Blanka Kutinová; Teo Sturm
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ON ALGEBRAIC CLOSURES OF COMPACT ELEMENTS*)

BLANKA KUTINOVÁ, TEO STURM, Praha

(Received August 13, 1976)

It will be shown that an algebraic closure operator \( f : L \to L \) on a complete lattice \( \mathcal{L} = (L; \leq) \) has the following property: if \( c \in L \) is compact in \( \mathcal{L} \), then \( f(c) \) is compact in \( (f(L); \leq) \). If the lattice \( \mathcal{L} \) is algebraic, then \( (f(L); \leq) \) is algebraic as well and an element is compact in \( (f(L); \leq) \) iff it is an image by \( f \) of an element which is compact in \( \mathcal{L} \). This implies, i.e., some well-known results concerning lattices of subalgebras and those of congruences on universal algebras.

1. Remark. We shall use the following current terminology (see [1], chap. II.1, or [2] e.g.). Let \( \mathcal{L} = (L; \leq) \) be a complete lattice. A subset \( A \subseteq L \) is called a closure system of \( \mathcal{L} \), if for every set \( X \subseteq A \), there is \( \inf X = \inf X \). If \( A \) is a closure system of \( \mathcal{L} \), then \( (A; \leq) \) is a complete lattice.

Let \( A \) be a closure system of \( \mathcal{L} \). A mapping \( f_A : L \to L \) defined by

\[
f_A(x) = \inf_{(A; \leq)} \{ y \in A \mid x \leq y \} \quad (x \in L)
\]

is called closure operator on \( \mathcal{L} \) corresponding to \( A \) [every closure operator \( g \) on \( \mathcal{L} \) corresponds to the closure system \( g(L) \) of \( \mathcal{L} \)].

A closure system \( A \) is called algebraic, if for every non-empty chain \( B \) in \( (A; \leq) \), it holds \( \sup B = \sup B \). A closure operator is called algebraic, if the corresponding closure system is an algebraic one.

An element \( c \in L \) is called compact in \( L \), if every set \( X \subseteq L \) such that \( c \leq \sup X \), contains a finite subset \( X' \subseteq X \) with \( c \leq \sup X' \). We say that a set \( B \) generates \( \mathcal{L} \) by joins, if for every element \( x \in L \) there exists a set \( B(x) \subseteq B \) such that \( x = \sup_B B(x) \).

A complete lattice \( \mathcal{L} \) is called algebraic, if the set of all its compact elements generates it by joins.

*) This paper has originated at the seminar “Algebraic Foundations of Quantum Theories”, directed by Prof. Jiří Fábera.
2. **Theorem.** Let $A$ be an algebraic closure system of a complete lattice $\mathcal{L} = (L; \leq)$. If $c$ is compact in $\mathcal{L}$, then $f_A(c)$ is compact in $(A; \leq)$.

**Proof.** Let $f_A(c)$ be not compact in $(A; \leq)$. Then there exists at least one set $X$ satisfying the following condition:

(1) $X \subseteq A$, $f_A(c) \subseteq \sup X$; no finite subset $X' \subseteq X$ satisfies relation $f_A(c) \subseteq \sup X'$.

Among all sets $X$ satisfying condition (1), consider the sets with the smallest cardinality; choose one of them and denote it by $Y$. The set $Y$ is infinite by (1). Following the Axiom of Choice, there exists such a well-ordering $\leq$, that the ordinal type of $(Y; \leq)$ is the initial ordinal of the cardinality $|Y|$. For every $y \in Y$ put

$$z(y) = \text{Df} \sup_{(A; \leq)} \{ x \mid x \in X, \ x \leq y \}.$$ 

Then for $y_1, y_2 \in Y$, $y_1 \leq y_2$, there is $z(y_1) \leq z(y_2)$, hence the set $Z = \text{Df} \{ z(y) \mid y \in Y \}$ together with $\leq$ forms a chain. With respect to the algebraicity of $A$, we get

$$\sup Z = \sup_{(A; \leq)} Z.$$ 

Since for every $y \in L$ there is $\{ x \mid x \in Y, x \leq y \} \subseteq Y$, then $z(y) \leq \sup Y$. This implies the relation $\sup Z \leq \sup Y$. On the other hand, there is for every $y \in Y$, $y \in \{ x \mid x \in Y, x \leq y \}$, hence $y \leq z(y)$.

