Miroslav Fiedler

Minimal sets of vectors which generate $\mathbb{R}^n$ with excess $k$


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MINIMAL SETS OF VECTORS WHICH GENERATE $R_n$
WITH EXCESS $k$

MIROSLAV FIEDLER, Praha
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It is the purpose of this note to give a simple proof of the fact that, for fixed integers $n \geq 1$ and $k \geq 0$, the smallest set of vectors in an $n$-dimensional real vector space $R_n$ which generates $R_n$ as its convex hull and preserves this property even after removing any $k$ of the vectors, has cardinality $n + 2k + 1$. An equivalent result is proved in [1].

1. Preliminaries. In the whole paper, $R_n$ will denote an $n$-dimensional real vector space. The cardinality of a set $S$ will be denoted by $\text{card } S$.

(1,1) Definition. We shall say that a finite set $S$ of vectors in $R_n$ generates $R_n$ if any vector in $R_n$ is a linear combination of vectors in $S$ with nonnegative coefficients.

(1,2) Lemma. Let $S = \{v_1, \ldots, v_N\}$ be a set of vectors in $R_n$. Then the following are equivalent:

(i) $S$ generates $R_n$;
(ii) $S$ contains a basis of $R_n$ and, there exist positive numbers $\alpha_1, \ldots, \alpha_N$ such that

$$\sum_{i=1}^{N} \alpha_i v_i = 0;$$

(iii) any open halfspace of $R_n$ contains at least one vector from $S$.

(1,3) Definition. Let $n \geq 1$, $k \geq 0$ be integers. We shall say that a finite set $S$ of vectors in $R_n$ generates $R_n$ with excess $k$ if for any subset $S' \subset S$ with $k$ elements, $S \setminus S'$ generates $R_n$.

The following assertion is an easy consequence of (1,2):

(1,4) A finite set $S$ of vectors in $R_n$ generates $R_n$ with excess $k$ iff every open halfspace of $R_n$ contains at least $k + 1$ vectors from $S$. 

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(1,5) Lemma. If \( y_1, \ldots, y_m \) (\( m \geq 2 \)) are mutually different numbers then

\[
\sum_{i=1}^{m} \frac{y_i^s}{\prod_{j \neq i} (y_i - y_j)} = 0 \quad \text{for} \quad s = 0, \ldots, m-2.
\]

Proof. By the Lagrange interpolation formula [2], any polynomial \( f(x) \) of degree at most \( m - 1 \) satisfies the identity

\[
f(x) = \sum_{i=1}^{m} \frac{f(y_i)}{\prod_{j \neq i} (y_i - y_j)} (x - y_j).
\]

Choosing \( f(x) = x^s, s \in \{0, \ldots, m-2\} \) and comparing the coefficients at \( x^{m-1} \) on both sides, we obtain the desired equalities.

2. Results. (2,1) Theorem. Let \( n \geq 1, k \geq 0 \) be integers. Let \( S \) be a set of vectors in \( \mathbb{R}^n \) which generates \( \mathbb{R}^n \) with excess \( k \). Then

\[
\text{card } S \leq n + 2k + 1
\]

and this bound is sharp for all \( k \) and \( n \).

Proof. Let \( S \) be a set of vectors in \( \mathbb{R}^n \) which generates \( \mathbb{R}^n \) with excess \( k \) and assume that

\[
\text{card } S \leq n + 2k.
\]

Then there exists a subset \( S_0 \subseteq S \) consisting of \( n - 1 \) linearly independent vectors. Let \( R_0 \) be the hyperplane spanned by the vectors in \( S_0 \). Since \( S \setminus S_0 \) contains at most \( 2k + 1 \) vectors, at least one of the open halfspaces of \( \mathbb{R}^n \) with boundary \( R_0 \) contains at most \( k \) vectors from \( S \). By (1,4), \( S \) does not generate \( \mathbb{R}^n \) with excess \( k \), a contradiction.

It remains to find, for any \( n \geq 1 \) and \( k \geq 0 \), a set of \( n + 2k + 1 \) vectors which generates \( \mathbb{R}^n \) with excess \( k \). This will be done in the following theorem.

(2,2) Theorem. Let \( n \geq 1, k \geq 0 \) be integers, let \( x_1 > x_2 > \ldots > x_{n+2k+1} \) be real numbers. Then the \( 2k + 1 \) row vectors

\[
(-1)^{s-1} \left( (x_s - x_{2k+2})^{-1}, \ldots, (x_s - x_{2k+n+1})^{-1} \right),
\]

\( s = 1, \ldots, 2k + 1 \),

together with the \( n \) unit vectors \((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)\), form a set which generates the space \( \mathbb{R}^n \) of all row vectors with excess \( k \).
Proof. Denote by $B = (b_{pi})$ the $(n + 2k + 1) \times n$ matrix with entries

$$b_{pi} = \left( (x_p - x_{2k+1+i}) \prod_{q=1}^{2k+1}(x_p - x_q) \right)^{-1}$$

if

$$p = 1, \ldots, 2k + 1, \quad i = 1, \ldots, n,$$

$$b_{pi} = \left( \prod_{q=1}^{2k+1}(x_p - x_q) \right)^{-1} \delta_{i,p-2k-1}$$

if

$$p = 2k + 2, \ldots, n + 2k + 1, \quad i = 1, \ldots, n,$$

where $\delta_{ik}$ are the Kronecker symbols.

