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MINIMAL SETS OF VECTORS WHICH GENERATE R_n
WITH EXCESS k

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It is the purpose of this note to give a simple proof of the fact that, for fixed integers $n \geq 1$ and $k \geq 0$, the smallest set of vectors in an n -dimensional real vector space R_n which generates R_n as its convex hull and preserves this property even after removing any k of the vectors, has cardinality $n + 2k + 1$. An equivalent result is proved in [1].

1. Preliminaries. In the whole paper, R_n will denote an n -dimensional real vector space. The cardinality of a set S will be denoted by $\text{card } S$.

(1,1) Definition. We shall say that a finite set S of vectors in R_n generates R_n if any vector in R_n is a linear combination of vectors in S with nonnegative coefficients.

The following lemma is well known:

(1,2) Lemma. Let $S = \{v_1, \dots, v_N\}$ be a set of vectors in R_n . Then the following are equivalent:

- (i) S generates R_n ;
- (ii) S contains a basis of R_n and, there exist positive numbers $\alpha_1, \dots, \alpha_N$ such that

$$\sum_{i=1}^N \alpha_i v_i = 0;$$

- (iii) any open halfspace of R_n contains at least one vector from S .

(1,3) Definition. Let $n \geq 1$, $k \geq 0$ be integers. We shall say that a finite set S of vectors in R_n generates R_n with excess k if for any subset $S' \subset S$ with k elements, $S \setminus S'$ generates R_n .

The following assertion is an easy consequence of (1,2):

(1,4) A finite set S of vectors in R_n generates R_n with excess k iff every open halfspace of R_n contains at least $k + 1$ vectors from S .

(1,5) Lemma. If y_1, \dots, y_m ($m \geq 2$) are mutually different numbers then

$$\sum_{i=1}^m \frac{y_i^s}{\prod_{\substack{j=1 \\ j \neq i}}^m (y_i - y_j)} = 0 \quad \text{for } s = 0, \dots, m - 2.$$

Proof. By the Lagrange interpolation formula [2], any polynomial $f(x)$ of degree at most $m - 1$ satisfies the identity

$$f(x) \equiv \sum_{i=1}^m \frac{f(y_i)}{\prod_{\substack{j=1 \\ j \neq i}}^m (y_i - y_j)} \prod_{j=1}^m (x - y_j).$$

Choosing $f(x) = x^s$, $s \in \{0, \dots, m - 2\}$ and comparing the coefficients at x^{m-1} on both sides, we obtain the desired equalities.

2. Results. (2,1) Theorem. Let $n \geq 1$, $k \geq 0$ be integers. Let S be a set of vectors in R_n which generates R_n with excess k . Then

$$\text{card } S \geq n + 2k + 1$$

and this bound is sharp for all k and n .

Proof. Let S be a set of vectors in R_n which generates R_n with excess k and assume that

$$\text{card } S \leq n + 2k.$$

Then there exists a subset $S_0 \subseteq S$ consisting of $n - 1$ linearly independent vectors. Let R_0 be the hyperplane spanned by the vectors in S_0 . Since $S \setminus S_0$ contains at most $2k + 1$ vectors, at least one of the open halfspaces of R_n with boundary R_0 contains at most k vectors from S . By (1,4), S does not generate R_n with excess k , a contradiction.

It remains to find, for any $n \geq 1$ and $k \geq 0$, a set of $n + 2k + 1$ vectors which generates R_n with excess k . This will be done in the following theorem.

(2,2) Theorem. Let $n \geq 1$, $k \geq 0$ be integers, let $x_1 > x_2 > \dots > x_{n+2k+1}$ be real numbers. Then the $2k + 1$ row vectors

$$\begin{aligned} &(-1)^{s-1} ((x_s - x_{2k+2})^{-1}, (x_s - x_{2k+3})^{-1}, \dots, (x_s - x_{2k+n+1})^{-1}), \\ &s = 1, \dots, 2k + 1, \end{aligned}$$

together with the n unit vectors $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 1)$, form a set which generates the space R'_n of all row vectors with excess k .

Proof. Denote by $B = (b_{pi})$ the $(n + 2k + 1) \times n$ matrix with entries

$$b_{pi} = ((x_p - x_{2k+1+i}) \prod_{\substack{q=1 \\ q \neq p}}^{2k+1} (x_p - x_q))^{-1}$$

if

$$p = 1, \dots, 2k + 1, \quad i = 1, \dots, n,$$

$$b_{pi} = \left(\prod_{q=1}^{2k+1} (x_p - x_q) \right)^{-1} \delta_{i, p-2k-1}$$

if

$$p = 2k + 2, \dots, n + 2k + 1, \quad i = 1, \dots, n,$$

where δ_{ik} are the Kronecker symbols.

