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SEQUENTIALLY COMPLETE SPACES

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In this paper we discuss sequential completeness of sequentially regular closure spaces. Sequential completions of convergence spaces are studied and several theorems on extensions of continuous mappings to sequentially complete convergence spaces are proved.

0. PRELIMINARIES

By a *closure operator* u for a set X we mean an operator which assigns to each subset A of X its closure $uA \subset X$ so that $u\emptyset = \emptyset$, $A \subset uA$, and $u(A \cup B) = uA \cup uB$. The pair (X, u) is a *closure space* (cf. [1]) and is often denoted simply by X. If u is idempotent, then X is a topological space. A *convergence space* (cf. [7]) is a closure space (X, λ) where the closure operator λ is induced by a sequential convergence on X, i.e. $\lambda A = \{x \mid x = \lim x_n, \bigcup(x_n) \subset A\}$. Let (X, u) be a closure space. The convergence of sequences in X is defined in the usual way, i.e. $\langle x_n \rangle$ converges to x iff each neighborhood of x contains x_n for all but finitely many $n \in N$, and throughout the paper we make a blanket assumption that in all spaces *every sequence converges to at most one point*. Denote by λ_u the corresponding closure for X. Clearly $(\lambda_u)_u = \lambda_u < u$, (X, u) is a *Fréchet space* iff $\lambda_u = u$ and X is topological, (X, u) is a *sequential space* iff $(\lambda_u)^{\omega_1} = u$ (μ^{ω_1} denotes the topological modification of a convergence closure operator μ, i.e. the finest topological closure operator coarser than μ), and a subset A is *sequentially closed* in X iff $\lambda_u A = A$. A subset A is said to be *sequentially dense* in X if $(\lambda_u)^{\omega_1} A = X$. We denote the set of all continuous or bounded continuous functions on X by $C(X)$, $C^*(X)$, respectively. F will denote a set of functions on X. However, if $F \subset C(X)$ and we want to stress that we deal with continuous functions we shall use C_0 instead of F. Denote $C_s((X, u)) = C((X, \lambda_u))$, i.e. $C_s(X)$ is the set of all sequentially continuous functions on X. Clearly $C(X) \subset C_s(X)$ and if X is a convergence or a sequential space, then $C(X) = C_s(X)$.

Definition 0.1. Let Y be a closure space, X a subspace of Y, and $F \subset C_s(X)$. The space X is said to be *sequentially F-embedded* in Y if each $f \in F$ has an extension $\bar{f} \in C_s(Y)$.

For $F \subset C(X)$ it follows that if X is F-embedded in Y, then it is also sequentially F-embedded in Y, and if X and Y are convergence or sequential spaces, then X is sequentially F-embedded in Y iff it is F-embedded in Y.

Definition 0.2. Let X be a set and $F \subset R^X$. A closure space (X, u) is said to be *F-sequentially regular* if the convergence of sequences in X is projectively generated by F, i.e. $\lim x_n = x$ iff for each $f \in F$ we have $\lim f(x_n) = f(x)$. If $F = C(X)$, then we simply say that X is *sequentially regular.*

Note that if a space is F-sequentially regular, then the uniqueness of sequential limits implies that F separates the points of the space.

Clearly, if X is F-sequentially regular, then $F \subset C_s(X)$. If $\lambda_u = u$, then we obtain the definition of F_0-sequential regularity from [8]. If, moreover, $F = C(X)$, then we have the sequential regularity defined in [7].

1. SEQUENTIALLY COMPLETE SPACES

Definition 1.1. Let X be a closure space and $F \subset R^X$. A sequence $\langle x_n \rangle$ of points of X is said to be *F-fundamental* if for each $f \in F$ the sequence $\langle f(x_n) \rangle$ converges in R.

Note that if $F \subset C_s(X)$, in particular if X is F-sequentially regular, then every convergent sequence in X is F-fundamental.

Lemma 1.2. *Every F-fundamental sequence in an F-sequentially regular closure space is either convergent or totally divergent.*

Proof. Let X be an F-sequentially regular closure space and let $\langle x_n \rangle$ be an F-fundamental sequence which is not totally divergent in X. Then there is a subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$ converging to a point x in X. Consequently, we have $\lim f(x'_n) = f(x)$ and therefore also $\lim f(x_n) = f(x)$. Since X is F-sequentially regular, we have $\lim x_n = x$.

