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LU FACTORIZATIONS

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I. INTRODUCTION

Suppose A is an $n \times n$ matrix over the complex field. The problem of factoring A into a product LU, where L is a lower-triangular matrix, and U is an upper-triangular matrix with specified diagonal, is of importance in solving systems of linear equations and in the construction of compact schemes for matrix inversion [1]. In fact, Gaussian elimination is concerned with effecting an LU factorization.

We shall concern ourselves with the following problem. Suppose A is an $n \times n$ matrix. What are necessary and sufficient conditions that A can be factored as LU, where U has a diagonal consisting entirely of ones?

II. *LU* FACTORIZATIONS

Let α and β be increasing sequences on $\{1, \ldots, n\}$. We shall use the following notation. $A(\alpha \mid \beta)$ denotes the minor of A with rows indexed by α and columns indexed by β. $A[\alpha \mid \beta]$ is the submatrix of A contained in rows α and columns β. A principal minor is written $A(\alpha)$, and a principal submatrix $A[\alpha]$. $\hat{\alpha}$ is the complement of α. Finally, $R(\cdot)$ denotes the range space.

Theorem 1. *An $n \times n$ matrix A has an LU-factorization with $u_{ii} = 1$ iff $A[n^\wedge]$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{u}_{ii} = 1$ and $A[n^\wedge \mid n] \in R(\tilde{L}) = R(A[n^\wedge])$.*

Proof. Assume A has an LU-factorization. Partition L, U, and A conformally so that

$$LU = \begin{pmatrix} L_{11} & Z^T \\ L_{21} & l \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ Z & 1 \end{pmatrix} = A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix}$$

where Z is a $1 \times (n-1)$ zero vector and L_{11}, U_{11}, A_{11} are $(n-1) \times (n-1)$ in dimension. Let $\tilde{L} = L_{11}$ and $\tilde{U} = U_{11}$, then $A[n^\wedge] = A_{11} = \tilde{L}\tilde{U}$ and $\tilde{L}U_{12} = A_{12}$, i.e. $A[n^\wedge \mid n] \in R(\tilde{L})$. Note that since $A[n^\wedge] = \tilde{L}\tilde{U}$, $\tilde{L} = A[n^\wedge]\,\tilde{U}^{-1}$ and $R(\tilde{L}) = R(A[n^\wedge])$. On the other hand assume $A[n^\wedge]$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{u}_{ii} = 1$ and partition A as above. Since $A_{12} = A[n^\wedge \mid n] \in R(\tilde{L})$, there exists V such that $\tilde{L}V = A_{12}$. Since $\tilde{U}$ is nonsingular, there exists an $1 \times (n-1)$ vector Y such that $Y\tilde{U} = A_{21}$. Choose $l = a_{nn} - YV$. Now let

$$L = \begin{pmatrix} \tilde{L} & Z^T \\ Y & l \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \tilde{U} & V \\ Z & l \end{pmatrix}.$$

Then L is lower triangular and U is upper tiruangular with $u_{ii} = 1$, and

$$LU = \begin{pmatrix} \tilde{L} & Z^T \\ Y & l \end{pmatrix}\begin{pmatrix} \tilde{U} & V \\ Z & l \end{pmatrix} = \begin{pmatrix} \tilde{L}\tilde{U} & \tilde{L}V \\ Y\tilde{U} & YV + l \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix} = A\,.$$

Theorem 2. *An $n \times n$ matrix A has an LU-factorization with $u_{ii} = 1$ iff $A[1, 2, \ldots, k \mid k+1] \in R(A[1, 2, \ldots, k])$ for $k = 1, 2, \ldots, n-1$.*

Proof. This statement is the result of iteration of the necessary and sufficient condition of Theorem 1. Since A has an LU-factorization, $A[1, \ldots, n-1 \mid n] \in R(A[1, \ldots, n-1])$ and $A[1, \ldots, n-1]$ has an LU-factorization. Now the same argument applies to $A[1, \ldots, n-1]$, obtaining $A[1, \ldots, n-2 \mid n-1] \in R(A[1, \ldots, n-2])$ and the process continuous until finally we have $A[1 \mid 2] \in R(A[1])$. Conversely $A[1]$ has an LU-factorization trivially and $A[1 \mid 2] \in R(A[1])$ implies that $A[1, 2]$ has an LU-factorization, which together with $A[1, 2 \mid 3] \in R(A[1, 2])$ implies, in turn, that $A[1, 2, 3]$ has an LU-factorization. The argument repeats until we obtain that A has an LU-factorization.

