 1)\} = \sigma \{(x) \times j(1)\} = \{x\} \times j(1). \]
\[ i(y) = \{y\} \times j(1). \]

Hence \( i(x) = i(y) \) implies \( x = y \).

Case 3. Let \( x \in H, y \in I \). Then
\[ i(x) = \{x\} \times j(1) \]
and
\[ i(y) = \bigcup_{h \in H} \{h\} \times j(h^{-1}y). \]
Hence \( i(x) \neq i(y) \) since \( H \) has more than one element.

Remark. Left zero semigroups \( (xy = x \text{ for all } x, y) \) are not embeddable in \( 2^G \) for any finite group \( G \). Hence the commutative property of the semigroup is important to the problem.

Remark. The structure of finite abelian \( z \)-semigroups was thoroughly discussed in [6] and [7] but we are still unable to solve the general problem.

References


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