We define an equivalence relation on the set of all F-fundamental sequences in an F-sequentially regular closure space as follows: $\langle x_n \rangle \sim \langle y_n \rangle$ whenever $\lim f(x_n) = \lim f(y_n)$ for each $f \in F$. The equivalence class containing $\langle x_n \rangle$ will be denoted by $[\langle x_n \rangle]$.

Lemma 1.3. *Let* $\langle y_n \rangle \in [\langle x_n \rangle]$, $\langle z_n \rangle \in [\langle x_n \rangle]$, *and* $\lim y_n = y$. *Then* $\lim z_n = y$.
The easy proof is omitted.

Corollary 1.4. *The set of all equivalence classes $[\langle x_n \rangle]$ is the union of two disjoint sets, one consisting of all equivalence classes containing a constant sequence and the other consisting of all equivalence classes containing only totally divergent sequences.*

Definition 1.5. Let X be a closure space and $F \subset R^X$. The space X is said to be *F-sequentially complete* if every F-fundamental sequence converges in X. If $F = C(X)$, then we simply say that X is *sequentially complete.*

Note that an F-sequentially complete closure space need not be F-sequentially regular. E.g. if $F = R^X$, then X is always F-sequentially complete but X is F-sequentially regular iff it is discrete.

Definition 1.6. Let X be a closure space and $F \subset R^X$. The space X is said to have the *property p with respect to F* if

(p) for every two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of points of X such that $(\lambda_u \bigcup(x_n)) \cap (\lambda_u \bigcup(y_n)) = \emptyset$ there is a function $f \in F$ such that $\lim f(x_n) = \lim f(y_n)$ does not hold.

If $F = C(X)$, then we simply say that X has the *property p.*

This property has been introduced in [2] for convergence spaces and $F \subset C(X)$. It has been studied in [3] in the special case when $F = C(X) \cap E^X$, where $E \subset R$.

Lemma 1.7. *Let X be a closure space which has the property p with respect to $F \subset C_s(X)$. Then:*

(i) *X is F-sequentially regular.*

(ii) *X has the property p with respect to each F', $F \subset F' \subset R^X$.*

(iii) *If $F = C(X)$ or $F = C_s(X)$, then X has the property p with respect to $C^*(X)$ or $C_s^*(X)$, respectively.*

Proof. (i) and (ii) follow immediately. (iii) was proved in [3] (Corollary 1.8) for convergence spaces and the proof can be easily extended to closure spaces.

Theorem 1.8. *Let (X, u) be an F-sequentially regular closure space. Then the following statements are equivalent.*

(i) *X is F-sequentially complete.*

(ii) *X has the property p with respect to F.*

(iii) *X is sequentially closed in every closure space (Y, v) in which it is sequentially F-embedded.*

Proof. non (iii) implies non (ii). If $z \in \lambda_v X - X$, then there is a one-to-one sequence $\langle z_n \rangle$, $z_n \in X$, converging to a point z in Y. We have $(\lambda_u \bigcup(z_{2n-1})) \cap (\lambda_u \bigcup(z_{2n})) = \emptyset$. Since X is sequentially F-embedded in Y, we have $\lim f(z_{2n-1}) = \lim f(z_{2n})$ for each $f \in F$.

non (ii) implies non (i). Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences of points of X such that $(\lambda_u \bigcup(x_n)) \cap (\lambda_u \bigcup(y_n)) = \emptyset$ and suppose that for each $f \in F$ we have $\lim f(x_n) = \lim f(y_n)$. For $n \in N$ put $z_{2n-1} = x_n$, $z_{2n} = y_n$. Then $\langle z_n \rangle$ is an F-fundamental sequence not converging in X.

non (i) implies non (iii). Let $\langle x_n \rangle$ be a totally divergent F-fundamental sequence in X. Denote by $x_0 = [\langle x_n \rangle]$ the equivalence class containing $\langle x_n \rangle$ and put $Y =$