Corollary 1. *If an $n \times n$ matrix A has an LU-factorization with $u_{ii} = 1$, then $A(1, 2, \ldots, k \mid 1, 2, \ldots, k-1, j) = 0$ whenever $A(1, 2, \ldots, k) = 0$ for $1 \leqq k \leqq n-1$ and $k < j \leqq n$. In particular, if $a_{11} = 0$, then the first row of A must be zero.*

Proof. If $A(1, 2, \ldots, k) = 0$, then there is a nontrivial linear relation between the columns of $A[1, \ldots, k]$, say $\sum_{i=1}^{k} C_i A[1, 2, \ldots, k \mid i] = 0$. Suppose $C_k = 0$. Then the first $k-1$ columns in $A[1, 2, \ldots, k]$ are linearly dependent and $A(1, \ldots, k \mid 1, \ldots, k-1, j) = 0$. If $C_k \neq 0$, then the k-th column in $A[1, 2, \ldots, k]$ depends on the first $k-1$ columns. Since it follows from the theorem that $A[1, 2, \ldots, k \mid j] \in R(A[1, 2, \ldots, k])$ for $k < j \leqq n$, $A[1, 2, \ldots, k \mid j]$ also depends on these $k-1$ columns and $A(1, 2, \ldots, k \mid 1, 2, \ldots, k-1, j) = 0$.

The converse of the last corollary is not true. The matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

satisfies the condition of the corollary but not that of Theorem 2 and hence cannot have an LU-factorization.

Theorem 3. *An $n \times n$ matrix A has an LU-factorization with $u_{ii} = 1$ iff $A[1, 2, \dots, k \mid k+1] \in R(A[1, 2, \dots, k])$ for $m \leqq k \leqq n-1$ where m is the smallest positive integer such that $A(1, 2, \dots, m) = 0$.*

Proof. Note that if $A(1, 2, \dots, k) \neq 0$, the columns of $A[1, 2, \dots, k]$ form a basis of k-space and $A[1, 2, \dots, k \mid k+1] \in R(A[1, 2, \dots, k])$ is automatically satisfied. Thus it follows from the Theorem 2 that A has an LU-factorization with $u_{ii} = 1$ iff $A[1, 2, \dots, k \mid k+1] \in R(A[1, 2, \dots, k])$ whenever $A(1, 2, \dots, k) = 0$. However, when A has an LU-factorization, $A(1, 2, \dots, k) = L(1, 2, \dots, k)\, U(1, 2, \dots, k) = \prod_{i=1}^{k} l_{ii}$. So if $A(1, 2, \dots, m) = 0$ where m is the smallest such integer, necessarily $l_{mm} = 0$ and $A(1, 2, \dots, k) = 0$ for $m \leqq k \leqq n-1$. Hence it is required that $A[1, 2, \dots, k \mid k+1] \in R(A[1, 2, \dots, k])$ for $m \leqq k \leqq n-1$.

Corollary 2. *If the proper leading principal minors of an $n \times n$ matrix A are nonzero, then A has an LU-factorization with $u_{ii} = 1$.*

This corollary is well-known (see, for example, [3]). Also we can obtain the following corollary, which is given in Gantmacher ([2], p. 35).

Corollary 3. *Let A be an $n \times n$ matrix of rank r and $A(1, 2, \dots, k) \neq 0$ for $k = 1, 2, \dots, r$. Then A has an LU-factorization in which the last $n - r$ columns of L are zero and $u_{ii} = 1$.*

Proof. Such a matrix A satisfies the conditions of the theorem. It is sufficient to show that $A[1, 2, \dots, k \mid k+1] \in R(A[1, 2, \dots, k])$ for $k = r+1, r+2, \dots, n-1$. Since $A(1, 2, \dots, r) \neq 0$, it follows that for $k > r$, the first r columns in $A[1, 2, \dots, k]$ are linearly independent. Suppose $A[1, 2, \dots, k \mid k+1] \notin R(A[1, 2, \dots, k])$; then A would have at least $r+1$ linearly independent columns, contradicting that rank $(A) = r$. Thus A has an LU-factorization with $u_{ii} = 1$. Note that since U is invertible, rank $(L) = r$. $A(1, 2, \dots, r) = \prod_{i=1}^{r} l_{ii} \neq 0$ implies that the first r columns of L are linearly independent. Any of the last $n - r$ columns of L being nonzero would imply that L has more than r linearly independent columns, contradicting that rank $(L) = r$.

