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ON ALGEBRAIC CLOSURES OF COMPACT ELEMENTS*)

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It will be shown that an algebraic closure operator $f : L \to L$ on a complete lattice $\mathscr{L} = (L; \leqq)$ has the following property: if $c \in L$ is compact in $\mathscr{L}$, then $f(c)$ is compact in $(f(L); \leqq)$. If the lattice $\mathscr{L}$ is algebraic, then $(f(L); \leqq)$ is algebraic as well and an element is compact in $(f(L); \leqq)$ iff it is an image by f of an element which is compact in $\mathscr{L}$. This implies, i.e., some well-known results concerning lattices of subalgebras and those of congruences on universal algebras.

1. Remark. We shall use the following current terminology (see [1], chap. II.1, or [2] e.g.). Let $\mathscr{L} = (L; \leqq)$ be a complete lattice. A subset $A \subseteqq L$ is called a *closure system* of $\mathscr{L}$, if for every set $X \subseteqq A$, these is $\inf_{(A;\leqq)} X = \inf_{\mathscr{L}} X$. If A is a closure system of $\mathscr{L}$, then $(A; \leqq)$ is a complete lattice.

Let A be a closure system of $\mathscr{L}$. A mapping $f_A : L \to L$ defined by

$$f_A(x) =_{\mathrm{Df}} \inf_{(A;\leqq)} \{y \in A \mid x \leqq y\} \quad (x \in L)$$

is called *closure operator* on $\mathscr{L}$ corresponding to A [every closure operator g on $\mathscr{L}$ corresponds to the closure system $g(L)$ of $\mathscr{L}$].

A closure system A is called *algebraic*, if for every non-empty chain B in $(A; \leqq)$, it holds $\sup_{(A;\leqq)} B = \sup_{\mathscr{L}} B$. A closure operator is called *algebraic*, if the corresponding closure system is an algebraic one.

An element $c \in L$ is called *compact* in $\boldsymbol{L}$, if every set $X \subseteqq L$ such that $c \leqq \sup_{\mathscr{L}} X$, contains a finite subset $X' \subseteqq X$ with $c \leqq \sup_{\mathscr{L}} X'$. We say that a set B *generates* $\mathscr{L}$ *by joins*, if for every element $x \in L$ there exists a set $B(x) \subseteqq B$ such that $x = \sup_{\mathscr{L}} B(x)$. A complete lattice $\mathscr{L}$ is called *algebraic*, if the set of all its compact elements generates it by joins.

*) This paper has originated at the seminar "Algebraic Foundations of Quantum Theories", directed by Prof. JIŘÍ FÁBERA.

2. Theorem. *Let A be an algebraic closure system of a complete lattice $\mathcal{L} = (L; \leqq)$. If c is compact in $\mathcal{L}$, then $f_A(c)$ is compact in $(A; \leqq)$.*

Proof. Let $f_A(c)$ be not compact in $(A; \leqq)$. Then there exists at least one set X satisfying the following condition:

(1) $X \subseteq A, f_A(c) \leqq \sup_{(A;\leqq)} X$; no finite subset $X' \subseteq X$ satisfies relation $f_A(c) \leqq \sup_{(A;\leqq)} X'$.

Among all sets X satisfying condition (1), consider the sets with the smallest cardinality; choose one of them and denote it by Y. The set Y is infinite by (1). Following the Axiom of Choice, there exists such a well-ordering $\preceq$, that the ordinal type of $(Y; \preceq)$ is the initial ordinal of the cardinality $|Y|$. For every $y \in Y$ put

$$z(y) =_{\mathrm{Df}} \sup_{(A;\leqq)} \{x \mid x \in X,\ x \preceq y\} .$$

Then for $y_1, y_2 \in Y, y_1 \preceq y_2$, there is $z(y_1) \leqq z(y_2)$, hence the set $Z =_{\mathrm{Df}} \{z(y) \mid y \in Y\}$ together with $\leqq$ forms a chain. With respect to the algebraicity of A, we get