$= X \cup (x_0)$. For each $f \in F$ define a function $\bar{f}$ on Y as follows: $\bar{f}(x) = f(x)$ for $x \in X$, $\bar{f}(x_0) = \lim f(x_n)$. Put $\bar{F} = \{\bar{f} \mid f \in F\}$. Define a closure v for Y as follows: $y \in vA$ if either $y \in u(A \cap X)$ or there is a sequence $\langle y_n \rangle$ in A such that for each $\bar{f} \in \bar{F}$ we have $\lim \bar{f}(y_n) = \bar{f}(y)$. Then (X, u) is a sequentially F-embedded subspace of (Y, v) and X is not sequentially closed in Y. Note that X is open and hence sequentially open in Y.

Let $\mathscr{P}$-space stand for one of the following: convergence space, sequential space, closure space, topological space. We have

Theorem 1.9. *Let (X, u) be a C_0-sequentially regular $\mathscr{P}$-space, $C_0 \subset C(X)$. Then the following statements are equivalent.*

(i) *X is C_0-sequentially complete.*
(ii) *X is sequentially closed in every $\mathscr{P}$-space Y in which it is sequentially C_0-embedded.*
(iii) *X is sequentially closed in every $\mathscr{P}$-space Y in which it is C_0-embedded.*

Proof. (i) implies (ii) by Theorem 1.8. Since $C_0 \subset C(X)$ and hence a C_0-embedded subspace is also sequentially C_0-embedded, it follows that (ii) implies (iii). It remains to prove that non (i) implies non (iii).

(1) Let (X, u) be a convergence space. Then the space (Y, v) constructed in the third part of the proof of Theorem 1.8 is a convergence space as well and the assertion follows immediately.

(2) Let (X, u) be a sequential space. Let (X, λ_u) be the associated convergence space. It follows from Theorem 3.3 and Theorem 3.6 in [3] that (X, u) is C_0-sequentially regular (complete) iff (X, λ_u) is C_0-sequentially regular (complete). We have just proved in (1) that there is a convergence space (Y, μ) such that (X, λ_u) is its open but not sequentially closed subspace which is sequentially C_0-embedded, and hence C_0-embedded, in (Y, μ). Let v be the topological modification of μ. The space (Y, v) is sequential and, X being open, (X, u) is a subspace of (Y, v). Clearly, (X, u) is C_0-embedded in (Y, v).

(3) Let (X, u) be a closure space. Let $\langle x_n \rangle$ be a totally divergent C_0-fundamental sequence in X. Denote by $x_0 = [\langle x_n \rangle]$ the equivalence class containing $\langle x_n \rangle$ and let $Y = X \cup (x_0)$. For each $f \in C_0$ define a function $\bar{f}$ on Y as follows: for $x \in X$ let $\bar{f}(x) = f(x)$ and let $\bar{f}(x_0) = \lim f(x_n)$. Denote by $\bar{C}_0$ the family of all such extensions. Define a closure operator v for Y as follows. Let $A \subset Y$. Then for $y \neq x_0$ let $y \in vA$ iff $y \in u(A \cap X)$ and let $x_0 \in vA$ iff for each $g \in \bar{C}_0$ we have $g(x_0) \in \operatorname{cl} g[A]$. It is easy to see that (X, u) is a C_0-embedded subspace of (Y, v).

(4) Let (X, u) be a topological space. Clearly it suffices to prove that the closure space (Y, v) constructed in (3) is a topological space, i.e. $v(vA) = vA$ for each subset A of Y. Let $A \subset Y$ and let $y \in v(vA)$, $y \neq x_0$. Then $y \in u((vA) \cap X) = u(u(A \cap X)) = u(A \cap X) \subset vA$. Now let $x_0 \notin vA$. Then there is $g \in \bar{C}_0$ such that $g(x_0) \notin \operatorname{cl} g[A]$,

and $vA = u(A \cap X) = uA$. Since $g \mid X \in C_0 \subset C(X)$, we have $g[uA] \in \operatorname{cl} g[A]$ and hence $\operatorname{cl} g[uA] \subset \operatorname{cl} g[A]$. Thus $g(x_0) \notin \operatorname{cl} g[uA] = \operatorname{cl} g[vA]$ which implies $x_0 \notin$ $\notin v(vA)$. Consequently $v(vA) = vA$. This completes the proof.