It is easily seen that the rows of the matrix $B$ are positive multiples of the vectors defined above.

By Lemma (1,5), the product

$$(1) \quad VB = 0$$

where $V = (v_{sp})$ is the $(2k + 1) \times (n + 2k + 1)$ "Vandermonde matrix" with entries

$$v_{sp} = x_p^{2k+1-x}, \quad p \in M = \{1, 2, \ldots, n + 2k + 1\},$$
$$x \in K = \{1, \ldots, 2k + 1\}.$$

On the other hand, there exists an $(n + 2k + 1) \times (2k + 1)$ matrix

$$Y = (y_{(j_1, \ldots, j_k)x})$$

where $(j_1, \ldots, j_k)$ is a combination of $k$ elements of the set of indices $M$ and $x \in K$, such that

$$(2) \quad YV = Z = (z_{(j_1, \ldots, j_k)p})$$

with

$$(3) \quad z_{(j_1, \ldots, j_k)p} = \prod_{s=1}^{k}(x_p - x_{j_s})^2, \quad p \in M.$$

Indeed, the numbers $y_{(j_1, \ldots, j_k)x}$ are the coefficients in the polynomial

$$\prod_{s=1}^{k}(x - x_{j_s})^2 = \sum_{x=1}^{2k+1-x} x^{2k+1-x} y_{(j_1, \ldots, j_k)x}.$$

From (1) and (2), we have

$$ZB = 0.$$
Since any matrix \(((p_i - q_j)^{-1}), i, j = 1, \ldots, s,\) is nonsingular whenever \(p_1, \ldots, p_s, q_1, \ldots, q_s\) are different from each other, any \(n\)-rows of the matrix \(B\) are linearly independent. By (ii) of (1,2) and (3), for every subset \(M' = \{j_1, \ldots, j_k\}\) of \(M\) having \(k\) elements the rows \(b_{(p)}\) of the matrix \(B\) with \(p \in M \setminus M'\) generate \(R^n_s\). Consequently, the rows of \(B\), and hence also the vectors given in the theorem, generate \(R^n_s\) with excess \(k\). The proofs of both Theorems (2,2) and (2,1) are complete.

(2.3) Corollary. Let \(n \geq 1\). Then there exists a sequence of systems \(S_0, S_1, S_2, \ldots\) of vectors in \(R^n\) with the following properties:

1° \(\text{card } S_k = n + 2k + 1\);
2° \(S_k \subset S_{k+1}, k = 0, 1, \ldots\);
3° \(S_k\) generates \(R^n\) with excess \(k\).

We shall add four more theorems which are, by (1,4), equivalent to (2,1).

(2.4) Theorem. Let \(n \geq 1, l \geq 1\) be integers. Let \(N\) non-zero vectors in \(R^n\) have the property that every open halfspace of \(R^n\) contains at least \(l\) of these vectors. Then

\[N \geq n + 2l - 1\]

and this bound is sharp for all \(n\) and \(l\).

(2.5) Theorem. Let \(n \geq 2, k \geq 1\) be integers. Let \(N\) non-zero vectors in \(R^n\) have the property that any non-zero vector of \(R^n\) forms an acute angle with at least \(k\) of the given vectors. Then \(N \geq n + 2k - 1\) and this bound is sharp for all \(n\) and \(k\).

Remark. This theorem can also be reformulated in terms of distance graphs introduced in [3].

(2.6) Theorem. Let \(p \geq 1, q \geq 1\) be integers. Let \(H\) be a finite system of \(N\) open halfspaces on a \(p\)-dimensional sphere \(S_p\) which covers \(S_p\) \(q\)-times, i.e. every point of \(S_p\) is contained in at least \(q\) halfspaces from \(H\). Then \(N \geq p + 2q\) and this bound is sharp for all \(p\) and \(q\).

(2.7) Theorem. Let \(m \geq 1, l \geq 1\) be integers. Let \(N\) points on an \(m\)-dimensional sphere \(S_m\) have the property that every open halfsphere of \(S_m\) contains at least \(l\) of the given \(N\) points. Then

\[N \geq m + 2l + 2\]

and this bound is sharp for all \(m\) and \(l\).

Considering the Gram matrix of such systems of vectors, we obtain the following formulation:
(2.8) Theorem. Let a positive semi-definite real \( n \times n \) matrix \( A \) have the property that whenever one row and column is added in such a way that the resulting matrix remains positive semidefinite and of the same rank as \( A \) then the new row contains at least \( k \) positive entries. Then the rank of \( A \) is at least \( n - 2k + 3 \) and this bound is sharp.

References


Author’s address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).