$$\sup_{\mathcal{L}} Z = \sup_{(A;\leqq)} Z .$$

Since for every $y \in L$ there is $\{x \mid x \in Y, x \preceq y\} \subseteq Y$, then $z(y) \leqq \sup_{(A;\leqq)} Y$. This implies the relation $\sup_{(A;\leqq)} Z \leqq \sup_{(A;\leqq)} Y$. On the other hand, there is for every $y \in Y$, $y \in \{x \mid x \in Y, x \preceq y\}$, hence $y \leqq z(y)$.

It yields

$$\sup_{\mathcal{L}} Z = \sup_{(A;\leqq)} Z = \sup_{(A;\leqq)} Y .$$

Following the choice of Y we have $f_A(c) \leqq \sup_{(A;\leqq)} Y$, thus $f_A(c) \leqq \sup_{\mathcal{L}} Z$, as well. There is $c \leqq f_A(c)$, hence $c \leqq \sup_{\mathcal{L}} Z$. The element c is compact in $\mathcal{L}$; then there exists a finite set $\{y_1, \ldots, y_n\} \subseteq Y$ with $c \leqq \sup_{\mathcal{L}} \{z(y_1), \ldots, z(y_n)\}$. Moreover, $(Z; \leqq)$ being a chain, $\{z(y_1), \ldots, z(y_n)\}$ contains the greatest element with respect to $\leqq$; denotes it by $z(y_i)$. It holds

$$c \leqq \sup_{\mathcal{L}} \{z(y_1), \ldots, z(y_n)\} = z(y_i) = \sup_{(A;\leqq)} \{x \mid x \in Y, x \preceq y_i\} . \tag{2}$$

f_A is a closure operator, specially it is an isotonic mapping of $\mathcal{L}$ to $\mathcal{L}$. This fact together with (2) yields

$$f_A(c) \leqq f_A(\sup_{(A;\leqq)} \{x \mid x \in Y, x \preceq y_i\}) = \sup_{(A;\leqq)} \{x \mid x \in Y, x \preceq y_i\} \tag{3}$$

(the elements of A are fixed by f_A). By assumption, the ordinal type of the chain $(Y; \preceq)$ is the smallest one of the cardinality $|Y|$; hence the cardinality of a proper section $\{x \in Y \mid x \preceq y\}$ is strictly smaller then $|Y|$. Following (3) and the choice of Y [as a set with the smallest cardinality satisfying (1)], one can choose a finite subset $Y' \subseteq \{x \in Y \mid x \preceq y_i\}$ with $f_A(c) \leqq \sup_{(A;\leqq)} Y'$. However, Y' being a finite subset of Y, Y does not satisfy condition (1). This contradiction yields the compacticity of $f_A(c)$ in $(A; \leqq)$. The theorem is proved.

3. Lemma. (WARD). *Let $\mathscr{L} = (L; \leqq)$ be a complete lattice and let $f : L \to L$ be a closure operator on $\mathscr{L}$.* Then

$$f(\sup_{\mathscr{L}} X) = \sup_{(f(L);\leqq)} X$$

for every $X \subseteq f(L)$.

Proof. See [5], p. 76, Theorem 15 e.g.*). [See also [6], Section 1.6.]

4. Lemma. *Let A be a closure system of a complete lattice $\mathscr{L} = (L; \leqq)$. If a set B generates $\mathscr{L}$ by joins, then $f_A(B)$ generates by joins the complete lattice $(A; \leqq)$.*

Proof. For every $x \in L$, there exists, $B(x) \subseteq B$ such that $x = \sup_{\mathscr{L}} B(x)$. The mapping $f_A : L \to L$ is a closure operator in $\mathscr{L}$, hence it holds

$$x = \sup_{\mathscr{L}} B(x) \leqq \sup_{\mathscr{L}} f_A(B(x)) \leqq f_A(\sup_{\mathscr{L}} f_A(B(x))) .$$

This yields

$$f_A(x) \leqq f_A f_A(\sup_{\mathscr{L}} f_A(B(x))) = f_A(\sup_{\mathscr{L}} f_A(B(x))) = \sup_{(A;\leqq)} f_A(B(x))$$

[the latest equality holds following Section 3: it is $A = f_A(L)$].