Remark 1.10. From the proof of Theorem 1.9 it follows that in (ii) and (iii) we can require the $\mathscr{P}$-space Y to be $\bar{C}_0$-sequentially regular, $\bar{C}_0 \mid X = C_0$ or sequentially regular, respectively, and to contain X as a sequentially dense open subset. A similar remark holds for Theorem 1.8.

Theorem 1.9 can be extended to completely regular spaces provided that C_0 determines the topology of the underlying space.

Theorem 1.11. *Let X be a completely regular space and $C_0 \subset C(X)$ a subset which determines the topology of X. Then the following statements are equivalent.*

(i) *X is C_0-sequentially complete.*
(ii) *X is sequentially closed in every completely regular space in which it is sequentially C_0-embedded.*
(iii) *X is sequentially closed in every completely regular space in which it is C_0-embedded.*

Proof. It is easy to see that X is C_0-sequentially regular. Thus (i) implies (ii) which implies (iii) by Theorem 1.9. The remaining implication follows also from Theorem 1.9. Indeed, the topological space (Y, v) constructed in the proof of Theorem 1.9, parts (3) and (4), is completely regular. This follows from the fact that $\bar{C}_0$ determines the topology of Y.

Let $\mathscr{P}$-space stand for one of the following: closure space, convergence space, topological space, sequential space, completely regular space. Using Lemma 1.7 we have

Corollary 1.12. *Let X be a sequentially regular $\mathscr{P}$-space. Then the following statements are equivalent.*

(i) *X is sequentially complete.*
(ii) *X has the property p.*
(iii) *X is sequentially closed in every $\mathscr{P}$-space in which it is C-embedded.*
(iv) *X is sequentially closed in every $\mathscr{P}$-space in which it is C^*-embedded.*

The following example shows that in Theorem 1.9, $\mathscr{P}$-space cannot stand for Fréchet space and that Theorem 1.11 fails if all spaces in question are assumed to be Fréchet.

Example 1.13. Let $X = \bigcup_{m=1} \bigcup_{n=0} (x_{mn})$. Define a topology u for X as follows: for $n > 0$ all points x_{mn} are isolated and for each x_{m0} the family of all sets of the form

$(x_{m0}) \cup (\bigcup_{m=i} (x_{mn}))$ is a local base at x_{m0}. For each $k: N \to N$ and $l \in N$ define $f: X \to \to \{0, 1\}$ as follows:

$$f(x) = 1 \quad \text{for} \quad x \in (\bigcup_{m=l} (x_{m0})) \cup (\bigcup_{m=l} \bigcup_{n=k(m)} (x_{mn})),$$

and

$$f(x) = 0 \text{ otherwise}.$$

Denote by C_0 the family of all such functions. Then:
(a) (X, u) is a Fréchet completely regular space.
(b) $C_0 \subset C(X)$ and C_0 determines the topology of X.
(c) (X, u) is C_0-sequentially regular.
(d) $\langle x_{m0} \rangle$ is a totally divergent C_0-fundamental sequence.
(e) If $\langle y_n \rangle$ is a totally divergent C_0-fundamental sequence in X, then $\langle y_n \rangle \in [\langle x_{m0} \rangle]$ and $y_n \in \bigcup_{m=1} (x_{m0})$ for all but finitely many $n \in N$.
(f) (X, u) is not C_0-sequentially complete.
(g) X is sequentially closed in every Fréchet space in which (X, u) is sequentially C_0-embedded.

Corollary 1.14. *Let X be a completely regular space. Then the following statements are equivalent.*
(i) *X is sequentially complete.*
(ii) *X is sequentially closed in the Hewitt realcompactification υX of X.*
(iii) *X is sequentially closed in the Čech-Stone compactification βX of X.*

Proof. Since a completely regular space is sequentially regular, (i) implies (ii) by Corollary 1.12.

non (i) implies non (ii). It follows from Theorem 1.11 that there is a completely regular space Y such that X is a sequentially dense C-embedded proper subspace of Y. Thus Y is homeomorphic to a subspace of υX and the homeomorphism leaves X pointwise fixed (cf. [4]). Consequently, X is not sequentially closed in υX.

