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On integration in Banach spaces, III


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ON INTEGRATION IN BANACH SPACES, III

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INTRODUCTION

Let $T$ and $S$ be non empty sets and let $\mathcal{P}$ and $\mathcal{Q}$ be $\delta$-rings of subsets of $T$ and $S$, respectively. Let $X$, $Y$ and $Z$ be real or complex Banach spaces, and let $m : \mathcal{P} \to L(X, Y)$ and $I : \mathcal{Q} \to L(Y, Z)$ be two operator valued measures countably additive in the strong operator topologies with finite semivariations $m^\wedge$ and $I^\vee$. In this part of our theory of integration we investigate the existence of the product measure $I \otimes m : \mathcal{P} \otimes \mathcal{Q} \to L(X, Z)$, countably additive in the strong operator topology, and the validity of a Fubini type theorem for $\mathcal{P} \otimes \mathcal{Q} -$ measurable functions $f : T \times S \to X$. Here $\mathcal{P} \otimes \mathcal{Q}$ denotes the smallest $\delta$-ring containing all rectangles $A \times B$, $A \in \mathcal{P}$, $B \in \mathcal{Q}$, and $(I \otimes m)(A \times B) = I(B)m(A)$. The main results of the paper, namely Theorems 1 and 15, were announced in [9].

In Theorem 1 we prove that the most natural condition: “for each $E \in \mathcal{P} \otimes \mathcal{Q}$ and each $x \in X$ the function $s \to m(E^c)x$, $s \in S$, is integrable with respect to $I^\vee$, is necessary and sufficient for the existence of the product measure $I \otimes m : \mathcal{P} \otimes \mathcal{Q} \to L(X, Z)$, and that if it is satisfied, then $(I \otimes m)(E)x = \int_S m(E^c)x \, dl$ for each $E \in \mathcal{P} \otimes \mathcal{Q}$ and each $x \in X$. As a consequence, in Theorem 3 we prove that the continuity of the semivariation $I^\vee$ on $\mathcal{Q}(B_n \in \mathcal{Q}, B_n \searrow \emptyset \Rightarrow I^\vee(B_n) \searrow 0$, see the $*$-Theorem in Section 1.1 in [6]) is sufficient for the existence of the product measure $I \otimes m$ on $\mathcal{P} \otimes \mathcal{Q}$, and the continuity of $I^\vee$ on $\mathcal{Q}$ and $m^\wedge$ on $\mathcal{P}$ imply the continuity of $(I \otimes m)$ on $\mathcal{P} \otimes \mathcal{Q}$. Results similar to Theorem 3 were obtained by different approaches and in various settings by M. DUCHON in [10]–[16] and CH. SWARTZ in [28], [29] and [30], see also [2], [4], [17], [18], [25], [28] and [32].

Using Theorem 1, in Theorems 4 and 5 we establish the validity of the Fubini theorem for functions which are uniform limits of $\mathcal{P} \otimes \mathcal{Q} -$ simple functions, particularly for elements of $C_0(T \times S, X)$.

Let the product measure $I \otimes m : \mathcal{P} \otimes \mathcal{Q} \to L(X, Z)$ exist and let the function $f : T \times S \to X$ be integrable with respect to $I \otimes m$. Then, as the very simple example at the beginning of § 2 shows, the function $t \to f(t, s)$, $t \in T$, need not be integrable with respect to $m$ for any $s \in S$, even if the variations of both $m$ and $I$ are bounded. Hence in a general Fubini type theorem we must suppose that for each $s \in S$ the
function \( t \to f(t, s), t \in T \), is integrable with respect to \( m \). Adopting this assumption, our main task is to establish the \( \mathcal{B} \)-measurability of the partial integral \( g_E, g_E(s) = \int_{E} f(\cdot, s) \, dm, s \in S \), for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \). Although the author did not succeed in solving this problem in general, in § 2 we establish the \( \mathcal{B} \)-measurability of \( g_E \) in the following important cases: 1) the semivariation \( m^* \) is continuous on \( \mathcal{P} \) (Theorem 9), 2) \( Y \) is a separable Banach space (Theorem 10), and 3) \( \mathcal{P} \) is generated by a countable family (Theorem 12). Further we prove the \( I \)-essential \( \mathcal{B} \)-measurability of \( g_E \), see Definition 2, which is also sufficient, in the following important cases: 4) \( Z \) is separable or is a dual of a separable Banach space, and 5) \( I \) is countably additive in the uniform operator topology on \( \mathcal{B} \), see Theorems 13 and 14. Note that case 5) includes the following important subcase 6): \( I : \mathcal{B} \to L(Y, Z) \) is given by an equality \( I(B)y = u(y, \gamma(B)) \), where \( u : Y \times Z_1 \to Z, Z_1 \) being a Banach space, is a separately continuous bilinear map and \( \gamma : \mathcal{B} \to Z_1 \) is a countably additive vector measure. Indeed, by the Uniform Boundedness Principle \( u \) is bounded on \( Y \times Z_1 \), hence \( I : \mathcal{B} \to L(Y, Z) \) is countably additive in the uniform operator topology.

Assuming the integrability of \( f(\cdot, s) \) with respect to \( m \) for each \( s \in S \), and the \( I \)-essential \( \mathcal{B} \)-measurability of \( g_E \) for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \), in § 3 we prove the Fubini theorem and an important special case of it. This special case includes the recent results of Theorems 8 and 9 from [16], where the integral of R. G. Bartle [3] is used.

Let \( \mathcal{D} \) be a \( \delta \)-ring of subsets of \( S \). We say that \( g : S \to Y \) is \( \mathcal{D} \)-measurable, if there is a sequence \( g_n, n = 1, 2, \ldots \) of \( \mathcal{D} \)-simple functions (on \( S \) with values in \( Y \)) such that \( g_n(s) \to g(s) \) for each \( s \in S \). In addition to the information about this measurability given in § 1 in Part I (from now on [6] will be referred to as Part I and [7] as Part II) see also [24]. If \( g : S \to Y \) is integrable with respect to \( I : \mathcal{B} \to L(Y, Z) \), then by \( \int_E g \, dl \) we understand the integral \( \int_D g \, dl \), where \( D = \{ s \in S; g(s) \neq 0 \} \in \mathcal{E}(\mathcal{B}) \).

We note that a nice and deep Radon-Nikodym theorem for our integral was proved by H. B. Maynard in [26, Theorem 5].

As is well known, to each countably additive vector measure on a \( \sigma \)-ring there is a finite non negative countably additive measure on that \( \sigma \)-ring with the same zero sets; for a short proof see [20, Theorem 3.10]. Such a measure is called a control measure for the given vector measure.

**Correction to Part I.** In the definition of \( \mu \) in the proof of Theorem 1 in Part I the vector measures \( E \to \int_E f_n \, dm, E \in \mathcal{E}(\mathcal{P}), n = 1, 2, \ldots \), must be replaced by their control measures.

1. PRODUCTS OF OPERATOR VALUED MEASURES

We shall use the notation and terminology introduced in Parts I and II and in Introduction. Let \( \mathcal{P}_0 \) and \( \mathcal{B}_0 \) be \( \delta \)-rings of subsets of \( T \) and \( S \), respectively, and let \( m : \mathcal{P}_0 \to L(X, Y) \) and \( l : \mathcal{B}_0 \to L(Y, Z) \) be operator valued measures countably
additive in the strong operator topologies. Then $\mathcal{P}$ denotes the greatest $\delta$-subring of $\mathcal{P}_0$, where the semivariation $m^\circ$ is finite. By $\mathcal{P}_2$ we denote the greatest $\delta$-subring of $\mathcal{P}_0$ where $m$ is countably additive in the uniform operator topology, and by $\mathcal{P}^\circ$ we denote the greatest $\delta$-subring of $\mathcal{P}_0$ (equivalently, of $\mathcal{P}$, see Corollary of Theorem 5 in Part II), where the semivariation $m^\circ$ is continuous. Similarly we have $\mathcal{S}$, $\mathcal{S}_2$, and $\mathcal{S}^\circ$.

For a class of sets $\mathcal{A}$, we denote by $\mathcal{S}(\mathcal{A})$ the smallest $\sigma$-ring containing $\mathcal{A}$, which we call the $\sigma$-ring generated by $\mathcal{A}$. Similarly we have $\mathcal{S}(\mathcal{A})$, the $\sigma$-ring generated by $\mathcal{A}$. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are $\delta$-rings of subsets of $T$ and $S$, respectively, then clearly $\mathcal{S}(\mathcal{D}_1 \otimes \mathcal{D}_2) = \mathcal{S}(\mathcal{D}_1) \otimes \mathcal{S}(\mathcal{D}_2)$. Further, for each $E \in \mathcal{S}(\mathcal{D}_1 \otimes \mathcal{D}_2)$ there are $A \in \mathcal{D}_1$ and $B \in \mathcal{D}_2$ such that $E \subseteq A \times B$. Finally, for $E \subseteq T \times S$ and $s \in S$ we put $E^s = \{t \in T; (t, s) \in E\}$.

Before proceeding to the next definition we note that the Hahn-Banach theorem and the uniqueness of the extension of a finite scalar measure from a ring to the generated $\sigma$-ring, see [21, §13], imply that if $n_1, n_2 : \mathcal{P}_0 \otimes \mathcal{P}_0 \rightarrow L(X, Z)$ are two operator valued measures countably additive in the strong operator topology such that $n_1(A \times B) = n_2(A \times B)$ for each $A \in \mathcal{P}_0$ and $B \in \mathcal{P}_0$, then they are identical on $\mathcal{P}_0 \otimes \mathcal{P}_0$ (Theorem E in §33 and Theorem D in §13 in [21] are also used).

Definition 1. We say that the product of measures $m : \mathcal{P}_0 \rightarrow L(X, Y)$ and $l : \mathcal{P}_0 \rightarrow L(Y, Z)$ exists on $\mathcal{P}_0 \otimes \mathcal{P}_0$, if there is a necessarily unique $L(X, Z)$ valued measure countably additive in the strong operator topology on $\mathcal{P}_0 \otimes \mathcal{P}_0$, which we denote by $l \otimes m$, such that $(l \otimes m)(A \times B) = l(B)m(A)$ for each $A \in \mathcal{P}_0$ and $B \in \mathcal{P}_0$.

Lemma 1. For each $x \in X$ let there be a countably additive $Z$-valued vector measure $\mu_x$ on $\mathcal{P}_0 \otimes \mathcal{P}_0$ such that $\mu_x(A \times B) = l(B)m(A)x$ for each $A \in \mathcal{P}_0$ and $B \in \mathcal{P}_0$. Then the product measure $l \otimes m$ exists on $\mathcal{P}_0 \otimes \mathcal{P}_0$.

Proof. For $E \in \mathcal{P}_0 \otimes \mathcal{P}_0$ and $x \in X$ put $(l \otimes m)(E)x = \mu_x(E)$. We have to prove

(a) $\mu_{x_1 + x_2}(E) = \mu_{x_1}(E) + \mu_{x_2}(E)$,

and

(b) $\lim_{x \to 0} \mu_x(E) = 0$, $x \in X$, for each $E \in \mathcal{P}_0 \otimes \mathcal{P}_0$, all $x_1, x_2 \in X$ and all scalars $\alpha, \beta$.

