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SOME REMARKS ON EULERIAN GRAPHS

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This paper was motivated by the results of L. NEBESKÝ [1] concerning Eulerian subgraphs of graphs. The graphs studied in it are always finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$.

L. Nebeský has defined the number $\mathrm{eul}(G)$ as the maximal number of vertices of an Eulerian subgraph of a graph G. If G contains no Eulerian subgraphs, then by definition $\mathrm{eul}(G) = 2$.

It is natural to ask what are the interrelations between $\mathrm{eul}(G)$ and the number of vertices of G. If we admit disconnected graphs or graphs with bridges, this question is trivial. (By a bridge we mean each edge which is contained in no circuit, therefore also an edge incident with a vertex of the degree 1.) To an arbitrary graph G we can add an arbitrary tree T which is vertex-disjoint with G and join exactly one vertex of G with a vertex of T by an edge. For the resulting graph G' we have $\mathrm{eul}(G') = \mathrm{eul}(G)$ and the number of vertices of G' can be arbitrarily large. Thus we shall study only connected graphs without bridges. First we shall give a definition.

Definition. A binary tree of a height h is the tree B_h defined as follows. Let $V_0 = \{u\}$, where u is an element, let V_i for $i = 1, \ldots, h$ be the set of all i-dimensional vectors, all of whose co-ordinates are equal to 0 or 1. The vertex set of B_h is $V(B_h) = \bigcup_{i=0}^{h} V_i$. The vertex u is adjacent to both the vertices from V_1; for $i = 1, \ldots, h - 1$ each vector from V_i is adajcent to all the vectors from V_{i+1} which are obtained from it by adding one co-ordinate at its end. No further edges are contained in B_h.

Now we shall pıove a theorem.

Theorem 1. *To an arbitrary positive number N there exists a connected graph G without bridges such that*

$$\frac{|V(G|}{\mathrm{eul}(G)} > N .$$

Proof. Take two graphs T_1, T_2 both isomorphic to the binary tree B_h. Any vertex corresponding to a vector from V_h in T_1 will be joined by an edge with the vertex of T_2 corresponding to the same vector. The resulting graph will be denoted by G_h. (For $h = 4$ this graph is in Fig. 1.)

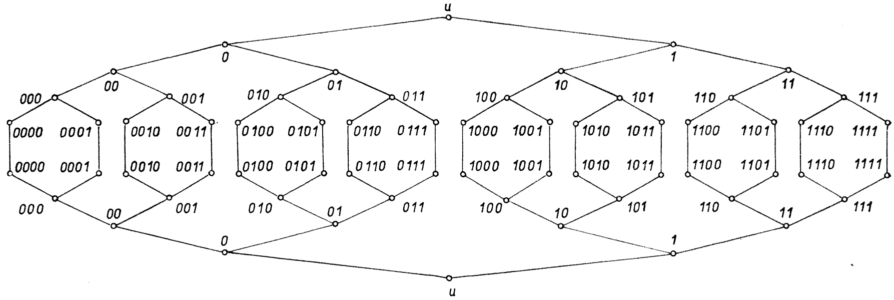

Fig. 1.

The graph G_h is connected and without bridges. Each of its vertices has a degree 2 or 3. Therefore, each Eulerian subgraph of G_h must be a connected graph in which all vertices have the degree 2; such a graph is a circuit. Hence $\text{eul}(G_h)$ is equal to the length of the longest circuit in G_h. This number is evidently equal to $4h + 2$. The number of vertices of G_h is equal to $2^{h+2} - 2$. We have

$$\frac{|V(G_h)|}{\text{eul}(G_h)} = \frac{2^{h+1} - 1}{2h + 1} .$$

For $h \to \infty$ this expression tends to infinity, therefore for each number N there exists h such that

$$\frac{|V(G_h)|}{\text{eul}(G_h)} > N ,$$

which was to be proved.

For various numerical characteristics of graphs (e.g. the chromatic number), an edge-critical graph is defined as a graph with the property that this characteristic is changed by deleting an arbitrary edge. Thus an edge-critical graph with respect to $\text{eul}(G)$ will be defined as a graph G with the property that for an arbitrary graph G' obtained from G by deleting an edge it is $\text{eul}(G') < \text{eul}(G)$. (Obviously $\text{eul}(G)$ cannot increase by deleting an edge from G.)