The equivalence of (i) and (iii) can be proved in the same way.

Let X be a completely regular space. Consider υX as a subspace of βX. From Corollary 1.14 follows the known result (cf. [2], [3], [5]) that a realcompact space is sequentially complete. Thus υX is sequentially closed in βX. Denote by σX the smallest sequentially closed subset of βX containing X. The next corollary is a generalization of Theorem 8 in [2].

Corollary 1.15. $X \subset \sigma X \subset \upsilon X \subset \beta X$.

Remark 1.16. The notion of F-sequential completeness generalizes several previous definitions of sequential completeness. When $F = C(X)$ and X is a sequentially regular convergence space we obtain $\mathscr{L}$-completeness defined in [6]. By Theorem 1.9

when $F \subset C(X)$ and X is an F-sequentially regular convergence (sequential) space we obtain C_0-sequential completeness defined in [3]. By Corollary 1.14 when $F = C^*(X)$ and X is a completely regular space we obtain sequential completeness defined in [5].

2. SEQUENTIAL COMPLETION

Now we shall consider C_0-sequentially regular convergence spaces. Since not all sequentially regular spaces are sequentially complete (see e.g. [6]) it is natural to consider a suitable sequentially complete convergence space into which a given space can be embedded as a sequentially dense subspace. The following definition has been introduced by J. Novák in [8].

Definition 2.1. Let (L, λ) be a C_0-sequentially regular convergence space. A convergence space (S, σ) is said to be a C_0-*sequential envelope* $\sigma_0(L)$ of (L, λ) if
(e_1) (L, λ) is a sequentially dense C_0-embedded subspace of (S, σ);
(e_2) (S, σ) is $\bar{C}_0(S)$-sequentially regular where $\bar{C}_0(S) = \{f \in C(S) \mid f|L \in C_0\}$; and
(e_3) there is no convergence space (S', σ') containing (S, σ) as a proper subspace and satisfying (e_1) and (e_2) with respect to (L, λ).

The C_0-sequential envelope is unique in the sense that if S_1 and S_2 are C_0-sequential envelopes of L, then there is a homeomorphism of S_1 onto S_2 that leaves L pointwise fixed (Theorem 5 in [8]) and we write $S_1 = S_2$.

The second part of the following theorem was announced in [2] (Theorem 6).

Theorem 2.2. *In Definition* 2.1, *the condition* (e_3) *is equivalent to either of the following conditions*:
(i) *S is sequentially closed in every convergence space in which it is $\bar{C}_0(S)$-embedded.*
(ii) *S is $\bar{C}_0(S)$-sequentially complete.*
Moreover, the conditions (e_2) *and* (e_3) *are equivalent to*
(iii) *S has the property p with respect to $\bar{C}_0(S)$.*

Proof. It is easy to see that (e_3) is equivalent to (i) and hence, by Theorem 1.9, to (ii). To prove the second assertion note that a convergence space having the property p with respect to $\bar{C}_0(S)$ is $\bar{C}_0(S)$-sequentially regular by Lemma 1.7. The statement then follows from the first assertion and Theorem 1.8.

Corollary 2.3. *Let (L, λ) be a C_0-sequentially regular convergence space. Then the following statements are equivalent.*
(i) $\sigma_0(L) = (L, \lambda)$.
(ii) *(L, λ) is C_0-sequentially complete.*

Proof. The assertion follows from Theorem 2.2 (condition (ii)).

Corollary 2.4. *$\sigma_0(L)$ is a C_1-sequential envelope of (L, λ) for each C_1 such that $C_0 \subset C_1 \subset C(\sigma_0(L)) \mid L$.*

Proof. The assertion follows from Theorem 2.2 (condition (iii)) and Lemma 1.7.

Corollary 2.5. *$\sigma_1(\sigma_0(L)) = \sigma_0(L)$ for each C_1 such that $\bar{C}_0(\sigma_0(L)) \subset C_1 \subset$ $\subset C(\sigma_0(L))$.*

Proof. The assertion follows from Theorem 2.2 (condition (iii)) and Lemma 1.7.