Denote by $\mathcal{R}$ the ring of all finite unions of pairwise disjoint rectangles $A \times B$, $A \in \mathcal{P}_0$, $B \in \mathcal{P}_0$, see Theorem E in §33 in [21]. We shall need the following fact:

(c) Let $z^* \in Z^*$ and let $E \in \mathcal{P}_0 \otimes \mathcal{P}_0$. Then the obvious inequality $|z^* \mu_x(E_1) - z^* \mu_x(E_2)| \leq v(z^* \mu_x, E_1 \Delta E_2)$, $E_1, E_2 \in \mathcal{P}_0 \otimes \mathcal{P}_0$, and Theorem D in §13 in [21] imply that for each $\varepsilon > 0$ there is a set $F \in \mathcal{R}$ such that $|z^* \mu_x(E) - z^* \mu_x(F)| < \varepsilon$.

Let $\alpha, \beta$ and $x_1, x_2$ be given. Then (a) is true for $E \in \mathcal{R}$, since $\mu_x(A \times B) = l(B)m(A)x$ for each $A \in \mathcal{P}_0$ and $B \in \mathcal{P}_0$, the values of $l$ and $m$ are linear operators and $\mu_x$ is additive. Thus by (c) and the Hahn-Banach theorem (a) is true for each $E \in \mathcal{P}_0 \otimes \mathcal{P}_0$.  

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To prove (b), let \( E \in \mathcal{P}_0 \otimes \mathcal{D} \) and take \( A \in \mathcal{P}_0 \) and \( B \in \mathcal{D} \) so that \( E \subset A \times B \). Let \( F \in \mathcal{R} \cap (A \times B) \). Without loss of generality we may suppose that \( F = \bigcup_{i=1}^{r} (A_i \times B_i), \ A_i \in \mathcal{P}_0, \ B_i \in \mathcal{D}, \ i = 1, \ldots, r, \) with pairwise disjoint \( B_i \). But then
\[
|z^* \mu_x(F)| \leq |\mu_x(F)| = \sum_{i=1}^{r} \mu_x(A_i \times B_i) = \sum_{i=1}^{r} \mu_x(A_i) \cdot |x| \cdot \|m\| \cdot \mathcal{I}^\infty(B)
\]
for each \( z^* \in Z^* \) with \( |z^*| \leq 1 \). Since \( B \in \mathcal{D} \), we have \( \mathcal{I}^\infty(B) < +\infty \). By Uniform Boundedness Principle we conclude \( \|m\| (A) = \sup_{|x| \leq 1} \|m(x)\| (A) = \sup_{|x| \leq 1} \sup_{|y| \leq 1} \psi(y^* m(\cdot) x, A) < +\infty \). Thus \( \lim_{x \to 0} |z^* \mu_x(F)| = 0 \) uniformly for \( F \in \mathcal{R} \cap (A \times B) \) and \( z^* \in Z^* \) with \( |z^*| \leq 1 \), hence using (c) we easily obtain (b) for each \( E \).

**Lemma 2.** Let \( \mathcal{D} \) be a \( \delta \)-ring of subsets of \( S \). Then:
1) for each \( E \in \mathcal{P}_0 \otimes \mathcal{D} \) and each \( x \in X \) the function \( s \to m(E^s) x, s \in S \), is bounded and \( \mathcal{D} \)-measurable,
2) for each \( E \in \mathcal{P}_2 \otimes \mathcal{D} \) the function \( s \to \|m(E^s)\|, s \in S \), is bounded and \( \mathcal{D} \)-measurable, and
3) for each \( E \in \mathcal{P}^- \otimes \mathcal{D} \) the function \( s \to m^\infty(E^s), s \in S \), is bounded and \( \mathcal{D} \)-measurable.

**Proof.** 1) Let \( E \in \mathcal{P}_0 \otimes \mathcal{D} \) and let \( x \in X \). Take \( A \in \mathcal{P}_0 \) and \( B \in \mathcal{D} \) so that \( E \subset A \times B \), and denote by \( \mathcal{M} \) the class of all sets \( M \in \mathcal{P}_0 \otimes \mathcal{D} \cap (A \times B) \) for which 1) holds. Then clearly \( \mathcal{M} \) contains the ring \( \mathcal{R} \cap (A \times B) \), where \( \mathcal{R} \) is the ring of all finite unions of pairwise disjoint rectangles \( A_1 \times B_1, A_1 \in \mathcal{P}_0, B_1 \in \mathcal{D} \). Since \( \sup_{M \in \mathcal{M}} \|m(M) x\| \leq \|m(x)\| (A) < +\infty \) for each \( M \in \mathcal{M} \), and since the \( \mathcal{D} \)-measurable functions are closed under the formation of pointwise limits of sequences, see Section 1.2 in Part I and Lemma 1.2 in [24], the countable additivity of \( m(\cdot) x \) on \( \mathcal{P}_0 \) implies that \( \mathcal{M} \) is a monotone class. Thus \( \mathcal{M} = \mathcal{P}_0 \otimes \mathcal{D} \cap (A \times B) \) by Theorem B in § 6 in [21], hence \( E \in \mathcal{M} \).

2) and 3) may be proved similarly using the continuity and finiteness of the semi-variations \( \|m\| \) on \( \mathcal{P}_2 \) and \( m^\infty \) on \( \mathcal{P}^- \), respectively.

**Theorem 1.** The product measure \( I \otimes m : \mathcal{P}_0 \otimes \mathcal{D} \to L(X, Z) \) exists if and only if for each \( E \in \mathcal{P}_0 \otimes \mathcal{D} \) and each \( x \in X \) the function \( s \to m(E^s) x, s \in S \), is integrable with respect to \( I \). In that case
\[
(I \otimes m)(E) x = \int_S m(E^s) x \, dl
\]
for each \( E \in \mathcal{P}_0 \otimes \mathcal{D} \) and each \( x \in X \).

**Proof.** Suppose that the product measure \( I \otimes m : \mathcal{P}_0 \otimes \mathcal{D} \to L(X, Z) \) exists and let \( x \in X \). Denote by \( \mathcal{D} \) the class of all sets \( D \in \mathcal{P}_0 \otimes \mathcal{D} \) for which the function

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$s \to m(\mathcal{D})x$, $s \in S$, is integrable with respect to $I$ and for which the equation (1) is valid. Then clearly $\mathcal{D}$ is a subring of $\mathcal{P}_o \otimes \mathcal{B}$ which contains all rectangles $A \times B$, $A \in \mathcal{P}_o$, $B \in \mathcal{B}$, hence we have to prove that $\mathcal{D}$ is a $\delta$-ring, see Theorem E in § 33 in [21]. Let $D_n \in \mathcal{D}$, $n = 1, 2, \ldots$, let $D_n \uparrow D$, and let $F \in \mathcal{E}(\mathcal{P}_o \otimes \mathcal{D})$. Then $m(D_n) \to m(D)x$ for each $s \in S$ by the countable additivity of the vector measure $m(\cdot) \colon \mathcal{P}_o \to Y$, hence the function $s \to m(D_n)x$, $s \in S$, is $\mathcal{B}$-measurable, see Section 1.2 in Part I and Lemma 2.2 in [24]. Further, (1) and the countable additivity of the vector measure $(I \otimes m)(\cdot) \colon \mathcal{P}_o \otimes \mathcal{B} \to Z$ imply that $\int m(D_n)x \, dl \to (I \otimes m)(D \cap F)x$ for each $F \in \mathcal{E}(\mathcal{P}_o \otimes \mathcal{B})$ for each $F \in \mathcal{E}(\mathcal{P}_o \otimes \mathcal{B})$. Thus by Theorem 16 in Part I the function $s \to m(D)x$, $s \in S$, is integrable with respect to $I$ and (1) is true for $D$. Hence $D \in \mathcal{D}$, so $\mathcal{D}$ is a $\delta$-ring. Since $x \in X$ was arbitrary, the necessary part of the first assertion and the second assertion of the theorem are proved.

Suppose now that for each $E \in \mathcal{P}_o \otimes \mathcal{B}$ and each $x \in X$ the function $s \to m(E)x$, $s \in S$, is integrable with respect to $I$. For $x \in X$ and $E \in \mathcal{P}_o \otimes \mathcal{B}$ put $\mu_x(E) = \int_s m(E)x \, dl$. Since $\mu_x(A \times B) = I(B)m(A)$ for each $A \in \mathcal{P}_o$, $B \in \mathcal{B}$, and $x \in X$, according to Lemma 1 it suffices to prove that for each $x \in X$, $\mu_x : \mathcal{P}_o \otimes \mathcal{B} \to Z$ is a countably additive vector measure. Let $x \in X$, and suppose that $E_n \in \mathcal{P}_o \otimes \mathcal{B}$, $n = 1, 2, \ldots$ are pairwise disjoint sets with $\bigcup_{n=1}^{\infty} E_n = E \in \mathcal{P}_o \otimes \mathcal{B}$. We have to show that $\mu_x(E) = \sum_{n=1}^{\infty} \mu_x(E_n)$, where the series converges unconditionally in $Z$. Take $A \in \mathcal{P}_o$ and $B \in \mathcal{B}$ so that $E \subseteq A \times B$, and consider the $\sigma$-ring $\mathcal{P}_o \otimes \mathcal{B} \cap (A \times B)$. Since $\mu_x : \mathcal{P}_o \otimes \mathcal{B} \cap (A \times B) \to Z$ is additive, by the Orlicz-Pettis theorem, see IV.10.1 in [19], it is sufficient to prove that $z^* \mu_x(E) = \sum_{n=1}^{\infty} z^* \mu_x(E_n)$ for each $z^* \in Z^*$, where the series converges unconditionally. Let $E_n$, $n = 1, 2, \ldots$ be any rearrangement of the sequence $E_n$, $n = 1, 2, \ldots$, and let $z^* \in Z^*$. Then for each $n = 1, 2, \ldots$ we have

$$|z^* \mu_x(E) - \sum_{i=1}^{n} z^* \mu_x(E_i)| = |z^* \mu_x\left(\bigcup_{i=n+1}^{\infty} E_i\right)| =$$

$$= \left|z^* \left(\int_S m\left(\bigcup_{i=n+1}^{\infty} E_i\right)x \, dl\right)\right| = \left|\int_S m\left(\bigcup_{i=n+1}^{\infty} E_i\right)x \, dl\right| \leq$$

$$\leq \int_B \|m(\cdot)x\|\left(\bigcup_{i=n+1}^{\infty} E_i\right) \, dv(z^*l, \cdot),$$

see the paragraph after Theorem 7 in Part I and Lemma 2.2. Since $\|m(\cdot)x\|\left(\bigcup_{i=n+1}^{\infty} E_i\right) \uparrow 0$ as $n \to +\infty$ for each $s \in S$ by the countable additivity of the vector measure $m(\cdot) \colon \mathcal{P}_o \to Y$, since $\|m(\cdot)x\|\left(\bigcup_{i=n+1}^{\infty} E_i\right) \leq \|m(\cdot)x\| (B) < +\infty$ for each $s \in S$ and $n = 1, 2, \ldots$, and since $v(z^*l, B) = z^*l (B) \leq |z^*|.l^* (B) < 482$.
< + ∞, see Example 5 in Section 1.1 in Part I, we conclude \( \int_B \| m(\cdot) \| (\bigcup_{i=n+1}^{\infty} E_i) \) 
\( \delta_0(z^* l, \cdot) \to 0 \) as \( n \to + \infty \) by the Lebesgue dominated convergence theorem. Thus 
\[ \sum_{i=1}^{n} z^* \mu_{\alpha}(E_i) \to z^* \mu_{\alpha}(E), \] 
which was to be shown. The theorem is proved.