It is easily seen that the rows of the matrix B are positive multiples of the vectors defined above.

By Lemma (1,5), the product

$$(1) \quad VB = 0$$

where $V = (v_{\alpha p})$ is the $(2k + 1) \times (n + 2k + 1)$ "Vandermonde matrix" with entries

$$v_{\alpha p} = x_p^{2k+1-\alpha}, \quad p \in M = \{1, 2, \dots, n + 2k + 1\}, \\ \alpha \in K = \{1, \dots, 2k + 1\}.$$

On the other hand, there exists an $\binom{n + 2k + 1}{k} \times (2k + 1)$ matrix

$$Y = (y_{(j_1, \dots, j_k)\alpha})$$

where (j_1, \dots, j_k) is a combination of k elements of the set of indices M and $\alpha \in K$, such that

$$(2) \quad YV = Z = (z_{(j_1, \dots, j_k)p})$$

with

$$(3) \quad z_{(j_1, \dots, j_k)p} = \prod_{s=1}^k (x_p - x_{j_s})^2, \quad p \in M.$$

Indeed, the numbers $y_{(j_1, \dots, j_k)\alpha}$ are the coefficients in the polynomial

$$\prod_{s=1}^k (x - x_{j_s})^2 = \sum_{\alpha=1}^{2k+1} x^{2k+1-\alpha} y_{(j_1, \dots, j_k)\alpha}.$$

From (1) and (2), we have

$$ZB = 0.$$

Since any matrix $((p_i - q_j)^{-1})$, $i, j = 1, \dots, s$, is nonsingular whenever $p_1, \dots, p_s, q_1, \dots, q_s$ are different from each other, any- n rows of the matrix B are linearly independent. By (ii) of (1,2) and (3), for every subset $M' = \{j_1, \dots, j_k\}$ of M having k elements the rows $b_{(p)}$ of the matrix B with $p \in M \setminus M'$ generate R'_n . Consequently, the rows of B , and hence also the vectors given in the theorem, generate R'_n with excess k . The proofs of both Theorems (2,2) and (2,1) are complete.

(2,3) Corollary. *Let $n \geq 1$. Then there exists a sequence of systems S_0, S_1, S_2, \dots of vectors in R_n with the following properties:*

- 1° $\text{card } S_k = n + 2k + 1$;
- 2° $S_k \subset S_{k+1}$, $k = 0, 1, \dots$;
- 3° S_k generates R_n with excess k .

We shall add four more theorems which are, by (1,4), equivalent to (2,1).

(2,4) Theorem. *Let $n \geq 1$, $l \geq 1$ be integers. Let N non-zero vectors in R_n have the property that every open halfspace of R_n contains at least l of these vectors. Then*

$$N \geq n + 2l - 1$$

and this bound is sharp for all n and l .

(2,5) Theorem. *Let $n \geq 2$, $k \geq 1$ be integers. Let N non-zero vectors in R_n have the property that any non-zero vector of R_n forms an acute angle with at least k of the given vectors. Then $N \geq n + 2k - 1$ and this bound is sharp for all n and k .*

Remark. This theorem can also be reformulated in terms of distance graphs introduced in [3].

(2,6) Theorem. *Let $p \geq 1$, $q \geq 1$ be integers. Let \mathcal{H} be a finite system of N open halfspheres on a p -dimensional sphere S_p which covers S_p q -times, i.e. every point of S_p is contained in at least q halfspheres from \mathcal{H} . Then $N \geq p + 2q$ and this bound is sharp for all p and q .*

(2,7) Theorem. *Let $m \geq 1$, $l \geq 1$ be integers. Let N points on an m -dimensional sphere S_m have the property that every open halfsphere of S_m contains at least l of the given N points. Then*

$$N \geq m + 2l + 2$$

and this bound is sharp for all m and l .

Considering the Gram matrix of such systems of vectors, we obtain the following formulation:

(2,8) Theorem. *Let a positive semi-definite real n by n matrix A have the property that whenever one row and column is added in such a way that the resulting matrix remains positive semidefinite and of the same rank as A then the new row contains at least k positive entries. Then the rank of A is at least $n - 2k + 3$ and this bound is sharp.*

References

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