The C_0-sequential envelope of a C_0-sequentially regular convergence space L can be constructed by embedding the space L into a suitable product of real lines (see [8]). Here we give a direct construction by successive adjoining of "ideal points" to the given space. The main reason is that this method will be used to prove Theorem 3.1. This method was developed in [6] for sequentially regular convergence spaces.

Theorem 2.6. *Let (L, λ) be a C_0-sequentially regular convergence space. Put $(L_0, \lambda_0) = (L, \lambda)$. For each $\xi \leqq \omega_1$ there is a convergence space (L_ξ, λ_ξ) with the following properties:*

(a) *(L_η, λ_η) is a subspace of (L_ξ, λ_ξ) for each $\eta \leqq \xi \leqq \omega_1$.*

(b) $L_\xi = \lambda_\xi^\xi L_0$.

(c) For each $\xi \leqq \omega_1$ the space (L_ξ, λ_ξ) is $\bar{C}_0(L_\xi)$-sequentially regular, where $\bar{C}_0(L_\xi) = \{f \in C(L_\xi) \mid f \mid L_0 \in C_0\}$.

(d) For each $\eta \leqq \xi$ the mapping $h_\eta(f) = f \mid L_\eta$ of $\bar{C}_0(L_\xi)$ into $\bar{C}_0(L_\eta)$ is one-to-one and onto.

(e) The space $(L_{\omega_1}, \lambda_{\omega_1})$ is $\bar{C}_0(L_{\omega_1})$-sequentially complete.

Proof. We shall use transfinite induction. Conditions (a) through (d) are clearly satisfied for $\xi = 0$. Suppose that the spaces (L_η, λ_η) with the required properties are already defined for each $\eta < \xi \leqq \omega_1$.

I. Let $\xi = \zeta + 1$. The space (L_ζ, λ_ζ) is $\bar{C}_0(L_\zeta)$-sequentially regular by (c). Let M_ζ be the set of all equivalence classes of $\bar{C}_0(L_\zeta)$-fundamental sequences in L_ζ which contain only totally divergent sequences. Put $L_\xi = L_\zeta \cup M_\zeta$. For each $f \in \bar{C}_0(L_\zeta)$ define a function $\bar{f}$ on L_ξ as follows: $\bar{f}(x) = f(x)$ for $x \in L_\zeta$ and $\bar{f}(x) = \lim f(x_n)$ for $x = [\langle x_n \rangle] \in M_\zeta$. Denote by $F = \{\bar{f} \mid f \in \bar{C}_0(L_\zeta)\}$ the family of all such extensions. Let λ_ξ be the convergence closure for L_ξ projectively generated by F, i.e. $x \in \lambda_\xi A$ iff there is a sequence $\langle x_n \rangle$ in A such that for each $\bar{f} \in F$ we have $\lim \bar{f}(x_n) = \bar{f}(x)$. It is easy to see that (L_ξ, λ_ξ) satisfies the conditions (a) through (d) and that $F = \bar{C}_0(L_\xi)$.

II. Let ξ be a limit ordinal. Let $L_\xi = \bigcup_{\eta < \xi} L_\eta$. Let $f \in C_0$ and $x \in L_\xi$. Then there is a least ordinal $\zeta < \xi$ such that $x \in L_\zeta$. By (d) there is a unique $g \in \bar{C}_0(L_\zeta)$ such that $f = g \mid L_0$. Put $\bar{f}(x) = g(x)$. Thus for each $f \in C_0$ we have defined a unique extension $\bar{f}$ on L_ξ. The convergence closure λ_ξ for L_ξ is defined in the same way as in case I. Again, it is easy to prove that the required conditions are satisfied and that $\{\bar{f} \mid f \in C_0\} = \bar{C}_0(L_\xi)$.

To prove the last statement let $\langle x_n \rangle$ be a $\bar{C}_0(L_{\omega_1})$-fundamental sequence. Let $\xi < \omega_1$ be the least ordinal such that $\bigcup(x_n) \subset L_\xi$. It follows from the construction of $L_{\xi+1}$ that $\langle x_n \rangle$ converges in $L_{\xi+1} \subset L_{\omega_1}$.