Let \( g : S \to Y \) be a \( \mathcal{B} \)-measurable function. In Definition 1 in Part II we defined its \( L_1 \)-norm \( I^\prec(\mathcal{B}, B \otimes \mathcal{A}, \mathcal{B}) \) on a set \( B \in \mathcal{B} \) (actually, it is in general only a \( L_1 \)-pseudonorm) by the equality \( I^\prec(\mathcal{B}, B) = \sup \{ \| h \| : h : S \to Y \text{ is } \mathcal{B} \text{-simple and } |h(s)| \leq |g(s)| \text{ for each } s \in S \} \). Obviously this definition is meaningful for any real valued function \( g \) on \( S \). What is more important, Theorems 1, 2, 3, 5 and 6 remain valid in this case, and if the functions considered are \( \mathcal{B} \)-measurable, then also the important Theorems 16 and 17 are valid. (We mean theorems from Part II.) In the following we shall use these facts freely.

From Theorem 1 and from the definitions we easily obtain

**Theorem 2.** Let the product measure \( I \otimes m : \mathcal{P}_0 \otimes \mathcal{B} \to L(X, Z) \) exist, let \( E \in \mathcal{P}_0 \otimes \mathcal{B} \) and let \( f : T \times S \to X \) be a \( \mathcal{P}_0 \otimes \mathcal{B} \)-measurable function. Then

\[ \| I \otimes m \| (E) \leq I^\prec(\| m \| (E'), S) \]

and

\[ (\bigotimes \mathcal{M})(f, E) \leq I^\prec(\mathcal{M}(f(\cdot, s), E'), S). \]

Particularly, \( \| I \otimes m \| (A \times B) \leq \| m \| (A), \) \( I^\prec(B) < + \infty, \) and \( (\bigotimes \mathcal{M})(A \times B) \leq \mathcal{M}(A), \) \( I^\prec(B) \) for each \( A \in \mathcal{P}_0 \) and \( B \in \mathcal{B} \). Hence \( I \otimes m \) is finite on \( \mathcal{P} \otimes \mathcal{B}. \)

**Theorem 3.** The product measure \( I \otimes m \) exists on \( \mathcal{P}_0 \otimes \mathcal{B}^\sim \), on \( \mathcal{P}_2 \otimes \mathcal{B}^\sim \) it is countably additive in the uniform operator topology, and its semivariation \( (\bigotimes \mathcal{M}) \) is continuous on \( \mathcal{P} \otimes \mathcal{B}^\sim \).

**Proof.** Let \( E \in \mathcal{P}_0 \otimes \mathcal{B}^\sim \) and let \( x \in X \). By Lemma 2.1 the function \( s \to m(E^s) x, s \in S, \) is bounded and \( \mathcal{B}^\sim \)-measurable. Since \( \{ s \in S, m(E^s) x = 0 \} \in \mathcal{B} \), since the semivariation \( I^\prec \) is continuous on \( \mathcal{B}^\sim \), by Theorem 5 from Part I the function \( s \to m(E^s) x, s \in S, \) is integrable. Since \( E \in \mathcal{P}_0 \otimes \mathcal{B}^\sim \) and \( x \in X \) were arbitrary, by Theorem 1 the product measure \( I \otimes m \) exists on \( \mathcal{P}_0 \otimes \mathcal{B}^\sim \).

It is easy to see that the product measure \( I \otimes m \) is countably additive in the uniform operator topology on \( \mathcal{P}_2 \otimes \mathcal{B}^\sim \) if and only if \( E_n \in \mathcal{P}_2 \otimes \mathcal{B}^\sim, n = 1, 2, \ldots \) and \( E_n \nrightarrow 0 \) imply that \( \| I \otimes m \| (E_n) \nrightarrow 0 \). Let \( E_n \in \mathcal{P}_2 \otimes \mathcal{B}^\sim, n = 1, 2, \ldots \) and let \( E_n \nrightarrow 0 \). By Lemma 2.2 the functions \( s \to m(E^s_n) x, s \in S, n = 1, 2, \ldots \) are bounded and \( \mathcal{B}^\sim \)-measurable. Since \( \{ s \in S, m(E^s_n) x = 0 \} \in \mathcal{B}^\sim \), they belong to \( \mathcal{B}_1(l), \) see Definition 4 and Theorem 1.c) in Part II. Since \( m \) is countably additive in the uniform operator topology on \( \mathcal{P}_2 \) and since \( E^s \in \mathcal{P}_2 \) for each \( s \in S \) and \( n = 1, 2, \ldots \), we obtain that \( \| m \| (E^s_n) \nrightarrow 0 \) as \( n \to + \infty \) for each \( s \in S \). Thus by Theorem 17 in Part II and Theorem 2 we have \( \| I \otimes m \| (E_n) \leq \| I \otimes \| (E^s_n), S \nrightarrow 0 \), which was to be shown.
The last assertion of the theorem may be proved similarly as the second assertion.

Denote by \( \mathfrak{I}_2(\mathcal{P} \otimes \mathcal{Q}) \) the closure of the set \( \mathfrak{I}_2(\mathcal{P} \otimes \mathcal{Q}) \) of all \( \mathcal{P} \otimes \mathcal{Q} \)-simple functions on \( T \times S \) with values in \( X \) in the sup norm \( \| \cdot \|_{T \times S} \), in the Banach space of all bounded \( X \) valued functions on \( T \times S \). For elements of \( \mathfrak{I}_2(\mathcal{P} \otimes \mathcal{Q}) \) we have the following Fubini type theorem.

**Theorem 4.** Let the product measure \( l \otimes m \) exist on \( \mathcal{P} \otimes \mathcal{Q} \), let \( f \in \mathfrak{I}_2(\mathcal{P} \otimes \mathcal{Q}) \) and let \( F \in \mathcal{P} \otimes \mathcal{Q} \) (if \( m^*(T) \cdot l^*(S) < +\infty \), then let \( F \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) \)). Then \( f \cdot \chi_F \) is integrable with respect to \( l \otimes m \), for each \( s \in S \) the function \( f(\cdot, s) \cdot \chi_F(\cdot, s) \) is integrable with respect to \( m \), for each \( E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) \) the function \( s \mapsto \int_{E^s} f(\cdot, s) \cdot \chi_F(\cdot, s) \cdot dm \) for each \( E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) \).

**Proof.** Let \( f_n \in \mathfrak{I}_2(\mathcal{P} \otimes \mathcal{Q}) \) be such that \( \| f_n - f \|_{T \times S} \to 0 \), \( n = 1, 2, \ldots \), and take \( A_0 \in \mathcal{P} \) and \( B_0 \in \mathcal{Q} \) so that \( F \subset A_0 \times B_0 \). (If \( m^*(T) \cdot l^*(S) < +\infty \), we take such \( A_0 \in \mathfrak{S}(\mathcal{P}) \) and \( B_0 \in \mathfrak{S}(\mathcal{Q}) \).) Then \( f_n(t, s) \to f(t, s) \) for each \( (t, s) \in T \times S \). If \( E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) \), then \( f_n \cdot \chi_E \in \mathfrak{I}_2(\mathcal{P} \otimes \mathcal{Q}) \) for each \( n = 1, 2, \ldots \). Thus by the definition of the semivariation \( l \otimes m \) and Theorem 2 we have

\[
\left| \int_E f_n \cdot \chi_E \, d(l \otimes m) - \int_E f_k \cdot \chi_E \, d(l \otimes m) \right| \leq \int_{E \cap F} (f_n - f_k) \, d(l \otimes m) \leq \left( \| f_n - f_k \|_{T \times S} \cdot (l \otimes m)(F) \right) \leq \left( \| f_n - f_k \|_{T \times S} \cdot m^*(A_0) \cdot l^*(B_0) \right)
\]

for each \( E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) \) and each \( n, k = 1, 2, \ldots \).

Since \( m^*(A_0) \cdot l^*(B_0) < +\infty \), we obtain by Theorem 7 from Part I that \( f \cdot \chi_E \) is integrable with respect to \( l \otimes m \), and

\[
\int_E f_n \cdot \chi_E \, d(l \otimes m) \to \int_E f \cdot \chi_E \, d(l \otimes m) \text{ for each } E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) .
\]

Let \( s \in S \). Then

\[
\left| \int_A f_n(\cdot, s) \cdot \chi_F(\cdot, s) \, dm - \int_A f_k(\cdot, s) \cdot \chi_F(\cdot, s) \, dm \right| \leq \left( \| f_n - f_k \|_{T \times S} \cdot m^*(A_0) \right)
\]

for each \( A \in \mathfrak{S}(\mathcal{P}) \) and each \( n, k = 1, 2, \ldots \).

Since \( m^*(A_0) < +\infty \), by Theorem 7 from Part I the function \( f(\cdot, s) \cdot \chi_F(\cdot, s) \) is integrable with respect to \( m \) and

\[
\int_A f_n(\cdot, s) \cdot \chi_F(\cdot, s) \, dm \to \int_A f(\cdot, s) \cdot \chi_F(\cdot, s) \, dm
\]

for each \( A \in \mathfrak{S}(\mathcal{P}) \); particularly,

\[
\int_{E^s} f_n(\cdot, s) \cdot \chi_F(\cdot, s) \, dm \to \int_{E^s} f(\cdot, s) \cdot \chi_F(\cdot, s) \, dm
\]

for each \( E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) \).
Let $E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B})$. Then using Theorem 14 from Part I we have

$$
\left| \int_B \int_{E^*} f_n(\cdot, s) \cdot \chi_F(\cdot, s) \, dm \, dl - \int_B \int_{E^*} f_0(\cdot, s) \cdot \chi_F(\cdot, s) \, dm \, dl \right| \leq \\
\leq \sup_{s \in B_0} \left| \int_{E^*} (f_n(\cdot, s) - f_0(\cdot, s)) \, dm \right| \cdot l^\wedge(B_0) \leq \\
\leq \|f_n - f_0\|_{T \times S} \cdot m^\wedge(A_0) \cdot l^\wedge(B_0) \quad \text{for each} \quad B \in \mathcal{E}(\mathcal{B}) \quad \text{and each} \quad n, k = 1, 2, \ldots.
$$

Since $m^\wedge(A_0) \cdot l^\wedge(B_0) < +\infty$, the relations (1) and (2) imply according to Theorem 16 from Part I ($\|f_n - f_k\|_{T \times S} \to 0$ as $n, k \to +\infty$) that the function $s \to \int_{E^*} f(\cdot, s) \cdot \chi_F(\cdot, s) \, dm$, $s \in S$, is integrable with respect to $l$ and that

$$
\int_S \int_{E^*} f_n(\cdot, s) \cdot \chi_F(\cdot, s) \, dm \, dl \to \int_S \int_{E^*} f(\cdot, s) \cdot \chi_F(\cdot, s) \, dm \, dl.
$$

It remains to observe that owing to Theorem 1

$$
\int_E f_n \cdot \chi_F \, dl \otimes m = \int_S \int_{E^*} f_n(\cdot, s) \cdot \chi_F(\cdot, s) \, dm \, dl
$$

for each $E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B})$ and each $n = 1, 2, \ldots$.

Let now $T$ and $S$ be locally compact Hausdorff topological spaces. By $\mathcal{B}_0(T)$, $\mathcal{B}_0(S)$ and $\mathcal{B}_0(T \times S)$ we denote the $\delta$-rings of relatively compact Baire subsets of $T$, $S$ and $T \times S$, respectively. According to Theorem E in § 51 in [21] we have $\mathcal{B}_0(T \times S) = \mathcal{B}_0(T) \otimes \mathcal{B}_0(S)$, and according to Theorem 8 in Part I we have $C_0(T \times S, X) = \mathcal{B}_0(\mathcal{B}_0(T \times S))$. Hence Theorem 4 yields immediately the following result:

**Theorem 5.** Let $T$ and $S$ be locally compact Hausdorff topological spaces, let $m : \mathcal{B}_0(T) \to L(X, Y)$ and $l : \mathcal{B}_0(S) \to L(Y, Z)$ be Baire operator valued measures countably additive in the strong operator topologies with $m^\wedge(T) \cdot l^\wedge(S) < +\infty$, let their product $l \otimes m$ exist on $\mathcal{B}_0(T) \otimes \mathcal{B}_0(S) = \mathcal{B}_0(T \times S)$ and let $f \in C_0(T \times S, X)$. Then $f$ is integrable with respect to $l \otimes m$, $f(\cdot, s)$ is integrable with respect to $m$ for each $s \in S$, for each $E \in \mathcal{E}(\mathcal{B}_0(T \times S))$ the function $s \to \int_{E^*} f(\cdot, s) \, dm$, $s \in S$, is integrable with respect to $l$, and

$$
\int_E f \, dl \otimes m = \int_S \int_{E^*} f(\cdot, s) \, dm \, dl
$$

for each $E \in \mathcal{E}(\mathcal{B}_0(T \times S))$.

