Jan Troják; Jiří Vanžura
A characterization of hyperspheres in the quaternionic space

*Československá matematická časopis*, Vol. 29 (1979), No. 2, 284–286

Persistent URL: [http://dml.cz/dmlcz/101604](http://dml.cz/dmlcz/101604)

**Terms of use:**

© Institute of Mathematics AS CR, 1979

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*
Let us consider the 4-dimensional euclidean space $\mathbb{R}^4$, which we shall identify in the natural way with the division algebra $\mathbb{H}$ of quaternions. The left multiplication by the quaternionic units $i, j, k$ induces on $\mathbb{R}^4$ three tensor fields $I_1, I_2, I_3$ of type $(1,1)$ satisfying

$$I_1I_j + I_jI_1 = -2\delta_{ij},$$

i.e. a quaternionic structure. At any point $x \in \mathbb{R}^4$ the tensors $I_1, I_2, I_3$ are orthogonal automorphisms of the tangent space $T_x(\mathbb{R}^4)$.

We shall investigate the structure induced on a 3-dimensional submanifold $M \subset \mathbb{R}^4$ by the quaternionic structure on $\mathbb{R}^4$. Let us suppose that $M$ is orientable and let us denote by $N$ the field of positive unit normals on $M$. From the above mentioned properties of the tensors $I_1, I_2, I_3$ it follows easily that

$$\langle I_iN, N \rangle = 0, \quad \langle I_iN, I_jN \rangle = \delta_{ij}$$

for any $i, j = 1, 2, 3$.

It enables us to define three orthonormal tangent vector fields $V_1 = I_1N, V_2 = I_2N, V_3 = I_3N$ on $M$ obtaining thus on $M$ a complete parallelism. We write

$$[V_1, V_2] = a_1V_1 + a_2V_2 + a_3V_3,$$

$$[V_2, V_3] = b_1V_1 + b_2V_2 + b_3V_3,$$

$$[V_3, V_1] = c_1V_1 + c_2V_2 + c_3V_3$$

with $a_i, b_i, c_i; i = 1, 2, 3$ being functions on $M$.

Taking for $M$ a hypersphere of radius $r$ we get

$$[V_1, V_2] = -\frac{2}{r}V_3, \quad [V_2, V_3] = -\frac{2}{r}V_1, \quad [V_3, V_1] = -\frac{2}{r}V_2.$$

The goal of the present note is to prove the following
Theorem. Let $M$ be a connected oriented 3-dimensional submanifold of $\mathbb{R}^4$ on which the complete parallelism $V_1, V_2, V_3$ satisfies

$$[V_1, V_2] = -\frac{2}{r} V_3, \quad [V_2, V_3] = -\frac{2}{r} V_1, \quad [V_3, V_1] = -\frac{2}{r} V_2.$$ 

Then $M$ is part of a hypersphere with radius $r$.

For the proof we shall need two lemmas.

**Lemma 1.** Let $\nabla$ denote the Levi-Civita connection on $M$ and let us write $\nabla_{I_iN}(I_jN) = \Gamma^k_{ij}I_kN$. Then

$$\Gamma^k_{ij} = -\frac{1}{r} \text{sgn}(1 2 3) \text{ if } (1 2 3) \text{ is a permutation and } \Gamma^k_{ij} = 0 \text{ otherwise}.$$ 

**Proof.** Using the basic properties of the Levi-Civita connection we can write the identities

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle +$$

$$+ \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle - \langle X, [Y, Z] \rangle,$$

and

$$\langle \nabla_X Y, Z \rangle - \langle Z, [X, Y] \rangle = \langle \nabla_Y X, Z \rangle$$

with any $X, Y, Z \in T(M)$. The identity (1) enables us to evaluate

$$\nabla_{I_iN}(I_jN), I_kN) = -\frac{1}{r} \{\langle I_3N, I_3N \rangle + \langle I_2N, I_2N \rangle - \langle I_1N, I_1N \rangle\},$$

i.e. $\langle \Gamma^k_{12}I_kN, I_3N \rangle = -1/r$. It follows that $\Gamma^3_{12} = -1/r$ and due to (2) we have $\Gamma^3_{21} = 1/r$. Similarly it can be shown that $\Gamma^1_{13} = \Gamma^1_{23} = 1/r, \Gamma^2_{13} = \Gamma^2_{31} = -1/r$. Furthermore,

$$\nabla_{I_1N}(I_2N), I_1N) = \frac{1}{2}(I_2N) \langle I_1N, I_1N \rangle = 0$$

implies $\Gamma^k_{11} = 0$ and using the same argument we get $\Gamma^k_{ij} = 0$ whenever at least two of the indices $i, j, k$ are equal.

**Lemma 2.** Let $b_{ij}$ denote the components of the second fundamental form of $M$ with respect to the basis $I_1N, I_2N, I_3N$. Then

$$b_{ij} = -\frac{1}{r} \delta_{ij}.$$ 

**Proof.** We denote by $\hat{\nabla}$ the canonical connection in $\mathbb{R}^4$. Using Lemma 1 and the Gauss formula

$$\hat{\nabla}_{I_iN}(I_jN) = \nabla_{I_iN}(I_jN) + b_{ij}N$$

285
we can evaluate

\[ \hat{\nabla}_{I_i} N = -\hat{\nabla}_{I_i} (I_1^2 N) = -I_1 \hat{\nabla}_{I_i} (I_1 N) = -I_1 (\nabla_{I_i} (I_1 N) + b_{11} N) = -b_{11} I_i N, \]

\[ \hat{\nabla}_{I_i} N = -\hat{\nabla}_{I_i} (I_2^2 N) = -I_2 \hat{\nabla}_{I_i} (I_2 N) = -I_2 (\nabla_{I_i} (I_2 N) + b_{12} N) = \frac{1}{r} I_i N - b_{12} I_2 N, \]

\[ \hat{\nabla}_{I_i} N = -\hat{\nabla}_{I_i} (I_3^2 N) = -I_3 \hat{\nabla}_{I_i} (I_3 N) = -I_3 (\nabla_{I_i} (I_3 N) + b_{13} N) = \frac{1}{r} I_i N - b_{13} I_3 N. \]

Comparing the right hand sides of the above equations we get \( b_{11} = -1/r, b_{12} = b_{13} = 0 \). Proceeding along the same lines we find easily \( b_{22} = b_{33} = -1/r, b_{23} = 0 \).

The proof of our theorem follows now easily from Lemma 2, which in fact says that every point of \( M \) is umbilical. See e.g. Theorem 5.1 in Chap. VII of [1].

Reference


Authors' addresses: J. TROJÁK, 186 00 Praha 8, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK); J. VANŽURA, 771 46 Olomouc, Leninova 26, ČSSR (Přírodovědecká fakulta UP).