Theorem 2. *A graph G is an edge-critical graph with respect to* $\text{eul}(G)$ *if and only if the following conditions are satisfied:*

(i) *G is either Eulerian, or obtained from an Eulerian graph by adding isolated vertices.*

(ii) *Each circuit C in G has the property that by deleting all edges of C from G, a graph with more connected components than G is obtained.*

Proof. Suppose that (i) does not hold. If G contains no Eulerian subgraph, then $\mathrm{eul}(G) = 2$ and, by definition of $\mathrm{eul}(G)$, this number is the smallest possible, therefore for each subgraph G' of G we have $\mathrm{eul}(G') = 2 = \mathrm{eul}(G)$. If G contains an Eulerian subgraph G_1, then $\mathrm{eul}(G_1) = |V(G_1)| < |V(G)|$ and there exists an edge e of G not belonging to G_1. After deleting e we obtain a graph G_2 which again contains G_1. Therefore $\mathrm{eul}(G_2) = \mathrm{eul}(G)$ and G is not edge-critical with respect to $\mathrm{eul}(G)$.

Suppose that (i) holds and (ii) does not hold. Let C be a circuit in G with the property that the graph G' obtained from G by deleting all edges of C has the same number of connected components as G. As (i) holds, the graph G contains a connected component G_0 which is an Eulerian graph; other connected components of G (if any) are isolated vertices. The circuit C must be contained in G_0 and by deleting all edges of C from G_0 a connected graph G_1 is obtained. The graph G_1 has the same number of vertices as G_0 and is also Eulerian. Namely, all vertices of G_0 have even degrees and the degree of any vertex in G_1 is either the same as in G_0 or obtained from it by subtracting 2; in both the cases it is even. Let e be an edge of C, let G_2 be the graph obtained from G by deleting e. The graph G_2 contains an Eulerian subgraph G_1, therefore $\mathrm{eul}(G_2) = |V(G_2)| = |V(G_0)| = \mathrm{eul}(G)$ and G is not edge-critical with respect to $\mathrm{eul}(G)$.

Now let (i) and (ii) hold. Let e be an edge of G, let u, v be its end vertices. The edge e is contained in G_0, where G_0 has the same meaning as above. Let G' (or G'_0) be the graph obtained from G (or G_0, respectively) by deleting e. Obviously $\mathrm{eul}(G') = \mathrm{eul}(G'_0) \leqq \mathrm{eul}(G_0) = \mathrm{eul}(G)$. Suppose $\mathrm{eul}(G'_0) = \mathrm{eul}(G_0)$. This means that there exists an Eulerian graph H which is a spanning subgraph of G'_0. The vertex u has an even degree in G_0 and hence an odd degree in G'_0. If this degree is 1, this vertex cannot belong to H, which is a contradiction. In the opposite case there exists an edge in G'_0 which is incident with u and does not belong to H; let its end vertex distinct from u be w_1. Let G_1 be the graph obtained from G'_0 by deleting e; the graph G_1 also contains H as a spanning subgraph. Now we repeat this consideration with G_1 instead of G_0 and w_1 instead of u; we obtain a vertex w_2 and a graph G_2. Thus we proceeed further and obtain the vertices w_i and the graphs G_i (which all contain H as a spanning subgraph) for positive integers i. As all considered graphs are finite, we must finish this procedure after a finite number of steps. Evidently, the procedure is finished when we obtain $w_i = v$ for some i. Then the vertices $u, w_1, \ldots, w_i = v$ are vertices of a closed trail T in G_0, the graph G_i is obtained from G_0 by deleting all edges of this trail. The trail T contains all edges of a certain circuit C in G_0 which contains e. The graph obtained from G_0 by deleting all edges of C contains H as a spanning subgraph, therefore it is connected, which is a contradiction to (ii). Therefore $\mathrm{eul}(G') = \mathrm{eul}(G'_0) < \mathrm{eul}(G_0) = \mathrm{eul}(G)$. As e was chosen arbitrarily, the graph G is edge-critical with respect to $\mathrm{eul}(G)$.

If we have an Eulerian graph which is not edge-critical with respect to eul(G), there exists an edge of this graph, by deleting of which we obtain a graph G_1 with eul(G_1) = eul(G). If G_1 is not edge-critical, we can continue in this way, until we obtain an edge-critical Eulerian graph which is a spanning subgraph of G. We have the following proposition.

Proposition. *Each Eulerian graph contains a spanning Eulerian subgraph which is edge-critical with respect to* eul(G).

An example of an Eulerian graph which is edge-critical with respect to eul(G) is a subdivision of an arbitrary Eulerian graph, in which each edge is subdivided. (This means that each edge is incident with a vertex of the degree 2.)

Reference

[1] *L. Nebeský:* Eulerian subgraphs of complementary graphs. Czech. Math. J. (to appear).

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