On the other hand, there is $y \leqq x$ for every $y \in B(x)$, thus $f_A(y) \leqq f_A(x)$, as well; hence

$$\sup_{\mathscr{L}} f_A(B(x)) \leqq f_A(x) .$$

Since f_A is isotonic and $f_A f_A = f_A$, we get, by Section 3, the inequality

$$\sup_{(A;\leqq)} f_A(B(x)) = f_A(\sup_{\mathscr{L}} f_A(B(x))) \leqq f_A f_A(x) = f_A(x) .$$

5. Theorem. *Let $\mathscr{L} = (L; \leqq)$ be an algebraic lattice and let A be an algebraic closure system of $\mathscr{L}$. Then $(A; \leqq)$ is an algebraic lattice.*

Proof follows immediately from Sections 2 and 4: Let C denote the set of compact elements in $\mathscr{L}$. Then C generates $\mathscr{L}$ by joins and, following Lemma 4, $f_A(C)$ generates $(A; \leqq)$ by joins. Moreover. all elements of $f_A(C)$ are compact in $(A; \leqq)$ by Theorem 2.

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*) It is easy to prove a more general statement: Let $\mathscr{L} = (L; \leqq)$ be a complete lattice and let $f: L \to L$ be a closure operator on $\mathscr{L}$. Then $f(\sup_{\mathscr{L}} X) = \sup_{(f(L);\leqq)} f(X)$ for every $X \subseteq L$.

6. Remark. Let $\mathscr{L} = (L; \leqq)$ be a complete lattice and let f_A be an algebraic closure operator on $\mathscr{L}$. The converse of this statement is, in general, not true, since lattice $(A; \leqq)$ could contain some compact elements, which are not f_A-images of any compact element in $\mathscr{L}$:

Consider the closed interval $L = \langle 0, 1 \rangle$ in the set of real numbers with the usual ordering, $A = \{0, 1\}$. Then A is an algebraic system of closed elements in $\mathscr{L}$ and

$$f_A(0) = 0\,, \quad f_A(x) = 1 \quad \text{for every} \quad x \in (0, 1\rangle\,.$$

Further, 1 is compact in $(A; \leqq)$ although no element of $(0, 1\rangle$ is compact in $\mathscr{L}$. As shown in the following statements 7–9, the converse of Theorem 2 is, all the same, in some special case, true.

7. Theorem. *Let $\mathscr{L} = (L; \leqq)$ be a complete lattice, $f : L \to L$ of a closure operator on $\mathscr{L}$ and let $(S; \leqq)$ be a join subsemilattice in the join semilattice $\mathscr{L}$. Further, let c denote a compact element of $(f(L); \leqq)$ and let there exist $C \subseteqq S$ with $c = f(\sup_{\mathscr{L}} C)$. Then there exists $s \in S$ such that $c = f(s)$.*

Proof. Denote by $A =_{\mathrm{Df}} f(L)$, $d =_{\mathrm{Df}} \sup_{\mathscr{L}} C$. There is $x \leqq f(x)$ for every $x \in C$, hence $d \leqq \sup_{\mathscr{L}} f(C)$. This yields, with respect to Lemma 3, the inequality

$$c = f(d) \leqq f(\sup_{\mathscr{L}} f(C)) = \sup_{(A;\leqq)} f(C)\,.$$

Since c is compact in $(A; \leqq)$, then there exists a finite non-empty set $C' \subseteqq C$ such that $c \leqq \sup_{(A;\leqq)} f(C')$. Denoting by $s =_{\mathrm{Df}} \sup_{\mathscr{L}} C'$, there is $s \in S$ since $(S; \leqq)$ is a join subsemilattice in $\mathscr{L}$. Moreover, for every $x \in C'$, there is $f(x) \leqq f(s)$, hence

$$c \leqq \sup_{(A;\leqq)} f(C') = f(\sup_{\mathscr{L}} (f(C')) \leqq ff(s) = f(s)\,.$$