This theorem may be combined with results on representation of bounded linear operators on spaces of the type $C_0(T, X)$, see [4] and [8], to obtain results about
bounded linear operators on \( C_0(T \times S, X) \) which are of the form \( Wf = U(Vf(\cdot, s)) \), \( f \in C_0(T \times S, X) \), where \( V : C_0(T, X) \to Y \) and \( U : C_0(S, Y) \to Z \). (The fact that \( Vf(\cdot, s) \in C_0(S, Y) \) for \( f \in C_0(T \times S, X) \) follows immediately from the boundedness of \( V \) and from the easily proved fact: Let \( f \in C_0(T \times S, X) \), let \( s \in S \) and \( \varepsilon > 0 \). Then there is an open neighbourhood \( O(s) \) of \( s \) such that \( |f(t, s) - f(t, s')| < \varepsilon \) for each \( t \in T \) and each \( s' \in O(s) \).

We present one such result for illustration.

**Corollary.** Let \( X \) be a reflexive Banach space and let \( V : C_0(T, X) \to Y \) and \( U : C_0(S, Y) \to Z \) be unconditionally converging bounded linear operators. Then \( W : C_0(T \times S, X) \to Z \) defined by the equality \( Wf = U(Vf(\cdot, s)) \), \( f \in C_0(T \times S, X) \), is weakly compact.

**Proof.** According to Theorem 3 in [8], \( V \) and \( U \) have representations \( Vg = \int_T g \, dm \), \( g \in C_0(T, X) \), and \( Uh = \int_S h \, dl \), \( h \in C_0(S, Y) \), where \( m : \mathcal{B}(\mathcal{B}_0(T)) \to L(X, Y) \) and \( l : \mathcal{B}(\mathcal{B}_0(S)) \to L(Y, Z) \) are operator valued measures, and the semivariations \( m^\wedge \) and \( l^\wedge \) are continuous on \( \mathcal{B}(\mathcal{B}_0(T)) \) and \( \mathcal{B}(\mathcal{B}_0(S)) \), respectively. According to Theorem 3 the product measure \( l \otimes m \) exists on \( \mathcal{B}(\mathcal{B}_0(T)) \otimes \mathcal{B}(\mathcal{B}_0(S)) = \mathcal{B}(\mathcal{B}_0(T \times S)) \), and its semivariation \( (l \otimes m) \) is continuous on \( \mathcal{B}(\mathcal{B}_0(T \times S)) \). By Theorem 5 we have \( Wf = \int_{T \times S} f \, dl \otimes m \), \( f \in C_0(T \times S, X) \).

Since \( X \) is a reflexive Banach space, the continuity of the semivariation \( (l \otimes m) \) on \( \mathcal{B}(\mathcal{B}_0(T \times S)) \) is a necessary and sufficient for the weak compactness of \( W \), see Remark 1 in [8]. The corollary is proved.

**Some special cases.**

1. Let \( Z \) contain no isomorphic copy of \( c_0 \). Then by the \( \ast \)-Theorem in Section 1.1 in Part I the semivariation \( l^\wedge \) is continuous on \( \mathcal{P} \). Thus by Theorem 1 the product measure \( l \otimes m \) exists on \( \mathcal{P}_0 \otimes \mathcal{P} \). By Theorem 2 the semivariation \( \hat{(l \otimes m)} \) is finite on \( \mathcal{P} \otimes \mathcal{P} \), hence by the \( \ast \)-Theorem it is continuous on \( \mathcal{P} \otimes \mathcal{P} \).

2. Let \( X \) be the space of scalars and let \( Y = Z \) be a commutative Banach algebra, or let \( X = Y = Z \) be a commutative Banach algebra, or let \( X = Y = Z \) and let \( l(B) m(A) = m(A) l(B) \) for each \( A \in \mathcal{P} \) and \( B \in \mathcal{P} \). Suppose further that the product measure \( l \otimes m \) exists on \( \mathcal{P} \otimes \mathcal{P} \). Then by Lemma 1 the product measure \( m \otimes l \) exists on \( \mathcal{P} \otimes \mathcal{P} = \mathcal{P} \otimes \mathcal{P} \) and is equal to \( l \otimes m \). Thus in this case

\[
\int_S \int_{E^t} f(\cdot, s) \cdot \chi_{E^t}(\cdot, s) \, dm \, dl = \int_T \int_{E^t} f(t, \cdot) \cdot \chi_{E^t}(t, \cdot) \, dl \, dm,
\]

in Theorem 4 and similarly

\[
\int_S \int_{E^t} f(\cdot, s) \, dm \, dl = \int_T \int_{E^t} f(t, \cdot) \, dl \, dm
\]

in Theorem 5.

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Results on the products of operator valued measures have applications in convolutions of vector measures, see for example [34], [23], [14].

2. MEASURABILITY OF THE PARTIAL INTEGRAL

Example. Let $T = S = \{1, 2, \ldots\}$, let $\mathcal{P} = \mathcal{B} = 2^T$, let $X$ be the space of real numbers, and let $Y = Z = c_0$. Let $m : 2^T \to L(X, c_0) = c_0$ and $l : 2^S \to L(c_0, c_0)$ be defined by the countable additivity from the following elementary values:

$$m(k) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \frac{k}{0, \ldots, 0, 1, 0, 0, \ldots} & \in c_0 \text{ if } k \text{ is odd,} \end{cases}$$
$$l(k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{k}{0, \ldots, 0, 1, 0, 0, \ldots} & \in c_0 \text{ if } k \text{ is even.} \end{cases}$$

Then clearly $m$ and $l$ are operator valued measures with bounded countably additive variations and their product $l \otimes m = m \otimes l$ exists and is identically equal to zero. Thus every function $f : T \times S \to X$ is integrable with respect to $l \otimes m$. Now it is easy to see that the function $f(t, s) = e^t$, $(t, s) \in T \times S$, is not integrable with respect to $m$ for any $s \in S = \{1, 2, \ldots\}$.

From this example it is clear that in a general Fubini theorem we must suppose that for a $\mathcal{P} \otimes \mathcal{B}$-measurable function $f : T \times S \to X$, the function $t \mapsto f(t, s)$, $t \in T$, is integrable with respect to the measure $m$ for each $s \in S$. Since a $\mathcal{P} \otimes \mathcal{B}$-measurable function is, by definition, a pointwise limit of a sequence of $\mathcal{P} \otimes \mathcal{B}$-simple functions, we conclude from Theorem A in § 34 [21] and from the fact that the $\mathcal{P}$-measurable functions are closed under the formation of pointwise limits of sequences, see Section 1.2 in Part I and Lemma 1.2 in [24], that the function $f(\cdot, s)$ is $\mathcal{P}$-measurable for each $s \in S$ provided $f : T \times S \to X$ is $\mathcal{P} \otimes \mathcal{B}$-measurable.

Let $f : T \times S \to X$ be a $\mathcal{P} \otimes \mathcal{B}$-measurable function and let $f(\cdot, s)$ be integrable with respect to $m$ for each $s \in S$. In this section we investigate the $\mathcal{B}$-measurability and the essential $l - \mathcal{B}$-measurability of the partial integral $g_E$, $g_E(s) = \int_{E^r} f(\cdot, s) \, dm$, $s \in S$, $E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B})$. In fact, $\mathcal{B}$ is replaced in Theorems 6–12 by an arbitrary $\delta$-ring $\mathcal{D}$ of subsets of $S$. Besides, we obtain results on the $\mathcal{D}$-measurability of the function $h_E$, $h_E(s) = m^r(f(\cdot, s), E^r)$, $s \in S$, and important results which are needed for the proof of the Fubini theorem in § 3.

**Theorem 6.** Let $\mathcal{D}$ be a $\delta$-ring of subsets of $S$ and let $f : T \times S \to X$ be a $\mathcal{P}^r \otimes \mathcal{D}$-measurable function. Then for each $E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D})$ the function $h_E$, $h_E(s) = m^r(f(\cdot, s), E^r)$, $s \in S$, is $\mathcal{D}$-measurable.
Proof. Let $E \in \mathcal{S}(\mathcal{P} \otimes \mathcal{D})$ and let $f_n$, $n = 1, 2, \ldots$ be a sequence of $\mathcal{P}^\sim \otimes \mathcal{D}$-simple functions such that $f_n(t, s) \to f(t, s)$ and $|f_n(t, s)| \to |f(t, s)|$ for each $(t, s) \in T \times S$, see Section 1.2 in Part I. According to Theorem 4 in Part II we have $m^\sim(f(\cdot, s), E^s) = \sup_{|y| \leq 1} \int_{|y|} f(t, s) \, d\mathcal{M}(y^* m, \cdot)$ for each $s \in S$. The same equality holds for each $f_n$, $n = 1, 2, \ldots$. Hence $m^\sim(f(\cdot, s), E^s) = \lim_{n \to \infty} m^\sim(f_n(\cdot, s), E^s)$ for each $s \in S$ by the Fatou lemma. Therefore it suffices to prove the theorem for each $\mathcal{P}^\sim \otimes \mathcal{D}$-simple function $f : T \times S \to X$.