Theorem 2.7. *Let (L, λ) be a C_0-sequentially regular convergence space. The space $(L_{\omega_1}, \lambda_{\omega_1})$ is a C_0-sequential envelope of (L, λ).*

Proof. (a), (b), (c) and (d) in Theorem 2.6 imply for $\xi = \omega_1$ that conditions (e_1) and (e_2) are satisfied. Since $(L_{\omega_1}, \lambda_{\omega_1})$ is $\bar{C}_0(L_{\omega_1})$-sequentially complete, the assertion follows from Theorem 2.2.

3. EXTENSION OF MAPPINGS

Theorem 3.1. *Let φ be a continuous mapping of a $C_1(L)$-sequentially regular convergence space (L, λ) into a $C_2(M)$-sequentially regular convergence space (M, μ). If $\varphi \circ C_2(M) \subset C_1(L)$, then there is a uniquely determined continuous mapping $\bar{\varphi}$ of $\sigma_1(L)$ into $\sigma_2(M)$ such that the diagram*

$$\begin{array}{ccc} L & \longrightarrow & M \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ \sigma_1(L) & \longrightarrow & \sigma_2(M) \end{array}$$

commutes.

Proof. Let $\sigma_1(L) = (S_1, \sigma_1)$ and $\sigma_2(M) = (S_2, \sigma_2)$. By Theorems 2.6 and 2.7 there is a transfinite sequence of spaces (L_ξ, λ_ξ), $\xi \leqq \omega_1$, such that $(L_0, \lambda_0) = (L, \lambda)$ and $(L_{\omega_1}, \lambda_{\omega_1}) = (S_1, \sigma_1)$. Using transfinite induction we shall prove that for each $\xi \leqq \omega_1$ there is a continuous mapping $\varphi_\xi : L_\xi \to S_2$ such that

(a) $\varphi_\xi \mid L_\eta = \varphi_\eta$ for each $\eta \leqq \xi$, and

(b) $\varphi_\eta \circ \bar{C}_2(S_2) \subset \bar{C}_1(L_\eta)$ for each $\eta \leqq \xi$.

We then put $\bar{\varphi} = \varphi_{\omega_1}$. Since $\bar{\varphi} \mid L = \varphi$ and L is sequentially dense in S_1, the uniqueness of $\bar{\varphi}$ follows by a standard topological argument (see e.g. Lemma 5 in [7]) and the theorem will be proved.

Let $\varphi_0 = \varphi$. The assertion clearly holds. Assume that it holds for all $\eta < \xi \leqq \omega_1$.

I. Let $\xi = \zeta + 1$. We have $L_\xi = L_\zeta \cup M_\zeta$ where M_ζ is the set of all equivalence classes containing totally divergent $\bar{C}_1(L_\zeta)$-fundamental sequences. Let $x = [\langle x_n \rangle] \in M_\zeta$. Since $\varphi_\zeta \circ \bar{C}_2(S_2) \subset \bar{C}_1(L_\zeta)$, the sequence $\langle \varphi_\zeta(x_n) \rangle$ is $\bar{C}_2(S_2)$-fundamental and hence converges in S_2. We define $\varphi_\xi(x) = \lim \varphi_\zeta(x_n)$. For $x \in L_\zeta$ we define $\varphi_\xi(x) = \varphi_\zeta(x)$. Clearly (a) is satisfied.

Let $g \in \bar{C}_2(S_2)$. By (a) we have $\varphi_\xi \circ g \mid L_\zeta = \varphi_\zeta \circ g \in \bar{C}_1(L_\zeta)$. It follows from the construction of (L_ξ, λ_ξ) that $\varphi_\xi \circ g \in \bar{C}_1(L_\xi)$, and hence (b) holds.