It yields

$$\sup Z = \sup_{(A; \leq)} Z = \sup Y.$$ 

Following the choice of $Y$ we have $f_A(c) \subseteq \sup Y$, thus $f_A(c) \subseteq \sup Z$, as well. There is $c \leq f_A(c)$, hence $c \leq \sup Z$. The element $c$ is compact in $\mathcal{L}$; then there exists a finite set $\{ y_1, \ldots, y_n \} \subseteq Y$ with $c \leq \sup \{ z(y_1), \ldots, z(y_n) \}$. Moreover, $(Z; \leq)$ being a chain, $\{ z(y_1), \ldots, z(y_n) \}$ contains the greatest element with respect to $\leq$; denotes it by $z(y_i)$. It holds

(2) $$c \leq \sup_{(A; \leq)} \{ z(y_1), \ldots, z(y_n) \} = z(y_i) = \sup_{(A; \leq)} \{ x \mid x \in Y, x \leq y_i \}.$$ 

$f_A$ is a closure operator, specially it is an isotonic mapping of $\mathcal{L}$ to $\mathcal{L}$. This fact together with (2) yields

(3) $$f_A(c) \leq f_A \left( \sup_{(A; \leq)} \{ x \mid x \in Y, x \leq y_i \} \right) = \sup_{(A; \leq)} \{ x \mid x \in Y, x \leq y_i \}$$

(the elements of $A$ are fixed by $f_A$). By assumption, the ordinal type of the chain $(Y; \leq)$ is the smallest one of the cardinality $|Y|$; hence the cardinality of a proper
section \( \{ x \in Y \mid x \leq y \} \) is strictly smaller than \( |Y| \). Following (3) and the choice of \( Y \) [as a set with the smallest cardinality satisfying (1)], one can choose a finite subset \( Y' \subseteq \{ x \in Y \mid x \leq y \} \) with \( f_A(c) \leq \sup Y' \). However, \( Y' \) being a finite subset of \( Y \), \( Y \) does not satisfy condition (1). This contradiction yields the compacticity of \( f_A(c) \) in \( (A; \leq) \). The theorem is proved.

3. **Lemma.** (Ward). Let \( \mathcal{L} = (L; \leq) \) be a complete lattice and let \( f : L \to L \) be a closure operator on \( \mathcal{L} \). Then

\[
f(\sup X) = \sup_{(f(L); \leq)} X
\]

for every \( X \subseteq f(L) \).

**Proof.** See [5], p. 76, Theorem 15 e.g.*. [See also [6], Section 1.6.]

4. **Lemma.** Let \( A \) be a closure system of a complete lattice \( \mathcal{L} = (L; \leq) \). If a set \( B \) generates \( \mathcal{L} \) by joins, then \( f_A(B) \) generates by joins the complete lattice \( (A; \leq) \).

**Proof.** For every \( x \in L \), there exists, \( B(x) \subseteq B \) such that \( x = \sup_{\not\in} B(x) \). The mapping \( f_A : L \to L \) is a closure operator in \( \mathcal{L} \), hence it holds

\[
x = \sup_{\not\in} B(x) \leq \sup_{\not\in} f_A(B(x)) \leq f_A(\sup_{\not\in} f_A(B(x)))
\]

This yields

\[
f_A(x) \leq f_A(\sup_{\not\in} f_A(B(x))) = f_A(\sup_{\not\in} f_A(B(x))) = f_A(\sup_{(A; \leq)} f_A(B(x)))
\]

[the latest equality holds following Section 3: it is \( A = f_A(L) \)].

On the other hand, there is \( y \leq x \) for every \( y \in B(x) \), thus \( f_A(y) \leq f_A(x) \), as well; hence

\[
\sup_{\not\in} f_A(B(x)) \leq f_A(x).
\]

Since \( f_A \) is isotonic and \( f_Af_A = f_A \), we get, by Section 3, the inequality

\[
\sup_{(A; \leq)} f_A(B(x)) = f_A(\sup_{\not\in} f_A(B(x))) \leq f_A f_A(x) = f_A(x).
\]

5. **Theorem.** Let \( \mathcal{L} = (L; \leq) \) be an algebraic lattice and let \( A \) be an algebraic closure system of \( \mathcal{L} \). Then \( (A; \leq) \) is an algebraic lattice.

**Proof** follows immediately from Sections 2 and 4: Let \( C \) denote the set of compact elements in \( \mathcal{L} \). Then \( C \) generates \( \mathcal{L} \) by joins and, following Lemma 4, \( f_A(C) \) generates \( (A; \leq) \) by joins. Moreover, all elements of \( f_A(C) \) are compact in \( (A; \leq) \) by Theorem 2.

*) It is easy to prove a more general statement: Let \( \mathcal{L} = (L; \leq) \) be a complete lattice and let \( f : L \to L \) be a closure operator on \( \mathcal{L} \). Then \( f(\sup X) = \sup_{(f(L); \leq)} f(X) \) for every \( X \subseteq L \).
6. Remark. Let \( \mathcal{L} = (L; \leq) \) be a complete lattice and let \( f_A \) be an algebraic closure operator on \( \mathcal{L} \). The converse of this statement is, in general, not true, since lattice \( (A; \leq) \) could contain some compact elements, which are not \( f_A \)-images of any compact element in \( \mathcal{L} \).