Let $f : T \times S \to X$ be a $\mathcal{P}^\sim \otimes \mathcal{D}$-simple function of the form $f = \sum_{i=1}^{r} x_i \cdot \chi_{E_i}$, $x_i \in X$, $E_i \in \mathcal{P}^\sim \otimes \mathcal{D}$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, r$, and let $E \in \mathcal{S}(\mathcal{P} \otimes \mathcal{D})$. Since $\mathcal{P}^\sim \otimes \mathcal{D} \cap \mathcal{S}(\mathcal{P} \otimes \mathcal{D}) = \mathcal{P}^\sim \otimes \mathcal{D}$, and since $E_i \in \mathcal{P}^\sim \otimes \mathcal{D}$, $i = 1, \ldots, r$, we may suppose without loss of generality that $E \in \mathcal{P}^\sim \otimes \mathcal{D}$. Take $A \in \mathcal{P}^\sim$ and $B \in \mathcal{D}$ so that $E \subset A \times B$. Let $x \in X$ and $|x| = 1$, and let $d : T \times S \to X$ be the $\mathcal{P}^\sim$-simple function defined by the equality $d = (\sum_{i=1}^{r} |x_i|) \cdot x \cdot \chi_A$. Then clearly $d \in L_1(m)$, see Theorem 1c) and Definition 4 in Part II. Denote by $\mathcal{R}$ the ring of all finite unions of pairwise disjoint rectangles $C \times D$, $C \in \mathcal{P}^\sim$ and $D \in \mathcal{D}$, see Theorem E in §33 [21]. If $F_i \in \mathcal{R} \cap (A \times B)$ for each $i = 1, \ldots, r$, then for $g = \sum_{i=1}^{r} x_i \cdot \chi_{F_i}$, the function $s \to m^\sim(g(\cdot, s), A)$, $s \in S$, is clearly $\mathcal{D}$-measurable. Denote by $\mathcal{M}_1$ the class of all sets $F_1 \in \mathcal{P}^\sim \otimes \mathcal{D} \cap (A \times B)$ for which the function $s \to m^\sim(g(\cdot, s), A)$, $s \in S$, is $\mathcal{D}$-measurable provided $g = \sum_{i=1}^{r} x_i \cdot \chi_{F_i}$ and $F_1, \ldots, F_r \in \mathcal{R} \cap (A \times B)$. Then $\mathcal{R} \cap (A \times B) \subset \mathcal{M}_1$, and since $|g(t, s)| \leq |g_0(t)|$ for each $(t, s) \in T \times S$, $\mathcal{M}_1$ is a monotone class by Theorem 17 from Part II. Thus $\mathcal{M}_1 = \mathcal{P}^\sim \otimes \mathcal{D} \cap (A \times B)$ by Theorem B in §6 [21]. Similarly, if $\mathcal{M}_2$ is the class of all sets $F_2 \in \mathcal{P}^\sim \otimes \mathcal{D} \cap (A \times B)$ for which the function $s \to m^\sim(g(\cdot, s), A)$, $s \in S$, is $\mathcal{D}$-measurable provided $g = \sum_{i=1}^{r} x_i \cdot \chi_{F_i}$, $F_1 \in \mathcal{M}_1$ and $F_2, \ldots, F_r \in \mathcal{R} \cap (A \times B)$, then $\mathcal{M}_2 = \mathcal{P}^\sim \otimes \mathcal{D} \cap (A \times B)$. Continuing in this way we obtain that $\mathcal{M}_r = \mathcal{P}^\sim \otimes \mathcal{D} \cap (A \times B)$, which was to be shown. The theorem is proved.

Let us remind that a subset $A \subset Y^*$ is called norming (or total) for $Y$ if $|y| = \sup_{y^* \in A} |y^* y|$ for each $y \in Y$, see Definition 2.8.1 in [22]. It is well known, see Theorem 2.8.5 in [22], that separable Banach spaces and their duals have countable norming sets.

Theorem 7. Let $\mathcal{D}$ be a $\delta$-ring of subsets of $S$, let $f : T \times S \to X$ be a $\mathcal{P} \otimes \mathcal{D}$-measurable function and let $Y$ have a countable norming set. Then for each $E \in \mathcal{S}(\mathcal{P} \otimes \mathcal{D})$ the function $h_E : h_E(s) = m^\sim(f(\cdot, s), E^s)$, $s \in S$, is $\mathcal{D}$-measurable.

Proof. Let $y_n^* \in Y^*$, $n = 1, 2, \ldots$ be a countable norming set for $Y$ and let $E \in \mathcal{S}(\mathcal{P} \otimes \mathcal{D})$. Then by Theorem 4 from Part II, $h_E(s) = m^\sim(f(\cdot, s), E^s) = \sup_{n}$
\[ \int_{E^{\infty}} f(\cdot, s) \, dv(y_{n}^{*} m, \cdot) \] for each \( s \in S \). Hence by Theorem A in §20 [21] it suffices to prove the \( \mathcal{D} \)-measurability of the function \( s \to \int_{E^{\infty}} f(\cdot, s) \, dv(y_{n}^{*} m, \cdot) \) \( s \in S \), for each \( n = 1, 2, \ldots \). But this follows immediately from Theorem 6, since by assumption the function \( f \) is \( \mathcal{P} \otimes \mathcal{D} \)-measurable, and since \( v(y_{n}^{*} m, \cdot) \) is a countably additive finite non-negative measure on \( \mathcal{P} \) for each \( n = 1, 2, \ldots \), see Example 5 in Part I.

**Theorem 8.** Let \( \mathcal{D} \) be a \( \delta \)-ring of subsets of \( S \), let \( f : T \times S \to X \) be a \( \mathcal{P} \otimes \mathcal{D} \)-measurable function and let \( f(\cdot, s) \in L_{1}(m) \) for each \( s \in S \) (see Part II). Then for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \) the functions \( g_{E} \) and \( g_{E}(s) = \int_{E} f(\cdot, s) \, dm \), \( s \in S \), and \( h_{E} \), \( h_{E}(s) = m^{\infty}(f(\cdot, s), E^{t}) \), \( s \in S \), are \( \mathcal{D} \)-measurable. If \( \mathcal{D} = \mathcal{P} \), if the product measure \( l \otimes m \) exists on \( \mathcal{P} \otimes \mathcal{D} \), and if \( h_{T \times S} \in L_{1}(l) \), then \( f \in L_{1}(l \otimes m) \).

**Proof.** Let \( f_{n}, n = 1, 2, \ldots \) be a sequence of \( \mathcal{P} \otimes \mathcal{D} \)-simple functions on \( T \times S \) such that \( f_{n}(t, s) \to f(t, s) \) and \( |f_{n}(t, s)| \to |f(t, s)| \) for each \( (t, s) \in T \times S \), see Section 1.2 in Part I. Then clearly \( f_{n}(\cdot, s) \in L_{1}(m) \) for each \( n = 1, 2, \ldots \) and each \( s \in S \), hence \( f \) is \( \mathcal{P} \otimes \mathcal{D} \)-measurable. Thus by Theorem 6 the function \( h_{E} \) is \( \mathcal{D} \)-measurable for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \). Further, according to Theorem 17 in Part II we have \( m^{\infty}(f(\cdot, s) - f_{n}(\cdot, s), T) \to 0 \) for each \( s \in S \). Let \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \) and put \( g_{n,E}(s) = = \int_{E} f_{n}(\cdot, s) \, dm \), \( s \in S \), \( n = 1, 2, \ldots \). Then according to Lemma 2.1 the functions \( g_{n,E}, n = 1, 2, \ldots \) are \( \mathcal{D} \)-measurable. Applying Corollary of Theorem 2 from Part II we obtain that \( |g_{n,E}(s) - g_{E}(s)| \leq m^{\infty}(f(\cdot, s) - f_{n}(\cdot, s), T) \to 0 \) as \( n \to \infty \). Thus \( g_{n,E}(s) \to g_{E}(s) \) for each \( s \in S \) which proves the \( \mathcal{D} \)-measurability of \( g_{E} \) since the \( \mathcal{D} \)-measurable functions are closed under the formation of pointwise limits of sequences, see Section 1.2 in Part I of Lemma 1.2 in [24].

Concerning the second assertion of the theorem we have to show that the \( L_{1} \)-pseudonorm \( (l \otimes m)(f, \cdot) \) is continuous on \( \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \). Let \( E_{k} \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}), k = 1, 2, \ldots \), and let \( E_{k} \to 0 \). Since by assumption \( f(\cdot, s) \in L_{1}(m) \) for each \( s \in S \), we have \( h_{E_{k}}(s) \to 0 \) for each \( s \in S \) by Theorem 17 in Part II. By assumption \( h_{T \times S} \in L_{1}(l) \), hence \( l^{*}(h_{E_{k}}, S) \to 0 \) again by Theorem 17 in Part II. Thus by Theorem 2 we have \( (l \otimes m)(f, E_{k}) \leq l^{*}(h_{E_{k}}, S) \to 0 \), which completes the proof of the theorem.

**Theorem 9.** Let \( \mathcal{D} \) be a \( \delta \)-ring of subsets of \( S \), let \( f : T \times S \to X \) be a \( \mathcal{P} \otimes \mathcal{D} \)-measurable function and let for each \( s \in S \) the function \( t \to f(t, s), t \in T \), be integrable with respect to \( m \). Then for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \) the function \( g_{E} : L_{1}(m) \to \mathbb{R} \) defined by \( g_{E}(s) = = \int_{E} f(\cdot, s) \, dm \), \( s \in S \), is \( \mathcal{D} \)-measurable.

**Proof.** Put \( F = \{(t, s) \in T \times S, f(t, s) \neq 0\} \). Then \( F \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \), hence there are \( A \in \mathcal{E}(\mathcal{P}^{\infty}) \) and \( B \in \mathcal{E}(\mathcal{D}) \) such that \( F \subset A \times B \). Take \( A_{n} \in \mathcal{P}^{\infty}, n = 1, 2, \ldots \) so that \( A_{n} \uparrow A \). Clearly \( F_{n} = \{(t, s) \in T \times S, f(t, s) < n\} \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \) and \( F_{n} \uparrow F, n = 1, 2, \ldots \). Now it is easy to see that \( H_{n} = (A_{n} \times B) \cap F_{n} \in \mathcal{P} \otimes \mathcal{E}(\mathcal{D}), H_{n} \uparrow F \) and \( f(\cdot, s), \chi_{H_{n}}(\cdot, s) \in L_{1}(m) \) for each \( n = 1, 2, \ldots \) and each \( s \in S \). Thus by Theorem 8 the functions \( g_{n,E} : g_{n,E}(s) = \int_{E} f(\cdot, s) \cdot \chi_{H_{n}}(\cdot, s) \, dm \), \( s \in S \), \( n = 1, 2, \ldots \) and
\( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \), are \( \mathcal{D} \)-measurable. Since the integrability of the function \( t \rightarrow f(t, s) \), \( t \in T \), for each \( s \in S \) implies that \( g_E(s) = \lim_{n \to \infty} g_{n,E}(s) \) for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \) and each \( s \in S \), the theorem is proved.

**Theorem 10.** Let \( \mathcal{D} \) be a \( \delta \)-ring of subsets of \( S \), let \( f : T \times S \to X \) be a \( \mathcal{P} \otimes \mathcal{D} \)-measurable function and let the function \( f(\cdot, s) \) be integrable with respect to \( m \) for each \( s \in S \). Then for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \) the function \( g_{E, s}(s) = \int_{E,s} f(\cdot, s) \, dm \), \( s \in S \), is weakly \( \mathcal{D} \)-measurable. Hence, if \( Y \) is a separable Banach space, then \( g_{E, s}(s) \) is \( \mathcal{D} \)-measurable for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \).

**Proof.** Let \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \) and let \( y^* \in Y \). Then \( y^* g_{E, s}(s) = \int_{E,s} f(\cdot, s) \, dy^* m \) for each \( s \in S \), see the paragraph after Theorem 7 in Part I. According to Example 5 in § 1 in Part I we have \( v(y^* m, A) = y^* \hat{m}(A) \leq |y^*| \cdot m^*(A) < +\infty \) for each \( A \in \mathcal{P} \), hence \( y^* m \) is continuous on \( \mathcal{P} \). Thus the \( \mathcal{D} \)-measurability of \( y^* g_{E, s} \) follows from Theorem 9. For the second assertion of the theorem see Theorem 3.5.3 in [22].

**Theorem 11.** Let \( \mathcal{D} \) be a \( \delta \)-ring of subsets of \( S \), let \( f : T \times S \to X \) be a \( \mathcal{P} \otimes \mathcal{D} \)-measurable function and let \( f(\cdot, s) \) be integrable with respect to \( m \) for each \( s \in S \). Let further

\[
 f_n = \sum_{i=1}^{r_n} x_{n,i} \cdot \chi_{E_{n,i}}, \quad x_{n,i} \in X, \quad E_{n,i} \in \mathcal{P} \otimes \mathcal{D}, \quad n = 1, 2, \ldots, \quad i = 1, \ldots, r_n,
\]

be a sequence of \( \mathcal{P} \otimes \mathcal{D} \)-simple functions such that \( f_n(t, s) \to f(t, s) \) for each \((t, s) \in T \times S\), and let \( X \) be the closed linear span of \( X_0 = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{r_n} x_{n,i} \) in \( X \). Then for each \( s \in S \) the function \( f(\cdot, s) \) is integrable with respect to the restricted measure \( m : \mathcal{P} \to L(X_1, Y) \) and the set of all finite sums of the form \( \sum_{j=1}^{r} m(A_j) x_j, A_j \in \mathcal{P}, x_j \in X_0, j = 1, \ldots, r \) is dense in the subset \( \{ \int_A f(\cdot, s) \, dm; A \in \mathcal{E}(\mathcal{P}), s \in S \} \) of \( Y \).