On the other hand, there is

$$s = \sup_{\mathscr{L}} C' \leqq \sup_{\mathscr{L}} C = d$$

and, following,

$$f(s) \leqq f(d) = c\,.$$

8. Lemma. *Let $\mathscr{L} = (L; \leqq)$ be a complete lattice and let C be the set of all compact elements in $\mathscr{L}$. Then $(C; \leqq)$ is a join subsemilattice of the join semilattice $\mathscr{L}$.*

Proof follows immediately from the definition of a compact element.

9. Theorem. *Let $\mathscr{L} = (L; \leqq)$ be an algebraic lattice and let A be an algebraic closure system of $\mathscr{L}$. Then an element c is compact in $(A; \leqq)$ iff it is a f_A-image of a compact element in $\mathscr{L}$.* [See also [9], Theorem 4.3.]

Proof. The statement is a direct consequence of Section 2, 7 and 8.

10. Remark. Let us denote by $E(A)$ the set of all equivalences on a given set A, i.e.

$$E(A) = \{\sigma \subseteqq A \times A \mid \mathrm{id}_A \subseteq \sigma,\ \sigma = \sigma^{-1},\ \sigma\sigma \subseteqq \sigma\}$$

(where $\mathrm{id}_A = \{(x. x) \mid x \in A\}$ denotes the identity relation on A). It is well-known that $(E(A); \subseteq)$ is an algebraic lattice and that $\sigma \in E(A)$ is compact in $(E(A); \subseteqq)$ iff it satisfies the following two conditions:

(i) *The set of all classes of A/σ* *) *with at least two points, is finite.*

(ii) *Every class of A/σ is finite.*

It is obvious that $(\exp A; \subseteq)$ is also an algebraic lattice, all the compact elements of which are precisely finite subsets of A.

11. Remark. Let $(A; F)$ be a universal algebra with the support A and the set F of operations with finite arity on A. Denote by $S(A; F)$ the set of all subalgebras of $(A; F)$ and by $K(A; F)$ the set of all congruences on $(A; F)$. It is well-known that $S(A; F)$ is an algebraic closure system of $(\exp A; \subseteq)$ (the corresponding closure operator will be denoted by f) and that $K(A; F)$ is an algebraic closure system of $(E(A); \subseteq)$ the corresponding closure operator will be denoted by g); see [1], chap. II.5 and II.6, e.g. This yields, with respect to Theorem 5, the well-known result that the lattices $(S(A; F); \subseteq)$ and $(K(A; F); \subseteq)$ are algebraic. Following Theorem 9 and Remark 10, the sets of all compact elements of those lattices are

$$f(\{X \in \exp A \mid |X| < \aleph_0\}),$$

$$g(\{\sigma \in E(A) \mid \sigma \text{ satisfies conditions (i) and (ii) of Section 7}\}).$$

12. Remark. If we consider the usuel category of ordered sets, then the theory analogous to the theory of lattices of congruences on universal algebra, is the theory of kernels**) of isotonic mappings – see [3], [4].

Let $(A; \leqq)$ be an ordered set. The $G(A; \leqq)$ denotes the set of all kernels of isotonic mappings defined on $(A; \leqq)$. There is proved, in paper [3], Section 22, that $G(A; \leqq)$ is an algebraic closure system of the algebraic lattice $(E(A); \subseteq)$. Then, by Sections 5 and 10 of other present paper, $(G(A; \leqq); \subseteq)$ is an algebraic lattice. (This result is obtained, in [4] Section 30, by an other way.)

