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NOTE ON HOMOMORPHISMS OF DIRECT PRODUCTS OF ALGEBRAS

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Let A be a non-void set and F a set of (algebraic) operations on A. An algebra (A, F) is said to be *without zero-divisors* if

(i) there exist $0 \in A$ and $\oplus \in F$ (where ar $\oplus = 2$) such that $a \oplus 0 = a = 0 \oplus a$ for each $a \in A$ and

(ii) at least one $\omega \in F$ (where $\omega \neq \oplus$) is regular on (A, F), i.e. ar $\omega = n \geqq 2$ and for each $a_1, \ldots, a_n \in A$ we have $a_1 \ldots a_n\omega = 0$ iff $a_i = 0$ for at least one $i \in \{1, \ldots, n\}$.

The element 0 is called a *zero* of (A, F).

I. CHAJDA in [1] has investigated homomorphisms of algebras, which are direct products of algebras without zero-divisors. In this note we shall show that in Theorem 9 of [1] and in its Corollary the author omits the following assumption:

(iii) $0 \ldots 0\omega = 0$ for arbitrary $\omega \in F$.

Let A, B be algebras of the same type. The algebras A, B are called *r-similar* if they are without zero-divisors and have the same set of regular operations. If $f(0) = 0$ for each $f \in \text{Hom}(A, B)$, then the r-similar algebras A, B are said to *super similar*. See [1].

Remark 1. The following example shows that there exist r-similar algebras A, B of the same type such that the zero mapping $o : A \to \{0\} \subset B$ is not a homomorphism of A into B. See Notation, p. 167, [1].

Example 1. By I we denote the set of all integers. Put $a \oplus b = a + b$, $a \bigcirc b = ab$ and $a * b = 1$ for every $a, b \in I$. It is clear that 0 is a zero of the algebra $\mathscr{X} = (I, F)$, where $F = \{\oplus, \bigcirc, *\}$; $\oplus$ fulfils (i), $\bigcirc$ fulfils (ii). This implies that the algebra $\mathscr{X}$ is without zero-divisors and so $\mathscr{X}$, $\mathscr{X}$ are r-similar.

Now we shall show that $\text{Hom}(\mathscr{X}, \mathscr{X}) = \{\text{id}_I\}$.

Indeed, if $\varphi \in \text{Hom}(\mathscr{X}, \mathscr{X})$, then $\varphi(1) = \varphi(1 * 1) = \varphi(1) * \varphi(1) = 1$ and so we can prove by induction that $\varphi(n) = n$ for every positive integer n. It is clear that $\varphi(0) = 0$ and so $\varphi(-n) = -\varphi(n) = -n$.

Remark 2. The following example shows that Theorem 9 [1] is not true.

Example 2. It follows from Example 1 that the algebras $\mathscr{X}, \mathscr{X}$ are super similar. By h we denote the projection of $\mathscr{X} \times \mathscr{X}$ onto the first factor $\mathscr{X}$. It is clear that $h \in \operatorname{Hom}(\mathscr{X} \times \mathscr{X}, \mathscr{X})$.

Now we shall show that there exists no matrix representing h.

On the contrary, let us assume that h is represented by a matrix $H = \|h_{i1}\|$, where $h_{i1} \in \operatorname{Hom}(\mathscr{X}, \mathscr{X})$ and $i = 1, 2$. It follows from Example 1 that $h_{i1} = \mathrm{id}_I$ and so $0 = h(0, 1) = h_{11}(0) \oplus h_{21}(1) = 0 + 1 = 1$, which is a contradiction.

Remark 3. The following example shows that Corollary to Theorem 9 [1] is false.

Example 3. Let $s = \operatorname{card} \operatorname{Hom}(\mathscr{X} \times \mathscr{X}, \mathscr{X})$, where $\mathscr{X}$ is the same as in Example 1 and 2. Since both projections of $\mathscr{X} \times \mathscr{X}$ onto $\mathscr{X}$ are homomorphisms, we have $s \geqq 2$. On the other hand, it follows from Example 1 that $\operatorname{card} \operatorname{Hom}(\mathscr{X}, \mathscr{X}) = 1$ and so $s \neq 1 = \prod_{j=1}^{m} (1 + \sum_{i=1}^{n} (p_{ij} - 1))$, where $m = 1$, $n = 2$ and $p_{11} = p_{21} = 1$.

Remark 4. Let A_i, B_j be super similar algebras for $i = 1, \ldots, n$; $j = 1, \ldots, m$ and $A = \prod_{i=1}^{n} A_i$, $B = \prod_{j=1}^{m} B_j$. If we define a matrix $H = \|h_{ij}\|$ representing a mapping h of A into B such that either $h_{ij} \in \operatorname{Hom}(A_i, B_j)$ or h_{ij} is a zero mapping of A_i into B_j, then h need not be a homomorphism nor a zero mapping. Compare with Theorem 8 of [1].

Example 4. Let h be a mapping of $\mathscr{X}$ into $\mathscr{X} \times \mathscr{X}$ (see Examples 1 and 2) represented by a matrix $H = \|h_{ij}\|$, where $j = 1, 2$ and $h_{11} = \mathrm{id}_I$, $h_{12} = 0$. Evidently $h(1) = (1, 0) \neq (0, 0)$ and so h is no zero mapping. We shall show that h is no homomorphism. On the contrary, let us suppose that $h \in \operatorname{Hom}(\mathscr{X}, \mathscr{X} \times \mathscr{X})$. Then $(1, 0) = h(1) = h(1 * 1) = h(1) * h(1) = (1, 0) * (1, 0) = (1 * 1, 0 * 0) = (1, 1)$. a contradiction.

References

[1] *I. Chajda:* Homomorphisms of direct products of algebras. Czech. Math. J. *28* (1978), 155–170.

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