**Proof.** In the proof of Theorem 15 in Part I we found, under the assumptions of the theorem and for each \( s \in S \), a set \( N(s) \in \mathcal{E}(\mathcal{P}) \), a sequence \( F_k(s) \in \mathcal{P} \) and a subsequence \( n_k(s), k = 1, 2, \ldots \), such that \( \lim_{k \to \infty} \int_A f_{n_k}(\cdot, s) \cdot \chi_{F_k(s) \cup N(s)}(\cdot, s) \, dm = \int_A f(\cdot, s) \, dm \) uniformly with respect to \( A \in \mathcal{E}(\mathcal{P}) \). It remains to observe that for each \( s \in S \) the integrals on the left hand side of the last equality are of the form \( \sum_{j=1}^{r} m(A_j) x_j \) with \( A_j \in \mathcal{P}, x_j \in X_0, j = 1, \ldots, r \). Note that the semivariation of the restricted measure \( m : \mathcal{P} \to L(X_1, Y) \) is less than or equal to the semivariation of \( m : \mathcal{P} \to L(X_1, Y) \), hence it is finite on \( \mathcal{P} \).

Using Theorem 10 we immediately have

**Corollary.** Let \( \mathcal{D} \) be a \( \delta \)-ring of subsets of \( S \), let \( f : T \times S \to X \) be a \( \mathcal{P} \otimes \mathcal{D} \)-measurable function, let the function \( f(\cdot, s) \) be integrable with respect to \( m \) for each
\[ s \in S \text{ the and let } \{ m(A) x ; A \in \mathcal{P} \} \text{ be a separable subset of } Y \text{ for each } x \in X. \text{ Then}
\]

1) \( \{ \int_{A} f(\cdot, s) \, dm ; A \in \mathcal{E}(\mathcal{P}), s \in S \} \) is a separable subset of \( Y \), and

2) \( \text{for each } E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \text{ the function } g_{E} = \int_{E} f(\cdot, s) \, dm, s \in S, \text{ is } \mathcal{D}-\text{measurable.} \)

**Theorem 12.** Let \( \mathcal{P} \) be generated by a countable family of subsets of \( T \), let \( \mathcal{D} \) be a \( \delta \)-ring of subsets of \( S \), let \( f : T \times S \to X \) be a \( \mathcal{P} \times \mathcal{D} \)-measurable function and let the function \( f(\cdot, s) \) be integrable with respect to \( m \) for each \( s \in S \). Then

1) \( \{ \int_{A} f(\cdot, s) \, dm ; A \in \mathcal{E}(\mathcal{P}), s \in S \} \) is a separable subset of \( Y \),

2) \( \text{for each } E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D}) \text{ the function } g_{E} = \int_{E} f(\cdot, s) \, dm, s \in S, \text{ is } \mathcal{D}-\text{measurable, and} \)

3) the function \( v, v(s) = \sup_{A \in \mathcal{E}(\mathcal{P})} | \int_{A} f(\cdot, s) \, dm |, s \in S, \text{ is finite valued and } \mathcal{D}-\text{measurable.} \)

**Proof.** Without loss of generality we may suppose that \( \mathcal{P} \) is generated by a countable ring \( \mathcal{R} = \{ R_{n}, n = 1, 2, \ldots \} \), see Theorem C in §5 [21].

1) and 2). According to Corollary of Theorem 11 it suffices to show that \( Y_{x} = \{ m(A) x ; A \in \mathcal{P} \} \) is a separable subset of \( Y \) for each \( x \in X \).

Let \( x \in X \). Put \( R_{n} = (R_{1} \cup \ldots \cup R_{n}) \cap \mathcal{R} \) and \( \mathcal{I}_{n} = \mathcal{E}(\mathcal{R}_{n}), n = 1, 2, \ldots \). Then clearly \( \mathcal{P} = \delta(\mathcal{R}) = \bigcup_{n=1}^{\infty} \mathcal{I}_{n} \). We will show that the set \( Y_{0} \) of all finite sums of the form \( \sum_{i=1}^{r} m(R_{n_{i}}) x \) is dense in \( Y_{x} \) (\( Y_{0} \) is clearly countable). Let \( A \in \mathcal{P} \). Then there is an \( n_{A} \) such that \( A \in \mathcal{I}_{n_{A}} \). Let \( \lambda_{n_{A}} : \mathcal{I}_{n_{A}} \to (0, +\infty) \) be a control measure for the vector measure \( m(\cdot) x : \mathcal{I}_{n_{A}} \to Y \). Then the desired assertion immediately follows from Theorem D in §13 [21] applied to \( \lambda_{n_{A}} \) and from the simple inequality

\[ | m(A_{1}) x - m(A_{2}) x | \leq | m(A_{1} - A_{2}) x | + | m(A_{2} - A_{1}) x | \leq 2 \left\| m(\cdot) x \right\| (A_{1} \Delta A_{2}), \]

\( A_{1}, A_{2} \in \mathcal{I}_{n_{A}} \).

3) Since \( A \to \int_{A} f(\cdot, s) \, dm, A \in \mathcal{E}(\mathcal{P}) \) is a countably additive vector measure on a \( \sigma \)-ring, \( v \) is finite valued, see IV.10.4 in [19]. By Theorem IV.10.5 in [19] and Theorem D in §13 [21] we have \( v(s) = \sup_{n} | \int_{R_{n}} f(\cdot, s) \, dm | \) for each \( s \in S \), hence 2) and Theorem A in §20 [21] imply the \( \mathcal{D} \)-measurability of \( v \).

**Theorem 13.** In the following cases: 1) \( X \) is separable, 2) \( Y \) has a countable norming set, and 3) \( \mathcal{E}(\mathcal{P}_{2}) \supseteq \mathcal{P} \); for each \( A \in \mathcal{E}(\mathcal{P}) \) there is a countably additive measure \( \lambda_{A} : \mathcal{E}(\mathcal{P}) \to (0, +\infty) \) such that \( C \in \mathcal{E}(\mathcal{P}), \lambda_{A}(A \cap C) = 0 \Rightarrow m^{*}(A \cap C) = 0 \).

**Proof.** Let \( A \in \mathcal{E}(\mathcal{P}) \) and take \( A_{n} \in \mathcal{P}, n = 1, 2, \ldots \) so that \( A_{n} \nearrow A \). Since \( m^{*}(C) = \sup_{|y^{*}| \leq 1} v(y^{*} m, C) \) for each \( C \in \mathcal{E}(\mathcal{P}) \), see Lemma 1 in [8], we have \( m^{*}(A \cap C) = 491 \).
\[
= \lim_{n \to \infty} m^\wedge(A_n \cap C) \text{ for each } C \in \mathcal{E}(P)\]. Suppose that the theorem is proved for each 
\(A \in P\), take countably additive measures \(\lambda_n : \mathcal{E}(P) \to (0, +\infty)\) so that \(C \in \mathcal{E}(P)\), \n\(\lambda_n(A_n \cap C) = 0 \Rightarrow m^\wedge(A_n \cap C) = 0, n = 1, 2, \ldots,\) and put

\[
\lambda_n(C) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(A_n \cap C)}{1 + \lambda_n(T)}, \quad C \in \mathcal{E}(P).
\]

Then clearly \(\lambda_n\) has the required properties. Consequently, it is sufficient to prove the theorem for each \(A \in P\).

1) Let \(A \in P\) and let \(x_k \in X, k = 1, 2, \ldots,\) be a dense subset of \(X\). For each \(k = 1, 2, \ldots\) let \(\lambda_k : A \cap \mathcal{E}(P) \to (0, +\infty)\) be a control measure for the vector measure \(m(\cdot) x_k : A \cap \mathcal{E}(P) \to Y\). Then clearly

\[
\lambda_k(C) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\lambda_k(A \cap C)}{1 + \lambda_k(A)},
\]

\(C \in \mathcal{E}(P)\), has the required properties.

2) Let \(A \in P\) and let \(y^*_k \in Y^*, k = 1, 2, \ldots\) be a countable norming set for \(Y\). Then \(m^\wedge(A \cap C) = \sup_k v(y^*_k m, A \cap C)\) for each \(C \in \mathcal{E}(P)\), see Lemma 1 in [8]. Now clearly it suffices to put

\[
\lambda_k(C) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{v(y^*_k m, A \cap C)}{1 + v(y^*_k m, A)}, \quad C \in \mathcal{E}(P).
\]

3) Similarly as at the beginning of the proof we may suppose that \(A \in P_2\). But then \(m : A \cap (P \otimes B) \to L(X, Y)\) is countably additive, hence a control measure for it has the required properties.

**Definition 2.** A function \(u : T \to X\) is called \(m\)-null if there is an \(N \in \mathcal{E}(P)\) with \(m^\wedge(N) = 0\) such that \(\{t \in T; u(t) = 0\} \subset N\). A function \(f : T \to X\) is called \(m\)-essentially \(P\)-measurable (integrable) if it can be written in the form \(f = g + u\), where \(g\) is \(P\)-measurable (integrable) and \(u\) is \(m\)-null. In the case \(f\) is \(m\)-essentially integrable we extend the integral defining \(\int_A f \, dm = \int_A g \, dm\) for each \(A \in \mathcal{E}(P)\).

Clearly our theory of integration extends with obvious modifications to \(m\)-essentially measurable (integrable) functions. Particularly, if \(f_n : T \to X, n = 1, 2, \ldots\) are \(m\)-essentially \(P\)-measurable and \(\lim_{n \to \infty} f_n(t) = f(t) \in X\) a.e. \(m\), then \(f\) is also \(f\)-essentially \(P\)-measurable. Hence in the theorems of our extended theory the limit function is automatically \(m\)-essentially \(P\)-measurable. Note also that the range of an \(m\)-null, hence also of an \(m\)-essentially \(P\)-measurable function, need not be separable.

**Theorem 14.** Let \(f : T \times S \to X\) be a \(P \otimes B\)-measurable function, let the function \(f(\cdot, s)\) be integrable with respect to \(m\) for each \(s \in S\), and for each \(B \in \mathcal{E}(B)\) let there
be a countably additive measure $\lambda_B : \mathcal{E}(\mathcal{B}) \to (0, +\infty)$ such that $D \in \mathcal{E}(\mathcal{B})$, $\lambda_B(B \cap D) = 0 \Rightarrow I^*(B \cap D) = 0$, see Theorem 13. Then for each set $E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B})$ the function $g_E, g_E(s) = \int_{E_f} f(\cdot, s) ds, s \in S$, is $I$-essentially $\mathcal{B}$-measurable.