Consider the closed interval \( I = \langle 0, 1 \rangle \) in the set of real numbers with the usual ordering, \( A = \{0, 1\} \). Then \( A \) is an algebraic system of closed elements in \( \mathcal{L} \) and

\[
 f_A(0) = 0, \quad f_A(x) = 1 \quad \text{for every} \quad x \in (0, 1).
\]

Further, 1 is compact in \( (A; \leq) \) although no element of \( (0, 1) \) is compact in \( \mathcal{L} \). As shown in the following statements 7–9, the converse of Theorem 2 is, all the same, in some special case, true.

7. Theorem. Let \( \mathcal{L} = (L; \leq) \) be a complete lattice, \( f : L \to L \) of a closure operator on \( \mathcal{L} \) and let \( (S; \leq) \) be a join subsemilattice in the join semilattice \( \mathcal{L} \). Further, let \( c \) denote a compact element of \( (f(L); \leq) \) and let there exist \( C \subseteq S \) with \( c = f(\sup C) \). Then there exists \( s \in S \) such that \( c = f(s) \).

Proof. Denote by \( A = \text{def} f(L), d = \text{def} \sup C \). There is \( x \leq f(x) \) for every \( x \in C \), hence \( d \leq \sup x \). This yields, with respect to Lemma 3, the inequality

\[
 c = f(d) \leq f(\sup C) = \sup (A; \leq)
\]

Since \( c \) is compact in \( (A; \leq) \), then there exists a finite non-empty set \( C' \subseteq C \) such that \( c \leq \sup f(C') \). Denoting by \( s = \text{def} \sup C' \), there is \( s \in S \) since \( (S; \leq) \) is a join subsemilattice in \( \mathcal{L} \). Moreover, for every \( x \in C' \), there is \( f(x) \leq f(s) \), hence

\[
 c \leq \sup f(C') = f(\sup f(C')) \leq ff(s) = f(s).
\]

On the other hand, there is

\[
 s = \sup C' \leq \sup C = d
\]

and, following,

\[
 f(s) \leq f(d) = c.
\]

8. Lemma. Let \( \mathcal{L} = (L; \leq) \) be a complete lattice and let \( C \) be the set of all compact elements in \( \mathcal{L} \). Then \( (C; \leq) \) is a join subsemilattice of the join semilattice \( \mathcal{L} \).

Proof follows immediately from the definition of a compact element.

9. Theorem. Let \( \mathcal{L} = (L; \leq) \) be an algebraic lattice and let \( A \) be an algebraic closure system of \( \mathcal{L} \). Then an element \( c \) is compact in \( (A; \leq) \) iff it is a \( f_A \)-image of a compact element in \( \mathcal{L} \). [See also [9], Theorem 4.3.]
Proof. The statement is a direct consequence of Section 2, 7 and 8.

10. Remark. Let us denote by $E(A)$ the set of all equivalences on a given set $A$, i.e.

$$E(A) = \{ \sigma \subseteq A \times A \mid \text{id}_A \subseteq \sigma, \sigma = \sigma^{-1}, \sigma \sigma \subseteq \sigma \}$$

(where \( \text{id}_A = \{ (x, x) \mid x \in A \} \) denotes the identity relation on $A$). It is well-known that $(E(A); \subseteq)$ is an algebraic lattice and that $\sigma \in E(A)$ is compact in $(E(A); \subseteq)$ iff it satisfies the following two conditions:

(i) The set of all classes of $A/\sigma$ with at least two points, is finite.

(ii) Every class of $A/\sigma$ is finite.

It is obvious that $(\exp A; \subseteq)$ is also an algebraic lattice, all the compact elements of which are precisely finite subsets of $A$.

11. Remark. Let $(A; F)$ be a universal algebra with the support $A$ and the set $F$ of operations with finite arity on $A$. Denote by $S(A; F)$ the set of all subalgebras of $(A; F)$ and by $K(A; F)$ the set of all congruences on $(A; F)$. It is well-known that $S(A; F)$ is an algebraic closure system of $(\exp A; \subseteq)$ (the corresponding closure operator will be denoted by $f$) and that $K(A; F)$ is an algebraic closure system of $(E(A); \subseteq)$ the corresponding closure operator will be denoted by $g$; see [1], chap. II.5 and II.6, e.g. This yields, with respect to Theorem 5, the well-known result that the lattices $(S(A; F); \subseteq)$ and $(K(A; F); \subseteq)$ are algebraic. Following Theorem 9 and Remark 10, the sets of all compact elements of those lattices are

$$f(\{ X \in \exp A \mid X < \aleph_0 \}),$$

$$g(\{ \sigma \in E(A) \mid \sigma \text{ satisfies conditions (i) and (ii) of Section 7} \}).$$

12. Remark. If we consider the usual category of ordered sets, then the theory analogous to the theory of lattices of congruences on universal algebra, is the theory of kernels** of isotonic mappings -- see [3], [4].