Theorem 4. *If an $n \times n$ matrix A has an LU-factorization with $u_{ii} = 1$, L and U are unique iff all proper leading principal minors of A are nonzero.*

Proof. Note that in the conformal partitioning

$$LU = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A ,$$

$L_{11}U_{11} = A_{11}$, i.e., if A has an LU-factorization with $u_{ii} = 1$, every leading principal submatrix of A does also. Since $A[1]$ is factored uniquely as $(a_{11})(1)$, it is sufficient, by mathematical induction, to show that whenever $A_k = A[1, 2, \ldots, k]$ has a unique factorization, the factorization in A_{k+1} is also unique. Consider the partitioning

$$A_{k+1} = \begin{pmatrix} A_k & C \\ R & a_{k+1,k+1} \end{pmatrix} = L_{k+1}U_{k+1} = \begin{pmatrix} L_k & 0 \\ Y & l \end{pmatrix} \begin{pmatrix} U_k & V \\ 0 & 1 \end{pmatrix}$$

where L_k, U_k are unique with $L_kU_k = A_k$. Note that for $k = 1, 2, \ldots, n-1$, L_k is invertible since both U_k and A_k are invertible. Now $C = L_kV$ and $YU_k = R$ imply Y and V are both unique, and $YV + l = a_{k+1,k+1}$ implies l is unique. Thus L_{k+1}, U_{k+1} are unique.

Conversely, in the above partitioning, $C = L_kV$ and V is unique imply that L_k is invertible. Hence $A_k = L_kU_k$ is invertible and $A(1, 2, \ldots, k) \neq 0$. This holds true for $k = 1, 2, \ldots, n-1$.

Theorem 5. *An $n \times n$ matrix A with $a_{11} = 0$ has an LU-factorization with $l_{ii} = 0$ and $u_{ii} = 1$ iff the first row of A is zero and $A[1^\wedge \mid n^\wedge]$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{u}_{ii} = 1$ and $A[1^\wedge \mid n] \in R(\tilde{L}) = R(A[1^\wedge \mid n^\wedge])$. Furthermore, such a factorization is unique iff all leading principal minors of $A[1^\wedge \mid n^\wedge]$ are nonzero.*

Proof. First, assume A has an LU-factorization. By Corollary 1 the first row of L is necessarily zero. Partition L, U and A conformally so that

$$LU = \begin{pmatrix} Z & 0 \\ L_{21} & Z^T \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ Z & 1 \end{pmatrix} = A = \begin{pmatrix} Z & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

where Z is an $1 \times (n-1)$ zero vector and L_{21}, U_{11}, A_{21} are all of dimension $(n-1) \times (n-1)$. Then $A[1^\wedge \mid n^\wedge] = A_{21} = L_{21}U_{11}$. Let $\tilde{L} = L_{21}$ and $\tilde{U} = U_{11}$ implies $A[1^\wedge \mid n^\wedge]$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{u}_{ii} = 1$. Also $A[1^\wedge \mid n] = A_{22} = L_{21}U_{12}$ implies $A[1^\wedge \mid n] \in R(L_{21}) = R(\tilde{L})$. Note that since $A_{21} = \tilde{L}\tilde{U}$ and $\tilde{U}$ is nonsingular, $\tilde{L} = A_{21}\tilde{U}^{-1}$ and $R(\tilde{L}) = R(A_{21}) = R(A[1^\wedge \mid n^\wedge])$. For the converse, partition A into $\begin{pmatrix} Z & 0 \\ A_{21} & A_{22} \end{pmatrix}$ as above. Then $A_{21} = A[1^\wedge \mid n^\wedge]$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{u}_{ii} = 1$. Since $A_{22} = A[1^\wedge \mid n] \in R(\tilde{L})$, there exists V such that $\tilde{L}V = A_{22}$. Let

$$L = \begin{pmatrix} Z & 0 \\ \tilde{L} & Z^T \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \tilde{U} & V \\ Z & 1 \end{pmatrix}.$$

Then L is lower triangular with $l_{ii} = 0$ and U is upper triangular with $u_{ii} = 1$, and

$$LU = \begin{pmatrix} Z & 0 \\ \tilde{L} & Z^T \end{pmatrix} \begin{pmatrix} \tilde{U} & V \\ Z & 1 \end{pmatrix} = \begin{pmatrix} Z & 0 \\ \tilde{L}\tilde{U} & \tilde{L}V \end{pmatrix} = \begin{pmatrix} Z & 0 \\ A_{21} & A_{22} \end{pmatrix} = A\,.$$

For the uniqueness statement, consider L, U and A partitioned as in the last equation. If all leading minors of A_{21} are nonzero, it follows from the Theorem 4 that $\tilde{L}$, $\tilde{U}$ are unique. Also A_{21} is invertible implies $\tilde{L}$ is invertible, so it follows from $\tilde{L}V = A_{22}$ that V is unique. Hence L and U are unique. On the other hand, assume that L, U are unique, then $\tilde{L}$, $\tilde{U}$ are unique. So by the last theorem all proper leading principal minors of $A_{21} = \tilde{L}\tilde{U}$ are nonzero. Also V is unique and $\tilde{L}V = A_{22}$ implies that $\tilde{L}$ is invertible. Hence A_{21} is invertible and det $A_{21} \neq 0$. This completes the proof.

Example.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 5 & 4 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

References

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