Proof. Let $E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B})$. Take $A \in \mathcal{E}(\mathcal{P})$ and $B \in \mathcal{E}(\mathcal{B})$ so that $E \subseteq A \times B$, and take the corresponding measure $\lambda_B : \mathcal{E}(\mathcal{B}) \to (0, +\infty)$. Let $f_n : T \to X$, $n = 1, 2, \ldots$ be a sequence of $\mathcal{P} \otimes \mathcal{B}$-simple functions such that $f_n(t, s) \to f(t, s)$ for each $(t, s) \in T \times S$, and let $X_t$ be the closed linear span of the union of their ranges in $X$. Then according to Theorem 11 we may replace $X$ by the separable space $X_t$. But then by Theorem 13-1), there is a countably additive measure $\mu_A : \mathcal{E}(\mathcal{P}) \to (0, +\infty)$ such that $C \in \mathcal{E}(\mathcal{P})$ and $\mu_A(A \cap C) = 0 \Rightarrow m^\cdot_1(A \cap C) = 0$, where $m^\cdot_1$ is the semivariation of the restricted measure $m : \mathcal{P} \to L(X_t, Y)$ (clearly $m^\cdot_1(C) \leq m^\cdot(\mathcal{P})$ for each $C \in \mathcal{E}(\mathcal{P})$). Obviously $F = \bigcup_{n=0}^\infty \{(t, s) \in T \times S; f_n(t, s) \neq 0\} \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) = \mathcal{E}(\mathcal{P}) \otimes \mathcal{E}(\mathcal{B})$, where $f_0 = f$. Since $\lambda_B \otimes \mu_A : \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \to (0, +\infty)$ is a countably additive measure, according to the Egoroff–Lusin theorem, see Section 1.4 in Part I, there is a set $N \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B})$, $N \subseteq F$, and a sequence $F_k \in \mathcal{P} \otimes \mathcal{B}$, $k = 1, 2, \ldots$ such that $(\lambda_B \otimes \mu_A)(N) = 0$, $F_k \supseteq F - N$, and on each $F_k$, $k = 1, 2, \ldots$ the sequence $f_n$, $n = 1, 2, \ldots$ converges uniformly to $f$. Clearly $g_E(s) = g_{E,(F-N)}(s) + g_{E,N}(s)$ for each $s \in S$. Owing to Theorem 4 each function $g_{E,N}(s)$, $k = 1, 2, \ldots$ is $\mathcal{B}$-measurable. Thus to prove the theorem it is now sufficient to prove that the function $g_{E,N}$ is $I$-null. Obviously $\{s \in S; g_{E,N}(s) \neq 0\} \subseteq B$. Since $0 = (\lambda_B \otimes \mu_A)(A \times B \cap N) = \int_B \mu_A(A \cap N^c) d\lambda_B$, there is a set $D \in \mathcal{E}(\mathcal{B})$ with $\lambda_B(B \cap D) = 0$ such that $\mu_A(A \cap N^c) = 0$ for each $s \in B - D$, see Theorem A in §36 [21]. But then $m^\cdot_1(A \cap N^c) = 0$, hence $g_{E,N}(s) = 0$ for each $s \in B - D$. Thus $\{s \in S; g_{E,N}(s) = 0\} \subseteq B \cap D$. However $I^*(B \cap D) = 0$, hence $g_{E,N}$ is $I$-null, which proves the theorem.

Remark 1. Let $\mathcal{D}$ be a $\delta$-ring of subsets of $S$, let $f : T \times S \to X$ be a $\mathcal{P} \otimes \mathcal{D}$-measurable function and let for each $s \in S$ the function $f(\cdot, s)$ be integrable with respect to $\mu$. Then the $\mathcal{D}$-measurability of the function $g_E, g_E(s) = \int_{E_f} f(\cdot, s) dm, s \in S$, for each $E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D})$, depends of course on the function $f$. Particularly, if the range of $f$ is relatively $\sigma$-compact in $X$, then Theorem 4 and Theorem 16 from Part I immediately imply the $\mathcal{D}$-measurability of $g_E$ for each $E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{D})$.

3. THE FUBINI THEOREM

For the proof of the general Fubini theorem we shall need also the following lemmas:

Lemma 3. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be $\delta$-rings of subsets of $T$ and $S$, respectively, and let $f : T \times S \to X$ be a $\mathcal{D}_1 \otimes \mathcal{D}_2$-measurable function. Then there are sequences $A_n \in \mathcal{D}_1$, $B_n \in \mathcal{D}_2$, $n = 1, 2, \ldots$ such that $f$ is $\delta(\{A_n \times B_n\}_{n=1}^\infty)$-measurable.

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Proof. According to the definition of a $\mathcal{D}_1 \otimes \mathcal{D}_2$-measurable function, see Section 1.2 in Part I, there is a sequence $f_k, k = 1, 2, \ldots$ of $\mathcal{D}_1 \otimes \mathcal{D}_2$-simple functions such that $f_k(t, s) \to f(t, s)$ for each $(t, s) \in T \times S$. Each $f_k$ is of the form $f_k = \sum_{i=1}^{r_k} x_{k,i} \cdot I_{E_k,i}$ with $x_{k,i} \in X, E_{k,i} \in \mathcal{D}_1 \otimes \mathcal{D}_2$, $E_{k,i} \cap E_{k,j} = \emptyset$ for $i \neq j, i, j = 1, \ldots, r_k$. Since $\mathcal{D}_1 \otimes \mathcal{D}_2$ is the smallest $\delta$-ring over all rectangles $A \times B, A \in \mathcal{D}_1, B \in \mathcal{D}_2$, the obviously valid $\delta$-version of Theorem D in § 5 [21] implies that for each couple $(k, i), k = 1, 2, \ldots, i = 1, \ldots, r_k$, there are sequences $A_{k,i,j} \in \mathcal{D}_1, B_{k,i,j} \in \mathcal{D}_2$, $j = 1, 2, \ldots$, such that $E_{k,i} \in \delta\{A_{k,i,j} \times B_{k,i,j}\}$. By a suitable enumeration of the countable set $\{(k, i, j); k = 1, 2, \ldots, i = 1, \ldots, r_k, j = 1, 2, \ldots\}$ we immediately obtain the required sequences $A_n \in \mathcal{D}_1, B_n \in \mathcal{D}_2, n = 1, 2, \ldots$.

The following lemma is an immediate consequence of the Orlicz-Pettis theorem, see Theorem 3.2.3 in [22] and Theorem IV.10.1 in [19].

**Lemma 4.** Let $z_{n,k} \in Z, k, n = 1, 2, \ldots$, let the series $\sum_{k=1}^{\infty} z_{n,k}$ be unconditionally convergent in $Z$ for each $n = 1, 2, \ldots$ and let for each $I_n \subset \{1, 2, \ldots\}$ the series $\sum_{n=1}^{\infty} \sum_{k \in I_n} z_{n,k}$ be unconditionally convergent in $Z$. Then the series $\sum_{n=1}^{\infty} \sum_{k \in I_n} z_{n,k}$ is unconditionally convergent in $Z$.

Using these lemmas we prove

**Lemma 5.** Let $f : T \times S \to X$ be a $\mathcal{P} \otimes \mathcal{Q}$-measurable function, let the function $f(\cdot, s)$ be integrable with respect to $m$ for each $s \in S$, and let the function $g_E, g_E(s) = \int_X f(\cdot, s) \, dm, s \in S$, be integrable with respect to $I$ for each $E \in \mathcal{Q}(\mathcal{P} \otimes \mathcal{Q})$. Then the set function $E \to \int_S \int_E f(\cdot, s) \, dm \, dl, E \in \mathcal{P}(\mathcal{Q} \otimes \mathcal{Q})$, is a countably additive $Z$-valued vector measure on $\mathcal{Q}(\mathcal{P} \otimes \mathcal{Q})$.

**Proof.** Let $E_k \in \mathcal{Q}(\mathcal{P} \otimes \mathcal{Q}), k = 1, 2, \ldots$, be pairwise disjoint and let $E_0 = \bigcup_{k=1}^{\infty} E_k$. We have to show that $\int_S \int_{E_0} f(\cdot, s) \, dm \, dl = \sum_{k=1}^{\infty} \int_S \int_{E_k} f(\cdot, s) \, dm \, dl$ in the sense of unconditional convergence. According to Theorem 16 in Part I it suffices to show that the series on the right hand side is unconditionally convergent in $Z$.

According to Lemma 3 there is a countable family $\mathcal{A} \subset \mathcal{P}$ such that $E_k \in \mathcal{Q}(\mathcal{A}) \otimes \mathcal{Q}(\mathcal{Q})$ for each $k = 0, 1, 2, \ldots$. Take $A \in \mathcal{Q}(\mathcal{A})$ and $B \in \mathcal{Q}(\mathcal{Q})$ so that $E_0 \subset A \times B$, and a sequence $B_n \in \mathcal{Q}, n = 0, 1, \ldots$ such that $B_n \not\subset B$ and $B_0 = \emptyset$. According to Theorem 12.3), the function $v, v(s) = \sup_{A \in \mathcal{Q}(\mathcal{A})} \int_{A \cap E_0} f(\cdot, s) \, dm, s \in S$, is finite valued and $\mathcal{G}$-measurable. Therefore $F_n = \{s \in S; 0 \leq v(s) < n\} \in \mathcal{Q}(\mathcal{Q})$ for each $n = 0, 1, \ldots$ and $F_n \not\subset A$. Put $G_n = B_n \cap F_n - B_{n-1} \cap F_{n-1}, n = 1, 2, \ldots$. Then $G_n, n = 1, 2, \ldots$ are pairwise disjoint elements of $\mathcal{Q}$ and $\bigcup_{n=1}^{\infty} G_n \subset B$. Put $z_{n,k} = = \int_{G_n} \int_{E_k} f(\cdot, s) \, dm \, dl, n, k = 1, 2, \ldots$. Using Lemma 4 we shall show that the
series \( \sum_{n,k=1}^{\infty} z_{n,k} \) is unconditionally convergent in \( Z \), and this will prove the lemma, since then by Theorem 16 from Part I we have
\[
\sum_{n,k=1}^{\infty} z_{n,k} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_{E_n} \int_{E_k} f(\cdot, s) \, dm \, dl
\]
\[
dm \, dl = \sum_{k=1}^{\infty} \int_{E_k} \int_{E_k} f(\cdot, s) \, dm \, dl.
\]
Hence it remains to verify the validity of the assumptions of Lemma 4.

Let \( n \) be fixed. We shall show that for each \( z^* \in Z^* \) the equality \( z^* \int_{E_n} \int_{E_0} f(\cdot, s) \, dm \, dl = \sum_{k=1}^{\infty} z^* z_{n,k} \) holds in the sense of unconditional convergence, and this by the Orlicz-Pettis theorem will prove the unconditional convergence of the series \( \sum_{k=1}^{\infty} z_{n,k} \) in \( Z \).

Since by assumption \( f(\cdot, s) \) is integrable with respect to \( m \) for each \( s \in S \), Theorem 16 from Part I immediately yields that \( \int_{E_0} \int_{E_0} f(\cdot, s) \, dm \, dl = \sum_{k=1}^{\infty} \int_{E_k} \int_{E_k} f(\cdot, s) \, dm \, dl \) in the sense of unconditional convergence in \( Z \), for each \( s \in S \).

From the definition of the function \( v \) it is clear that \( \left| \sum_{k \in K} \int_{E_k} f(\cdot, s) \, dm \right| \leq v(s) \) for each \( s \in S \) and each \( K \subseteq \{1, 2, \ldots\} \). Thus for any finite \( K \subseteq \{1, 2, \ldots\} \) we have, see Theorem 14 in Part I, that \( \left| \sum_{k \in K} \int_{E_k} f(\cdot, s) \, dm \, dl \right| \leq \left| z^* \cdot \left| \int_{G_n} \left( \sum_{k \in K} \int_{E_k} f(\cdot, s) \, dm \right) \, dl \right| \leq \left| z^* \right| \cdot \sup_{s \in G_n} \left| \sum_{k \in K} \int_{E_k} f(\cdot, s) \, dm \right| \cdot l^*(G_n) \leq \left| z^* \right| \cdot \sup_{s \in G_n} v(s) \cdot l^*(B_n) \leq \left| z^* \right| \cdot n \cdot l^*(B_n) < +\infty \). Hence the series \( \sum_{k=1}^{\infty} \int_{G_n} \int_{E_k} f(\cdot, s) \, dm \, dl = \sum_{k=1}^{\infty} \int_{G_n} \int_{E_k} f(\cdot, s) \, dm \, dl \) is unconditionally convergent in \( Z \), hence by Theorem 16 from Part I
\[
\sum_{k=1}^{\infty} z^* z_{n,k} = \sum_{k=1}^{\infty} \int_{G_n} \int_{E_k} f(\cdot, s) \, dm \, dl = \sum_{k=1}^{\infty} \int_{G_n} \int_{E_k} f(\cdot, s) \, dm \, dl = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} z_{n,k} \right)
\]
is unconditionally convergent in \( Z \). Thus the assumptions of Lemma 4 are satisfied, which was to be shown.