Let $(A; \leq)$ be an ordered set. The $G(A; \leq)$ denotes the set of all kernels of isotonic mappings defined on $(A; \leq)$. There is proved, in paper [3], Section 22, that $G(A; \leq)$ is an algebraic closure system of the algebraic lattice $(E(A); \subseteq)$. Then, by Sections 5 and 10 of other present paper, $(G(A; \leq); \subseteq)$ is an algebraic lattice. (This result is obtained, in [4] Section 30, by an other way.)

*) If $A = 0$, then $A/\sigma$ has the usual sens; see [1], Chap. I.3. e.g. If $A = 0$, then $\sigma = 0$ and it is natural to define $A/\sigma = \{ 0 \}$ (see [3], Section 4).

**) If $g : X \to Y$, then the kernel ker $g$ of the mapping $g$ is this equivalence on $X$, which is defined by

$$(x, y) \in \ker g \iff x, y \in X \quad \text{and} \quad g(x) = g(y).$$

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By Sections 18, 45 and 49 of [3], the closure operator

$$\sigma \mapsto \sigma_\leq \quad [\sigma \in E(A)]$$

corresponding to the algebraic closure system \(G(A; \leq)\) of \((E(A); \leq)\), is given by

$$\leq_\sigma = \bigcup_{n=0}^{\infty} \sigma \circ \sigma \circ (\leq \circ \sigma)^n, \quad \sigma_\leq = \sigma \cap (\leq_\sigma)^{-1}.$$  \(4\)

This imply, with respect to Section 10, that the images of compact elements in \((E(A); \leq)\) by the closure operator \(\sigma \mapsto \sigma_\leq\) are precisely those equivalences \(\tau \in G(A; \leq)\), which satisfy the following three conditions:

(iii) For every \(X \in A/\tau\), the set of all elements of \(X\), which are either maximum or minimum in \((X; \leq)\), is finite.

(iv) For every \(X \in A/\tau\), every maximal chain in the ordered set \((X; \leq)\) has both upper and lower bound.

(v) The set of elements of \(A/\tau\) having at least two elements, is finite.

Suppose \(\tau \in G(A; \leq)\) satisfy conditions (iii)–(v). Denoting by \(\mathcal{F} = \{X \in A/\tau\} \quad 1 < |X|\) and by \(X^*\) the set of all both maximal and minimal elements in \((X; \leq)\) for every \(X \in \mathcal{F}\), put

$$\tau^* = \text{id}_A \cup \bigcup \{X^* \times X^* \mid X \in \mathcal{F}\}.$$  The proof of the fact, that every image of a compact element in \((E(A); \leq)\) by the closure operator \(\sigma \mapsto \sigma_\leq\) satisfies conditions (iii)–(v), is easy and it is left to the reader [use (i), (ii) and (4)].

Hence, from this and with respect to Theorem 9, it follows that the compact elements in \((G(A; \leq); \leq)\) are fully characterized by conditions (iii)–(v); this is also proved, by different way, in paper [4], Section 28.

13. Remark. Let \(V\) be a linear space over a linearly ordered field \(T\). It is obvious that the system \(\mathcal{X}\) of all convex subsets of \(V\) is an algebraic closure system of \((\exp V; \leq)\). Hence \((\mathcal{X}; \leq)\) is an algebraic lattice (see Section 5 and 10) and \(M \in \mathcal{X}\) is compact in \((\mathcal{X}; \leq)\) iff it is a simplex, i.e. a convex hull of a finite subset of \(V\) (see Section 9 and 10).

14. Remark.* This work appears with a great delay. Some papers that follow resultly here achieved have been yet published or will be published almost simultaneously. We would like to mention above all the works [6], [7] and [9] where the sentences 2, 5, 7 and 9 are generalized by different ways, and the paper [8], dealing — among others  — with characterization of \(m\)-compact elements in \((G(A; \leq); \leq)\) for arbitrary infinite cardinal \(m\).

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*) Included after the final version.
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Author's address: Blanka Kutinová, Spálená 51, 113 02 Praha 1, ČSSR (SNTL - Nakladatelství technické literatury); Teo Sturm, Suchbátarová 2, 166 27 Praha 6, ČSSR (Elektrotechnická fakulta ČVUT).

*) Included after the final version.