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*) If $A = \emptyset$, then A/σ has the usuel sens; see [1], Chap. I.3. e.g. If $A = \emptyset$, then $\sigma = \emptyset$ and it is natural to define $A/\sigma = \{\emptyset\}$ (see [3], Section 4).

**) If $g : X \to Y$, then the kernel $\ker g$ of the mapping g is this equivalence on X, which is defined by

$$(x, y) \in \ker g \Leftrightarrow_{\mathrm{Df}} x,\ y \in X \quad \text{and} \quad g(x) = g(y).$$

By Sections 18, 45 and 49 of [3], the closure operator

$$\sigma \mapsto \sigma_{\leqq} \quad [\sigma \in E(A)]$$

corresponding to the algebraic closure system $G(A; \leqq)$ of $(E(A); \subseteq)$, is given by

$$(4) \qquad \leqq_\sigma =_{\mathrm{Df}} \bigcup_{n=0}^{\infty} \sigma \circ (\leqq \circ \sigma)^n , \quad \sigma_{\leqq} =_{\mathrm{Df}} \leqq_\sigma \cap (\leqq_\sigma)^{-1} .$$

This imply, with respect to Section 10, that the images of compact elements in $(E(A); \subseteq)$ by the closure operator $\sigma \mapsto \sigma_{\leqq}$ are precisely those equivalences $\tau \in G(A; \leqq)$, which satisfy the following three conditions:

(iii) *For every $X \in A/\tau$, the set of all elements of X, which are either maximum or minimum in $(X; \leqq)$, is finite.*

(iv) *For every $X \in A/\tau$, every maximal chain in the ordered set $(X; \leqq)$ has both upper and lower bound.*

(v) *The set of elements of A/τ having at least two elements, is finite.*

Suppose $\tau \in G(A; \leqq)$ satisfy conditions (iii)–(v). Denoting by $\mathscr{T} =_{\mathrm{Df}} \{X \in A/\tau \mid 1 < |X|\}$ and by X^* the set of all both maximal and minimal elements in $(X; \leqq)$ for every $X \in \mathscr{T}$, put

$$\tau^* =_{\mathrm{Df}} \mathrm{id}_A \cup \bigcup \{X^* \times X^* \mid X \in \mathscr{T}\} .$$

Then τ^* is compact in $(E(A); \subseteq)$. Moreover, (4) yields $\tau^*_{\leqq} = \tau$. The proof of the fact, that every image of a compact element in $(E(A); \subseteq)$ by the closure operator $\sigma \mapsto \sigma_{\leqq}$ satisfies conditions (iii)–(v), is easy and it is left to the reader [use (i), (ii) and (4)].

Hence, from this and with respect to Theorem 9, it follows that the compact elements in $(G(A; \leqq); \subseteq)$ are fully characterized by conditions (iii)–(v); this is also proved, by different way, in paper [4], Section 28.

13. Remark. Let V be a linear space over a lineary ordered field T. It is obvious that the system $\mathscr{K}$ of all convex subsets of V is an algebraic closure system of $(\exp V; \subseteq)$. Hence $(\mathscr{K}; \subseteq)$ is an algebraic lattice (see Section 5 and 10) and $M \in \mathscr{K}$ is compact in $(\mathscr{K}; \subseteq)$ iff it is a simplex, i.e. a convex hull of a finite subset of V (see Section 9 and 10).

14. Remark.*) This work appears with a great delay. Some papers that follow resulty here achieved have been yet published or will be published almost simultaneously. We would like to mention above all the works [6], [7] and [9] where the sentences 2, 5, 7 and 9 are generalized by different ways, and the paper [8], dealing – among others – with characterization of $\mathfrak{m}$-compact elements in $(G(A; \leqq); \subseteq)$ for arbitrary infinite cardinal $\mathfrak{m}$.

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*) Included after the final version.

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