**Lemma 6.** Let \( f : T \to X \) be a \( \mathcal{P} \)-measurable function. Then there is a countably additive measure \( \lambda : \mathcal{G}(\mathcal{P}) \to (0, +\infty) \) such that \( N \in \mathcal{G}(\mathcal{P}) \), \( \lambda(N) = 0 \Rightarrow f \cdot \chi_N \) is integrable with respect to \( m \) and \( \int_N f \, dm = 0 \).

**Proof.** Let \( f_n : T \to X \), \( n = 1, 2, \ldots \), be a sequence of \( \mathcal{P} \)-simple functions such that \( f_n(t) \to f(t) \) for each \( t \in T \). To each vector measure \( A \to \int_A f_n \, dm \), \( A \in \mathcal{G}(\mathcal{P}), n = 495 \)
\[ \lambda(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(A)}{1 + \lambda_n(T)}, \quad A \in \mathcal{S}(\mathcal{P}). \]

**Theorem 15.** (The Fubini theorem.) Let the product measure \( I \otimes m : \mathcal{P} \otimes \mathcal{B} \to \mathcal{L}(X, Z) \) exist and let \( f : T \times S \to X \) be a \( \mathcal{P} \otimes \mathcal{B} \)-measurable function. Let further the function \( f(\cdot, s) \) be integrable with respect to \( m \) for each \( s \in S \), and let for each set \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \) the function \( g_E, g_E(s) = \int_{E_s} f(\cdot, s) \, dm, s \in S, \) be essentially \( \mathcal{B} \)-measurable. Then the following conditions are equivalent:

a) \( f \) is integrable with respect to \( I \otimes m \), and

b) \( g_E \) is essentially integrable with respect to \( I \) for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \), and if they hold, then

\[ \int_{E_s} f \, d(I \otimes m) = \int_S \int_{E_s} f(\cdot, s) \, dm \, dl \text{ for each } E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}). \]

**Proof.** Without loss of generality we may suppose that \( g_E \) is \( \mathcal{B} \)-measurable for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \). Let \( f_n : T \to X, n = 1, 2, \ldots \) be a sequence of \( \mathcal{P} \otimes \mathcal{B} \)-simple functions such that \( f_n(t, s) \to f(t, s) \) and \( ||f_n(t, s) - f(t, s)|| \to 0 \) for each \( (t, s) \in T \times S \). For each vector measure \( E \to \int_E f_n \, d(I \otimes m), E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}), n = 1, 2, \ldots, \) take a control measure \( \lambda_n : \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \to (0, +\infty) \) and put

\[ \lambda(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(E)}{1 + \lambda_n(T)}, \quad E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}). \]

Let \( X_1 \) be the closed linear span of the set \( \{f_n(t, s); (t, s) \in T \times S, n = 1, 2, \ldots\} \). Then \( X_1 \) is a separable Banach space, and according to Theorem 11 we may replace \( X \) by \( X_1 \), hence we may suppose that \( X \) is a separable Banach space.

Take \( A_0 \in \mathcal{E}(\mathcal{P}) \) and \( B_0 \in \mathcal{E}(\mathcal{B}) \) so that \( F = \{(t, s) \in T \times S; f(t, s) = 0\} \subset A_0 \times B_0 \). Then by Theorem 13-1) there is a countably additive measure \( \gamma_{A_0} : \mathcal{E}(\mathcal{P}) \to (0, +\infty) \) such that \( C \in \mathcal{E}(\mathcal{P}), \gamma_{A_0}(A_0 \cap C) = 0 \Rightarrow m^*(A_0 \cap C) = 0 \).

Let \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \). By assumption the function \( g_E, g_E(s) = \int_{E_s} f(\cdot, s) \, dm, s \in S, \) is \( \mathcal{B} \)-measurable. Hence by Lemma 6 there is a countably additive \( \omega_E : \mathcal{E}(\mathcal{B}) \to (0, +\infty) \) such that \( D \in \mathcal{E}(\mathcal{B}), \omega_E(D) = 0 \) implies that \( g_E \cdot \chi_D \) is integrable with respect to \( I \) and \( \int_D g_E \, dl = 0 \).

Put \( \mu_E(G) = \lambda(G) + (\omega_E \otimes \gamma_{A_0})(G), G \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}). \) Then we conclude from the above and from Theorem A in § 36 [21] that if \( N \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \) and \( \mu_E(N) = 0 \), then the function \( f \cdot \chi_{E \cap N} \) is integrable with respect to \( I \otimes m \), the function \( g_E \cdot \chi_{E \cap N} \) is integrable with respect to \( I \), and \( \int_{E \cap N} f \, d(I \otimes m) = \int_S g_{E \cap N} \, dl = 0 \).

According to the Egoroff-Lusin theorem, see Section 1.4 in Part I, there is an \( N \in \mathcal{E}(\mathcal{P} \otimes \mathcal{B}) \) with \( \mu_E(N) = 0 \) and a sequence \( F_k \in \mathcal{P} \otimes \mathcal{B}, k = 1, 2, \ldots \), such that \( F_k \uparrow F - N \) and on each \( F_k, k = 1, 2, \ldots \), the sequence \( f_n, n = 1, 2, \ldots \), converges uniformly to \( f \). Thus by Theorem 4 the function \( f \cdot \chi_{E \cap F_k} \) is integrable with respect
to \( l \otimes m \) for each \( k = 1, 2, \ldots \), the function \( g_{E \cap F_k} \) is integrable with respect to \( l \), and

\[
\int_{G \cap E \cap F_k} f \, dl \otimes m = \int_S g_{E \cap F_k \cap G} \, dl = \\
= \int_S \int_{(E \cap F_k \cap G)^*} f(\cdot, s) \, dm \, dl \quad \text{for each} \quad G \in \mathcal{E}(\mathcal{P} \otimes \mathcal{G}) .
\]

Since by assumption, the function \( f(\cdot, s) \) is integrable with respect to \( m \) for each \( s \in S \), we have

\[
g_{E \cap F_k}(s) = \int_{(E \cap F_k)^*} f(\cdot, s) \, dm \rightarrow \int_{(E \cap (F - N))^*} f(\cdot, s) \, dm = \\
= g_{E \cap (F \cap N)}(s) = g_{E - N}(s) \quad \text{for each} \quad s \in S .
\]

\( a) \Rightarrow b) \) and (F). Suppose that \( f \) is integrable with respect to \( l \otimes m \), and let \( B \in \mathcal{E}(\mathcal{G}) \). Then

\[
\int_B g_{E \cap F_k} \, dl = \int_{(A_0 \times B) \cap E \cap F_k} f \, dl \otimes m \\
\rightarrow \int_{(A_0 \times B) \cap (F - N) \cap E} f \, dl \otimes m = \int_{(A_0 \times B) \cap E} f \, dl \otimes m .
\]

Thus by Theorem 16 from Part I, (2) and (3) imply that the function \( g_{E - N} \), hence also \( g_E \), is integrable with respect to \( l \) and that \( \int_B g_E \, dl = \int_B g_{E - N} \, dl = \int_{(A_0 \times B) \cap E} f \, dl \otimes m \) for each \( B \in \mathcal{E}(\mathcal{G}) \). Taking \( B = B_0 \) we have also the equality (F).

\( b) \Rightarrow a) \) and (F). Suppose now that \( g_E \) is integrable with respect to \( l \) for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{G}) \). Take \( E = A_0 \times B_0 \) in the proof of \( a) \Rightarrow b) \) and (F) above. Then \( f \cdot \chi_{F_k} = f \cdot \chi_{(A_0 \times B_0) \cap F_k} \) is integrable with respect to \( l \otimes m \) for each \( k = 1, 2, \ldots \), and

\[
(f \cdot \chi_{F_k})(t, s) \rightarrow (f \cdot \chi_{F - N})(t, s) \quad \text{for each} \quad (t, s) \in T \times S .
\]

Since by Lemma 5 the set function \( G \rightarrow \int_S g_G \, dl \) \( G \in \mathcal{E}(\mathcal{P} \otimes \mathcal{G}) \) is a countably additive vector measure, by (1) we have

\[
\int_G f \cdot \chi_{F_k} \, dl \otimes m = \int_G f \, dl \otimes m = \\
= \int_S g_{(A_0 \times B_0) \cap F_k \cap G} \, dl = \int_S g_{F_k \cap G} \, dl \rightarrow \int_S g_{G \cap (F - N)} \, dl = \int_S g_G \, dl .
\]

According to Theorem 16 from Part I, (4) and (5) imply the integrability of \( f \) with respect to \( l \otimes m \) and the equality (F). The theorem is proved.
From Theorems 3, 13-3), 14, 15, and from Theorems 5 and 14 from part I we immediately obtain

**Theorem 16.** Let \( f : T \times S \to X \) be a bounded \( \mathcal{P} \otimes \mathcal{Q} \)-measurable function, let \( m^*(T) < +\infty \), let the function \( f(\cdot, s) \) be integrable with respect to \( m \) for each \( s \in S \) (if \( \mathcal{P}^\sim = \mathcal{P} = \mathcal{E}(\mathcal{P}) \), then by Theorem 5 from Part I this is always true), and let \( \mathcal{Q}^\sim = \mathcal{Q} = \mathcal{E}(\mathcal{Q}) \). Then the product measure \( l \otimes m : \mathcal{P} \otimes \mathcal{Q} \to L(X, Z) \) exists, the function \( g_E, g_{E^i}(s) = \int_E f(\cdot, s) \, dm, \, s \in S \), is essentially integrable with respect to \( l \) for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{Q}) \), the function \( f \) is integrable with respect to \( l \otimes m \), and \( \int_E f(\cdot, s) \, dm \, dl \) for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{Q}) \).

**Remark 2.** Let the product measure \( l \otimes m : \mathcal{P} \otimes \mathcal{Q} \to L(X, Z) \) exist, let \( f : T \times S \to X \) be integrable with respect to \( l \otimes m \), and let the function \( f(\cdot, s) \) be integrable with respect to \( l \otimes m \) for each \( s \in S \). Then it is clear from the proof of Theorem 15, that if \( \mu_E \) is replaced in this proof by the measure \( \lambda \) defined there, then there is a set \( N \in \mathcal{E}(\mathcal{P} \otimes \mathcal{Q}) \) such that \( g_{E-N} \) is integrable with respect to \( l \) for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{Q}) \) and \( \int_E f(\cdot, s) \, dl \otimes m = \int_E g_{E-N} \, dl \otimes m = \int_S g_{E-N} \, dl \) for each \( E \in \mathcal{E}(\mathcal{P} \otimes \mathcal{Q}) \). (Using Theorem 13-1) we may take \( N \in \mathcal{E}(\mathcal{P} \otimes \mathcal{Q}) \) such that \( (l \otimes m)(N) = 0 \). However, as Example at the beginning of § 2 shows, it may happen that \( N = T \times S \).

**References**


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