Finally, let $\lim z_n = z$ in L_ξ and let $f \in \bar{C}_2(S_2)$. By (b) we have $\varphi_\xi \circ f \in \bar{C}_1(L_\xi)$ and therefore $\lim (\varphi_\xi \circ f)(z_n) = (\varphi_\xi \circ f)(z)$. Hence $\langle f(\varphi_\xi(z_n))\rangle$ converges to $f(\varphi_\xi(z))$ for each $f \in \bar{C}_2(S_2)$. Since S_2 is $\bar{C}_2(S_2)$-sequentially regular, $\langle \varphi_\xi(z_n)\rangle$ converges to $\varphi_\xi(z)$ in S_2. This proves the continuity of φ_ξ.

II. Let ξ be a limit ordinal. Since $L_\xi = \bigcup_{\eta<\xi} L_\eta$, the mapping φ_ξ defined by $\varphi_\xi(x) = \varphi_\eta(x)$ for $x \in L_\eta$ is a well-defined mapping on L_ξ and satisfies (a). By the assumption, $\varphi_\eta \circ f \in \bar{C}_1(L_\eta)$ for each $f \in \bar{C}_2(S_2)$ and each $\eta < \xi$. It follows from the construction of (L_ξ, λ_ξ) that also $\varphi_\xi \circ f \in \bar{C}_1(L_\xi)$ for each $f \in \bar{C}_2(S_2)$ and hence (b) holds. The continuity of φ_ξ can be proved in the same way as in the case of an isolated ordinal.

For $M = L$, $\varphi = id$, we obtain Theorem 6 from [8] as a special case.

From Theorem 3.1 and Corollary 2.3 we obtain the following

Corollary 3.2. *Let φ be a continuous mapping of a $C_1(L)$-sequentially regular convergence space (L, λ) into a $C_2(M)$-sequentially regular convergence space (M, μ), and let (M, μ) be $C_2(M)$-sequentially complete. If $\varphi \circ C_2(M) \subset C_1(L)$, then there is a uniquely determined continuous mapping $\bar{\varphi}$ of $\sigma_1(L)$ into M such that the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & M \\ {\scriptstyle id}\downarrow & & \uparrow{\scriptstyle \bar{\varphi}} \\ \sigma_1(L) & \text{———————} & \end{array}$$

commutes.

Example 3.3. To see that the condition in Theorem 3.1 is not necessary consider a sequentially regular convergence space (L, λ) which is not pseudocompact (e.g. put $L = R$). Put $(M, \mu) = (L, \lambda)$, $C_1(L) = C^*$, $C_2(L) = C$, $\varphi = id$. It was proved in [2] that $\sigma_1(L) = \sigma_2(L)$. The assertion of the theorem follows, while clearly $\varphi \circ C_2(L) = C \not\subset C^* = C_1(L)$.

To obtain a condition both sufficient and necessary we have to consider also the family $C(\sigma_1(L))$. We have the following

Theorem 3.4. *Let φ be a continuous mapping of a $C_1(L)$-sequentially regular convergence space (L, λ) into a $C_2(M)$-sequentially regular convergence space (M, μ). Let $C_3(L) = \{f \mid f = g \mid L,\ g \in C(\sigma_1(L))\}$. There is a uniquely determined continuous mapping $\bar{\varphi}$ of $\sigma_1(L)$ into $\sigma_2(M)$ such that the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & M \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ \sigma_1(L) & \xrightarrow{\bar{\varphi}} & \sigma_2(M) \end{array}$$

commutes if and only if $\varphi \circ C_2(M) \subset C_3(L)$.

Proof. Since $C_1(L) \subset C_3(L) \subset C(L)$, it follows from Definition 0.2 that L is $C_3(L)$-sequentially regular. By Corollary 2.4 we have $\sigma_3(L) = \sigma_1(L)$. The sufficiency of the condition follows from Theorem 3.1.

To prove the necessity assume that, on the contrary, there is $h \in C_2(M)$ such that $\varphi \circ h \notin C_3(L)$. Then the function $f = \varphi \circ h$ cannot be continuously extended to $\sigma_3(L) = \sigma_1(L)$. On the other hand, h can be continuously extended to $\bar{h} \in C(\sigma_2(M))$ and it follows that $\bar{\varphi} \circ \bar{h}$ is a continuous extension of $\varphi \circ h$. This is a contradiction.

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