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DECOMPOSITIONS OF HOMOMORPHISMS

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Some sufficient conditions for direct decompositions of homomorphisms were investigated for suitable algebras in [5] and [6] and for lattices and the so called *weakly associative lattices* in [7]. The aim of this paper is to derive new conditions for direct decompositions of homomorphisms on universal algebras by using the well-known conditions for direct decompositions of congruences derived by G. A. FRASER and A. HORN in [2] and the results on the Unique Factorization Property derived by G. BIRKHOFF (see [1], [3], [4]).

Let $\prod_{i=1}^{n} A_i$ be the direct product of given algebras $A_1, \ldots, A_n$ of the same type and let pr_i be the projection of $\prod_{i=1}^{n} A_i$ onto the i-th factor A_i. If σ is a permutation of the subscript set $\{1, \ldots, n\}$, denote by $\varkappa(\sigma)$ the isomorphism of $\prod_{i=1}^{n} A_i$ onto $\prod_{i=1}^{n} A_{\sigma(i)}$ given by the rule

$$\varkappa(\sigma)(a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$

i.e., $\varkappa(\sigma)$ permutes only the direct factors, which is not essential for the direct product representation.

Let A_i, B_i (for $i = 1, \ldots, n$) be algebras of the same type and let h_i be a homomorphism of A_i into B_i. The mapping $h = \prod_{i=1}^{n} h_i$ of $A = \prod_{i=1}^{n} A_i$ into $B = \prod_{i=1}^{n} B_i$ introduced by

$$pr_i\, h(a) = h_i(pr_i a)$$

for every $a \in A$ and each $i = 1, \ldots, n$ is called the *direct product of homomorphisms* h_i (see e.g. G. Grätzer, Universal Algebra, p. 127).

With respect to the above remark, let us introduce conversely: a homomorphism h of A into B is said to be *direct by decomposable,* provided $h \,.\, \varkappa(\sigma) = \prod_{i=1}^{n} h_i$ for a suitable permutation σ of $\{1, \ldots, n\}$, where h_i is a homomorphism of A_i into $B_{\sigma(i)}$ for all $i = 1, \ldots, n$, and $\varkappa(\sigma)$ is an isomorphism of $\prod_{i=1}^{n} B_i$ onto $\prod_{i=1}^{n} B_{\sigma(i)}$.

Lemma. *Let B and $A_1, \ldots, A_n$ be algebras of the same type and let h be a homomorphism of $A = \prod_{i=1}^{n} A_i$ onto B inducing a congruence Θ_h on A. If $\Theta_h = \prod_{i=1}^{n} \Theta_i$, where Θ_i is a congruence on A_i, then there exist algebras $B_1^*, \ldots, B_n^*$ and homomorphisms h_i of A_i onto B_i^* such that $B \cong \prod_{i=1}^{n} B_i^*$ (by the isomorphism f) and $h\,.\,f = \prod_{i=1}^{n} h_i$.*

Proof. $B = h(\prod_{i=1}^{n} A_i) \cong \prod_{i=1}^{n} A_i/\Theta_h = \prod_{i=1}^{n} A_i/\prod_{i=1}^{n} \Theta_i = \prod_{i=1}^{n} (A_i/\Theta_i)$ (see e.g. [1], p. 140). Denote $B_i^* = A_i/\Theta_i$, hence $B \overset{f}{\cong} \prod_{i=1}^{n} B_i^*$ and clearly $h\,.\,f = \prod_{i=1}^{n} h_i$, where $h_i : : A_i \to A_i/\Theta_i$ is a canonical homomorphism. Q.E.D.

An algebra is said to have the *Unique Factorization Property over the class $\mathscr{M}$* (briefly $\mathscr{M}$-UFP), provided A is isomorphic with a direct product of algebras from $\mathscr{M}$ and this representation is unique up to isomorphism, i.e. provided the condition

$$A \cong \prod_{i \in I} A_i \cong \prod_{j \in J} B_j\,, \quad I = \{1, \ldots, n\}\,, \quad J = \{1, \ldots, m\}$$

(where card $A_i > 1$, card $B_j > 1$ for every subscript i, j) with $A_i, B_j \in \mathscr{M}$ for $i \in I$, $j \in J$ always implies the existence of a bijection σ of I onto J with $A_i \cong B_{\sigma(i)}$ for all $i \in I$. If $\mathscr{M}$ is the class of directly irreducible algebras (of the same type as A), A is said briefly to have the *Unique Factorization Property*. (For this definition, see e.g. [3] and [8]).

The concepts involved can be relativized by the following

Definition. Let $\mathscr{C}$ be a class of algebras of the same type and $\mathscr{M}$ its subclass. We say that:

(i) *$\mathscr{C}$ has $\mathscr{M}$-directly decomposable congruences* provided for every congruence Θ on each $A \in \mathscr{C}$ with $A = \prod_{i=1}^{n} A_i$ for $A_1, \ldots, A_n \in \mathscr{M}$ there exists Θ_i on A_i such that $\Theta = \prod_{i=1}^{n} \Theta_i$.

(ii) *$\mathscr{C}$ has $\mathscr{M}$-directly decomposable homomorphisms* provided for every $A, B \in \mathscr{C}$, $A = \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$ with $A_i, B_i \in \mathscr{M}$, every homomorphism h of A onto B is directly decomposable.

(iii) *$\mathscr{C}$ has $\mathscr{M}$*-UFP provided every $A \in \mathscr{C}$ has $\mathscr{M}$-UFP.

Theorem 1. *Let $\mathscr{C}$ be a class of algebras of the same type and $\mathscr{M}$ its arbitrary subclass closed under homomorphic images. Then the implication* (a) $\Rightarrow$ (*b*) *is satisfied, where*:

(a) *$\mathscr{C}$ has $\mathscr{M}$-directly decomposable congruences and $\mathscr{M}$-*UFP;

(b) *$\mathscr{C}$ has $\mathscr{M}$-directly decomposable homomorphisms.*

Proof. Suppose $A, B \in \mathscr{C}$, $A = \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$ with $A_i, B_i \in \mathscr{M}$ and let h be a homomorphism of A onto B. Let Θ_h be a congruence induced by h on A. By (a), $\Theta_h = \prod_{i=1}^{n} \Theta_i$, where Θ_i is a congruence on A_i. By Lemma, there exist $B_i^* \in \mathscr{C}$ ($i = 1, \ldots, n$) and homomorphisms h_i^* of A_i onto B_i^* with $B \cong \prod_{i=1}^{n} B_i^*$ (an isomorphism f) and $h \,.\, f = \prod_{i=1}^{n} h_i$. Since $\mathscr{M}$ is closed under homomorphic images and $B_i^* = h_i^*(A_i)$, also $B_i^* \in \mathscr{M}$ for $i = 1, \ldots, n$. As $\mathscr{C}$ has $\mathscr{M}$-UFP, there exists a permutation σ of $\{1, \ldots, n\}$ such that $B_i \cong B_{\sigma(i)}^*$. Denote by g_i this isomorphism $B_{\sigma(i)}^* \cong B_i$ and put $h_i = h_i^* \,.\, g_i$. Then clearly

$$h = h \,.\, f \,.\, \prod_{i=1}^{n} g_i = \prod_{i=1}^{n} h_i^* \,.\, \prod_{i=1}^{n} g_i = \prod_{i=1}^{n} (h_i^* \,.\, g_i) = \prod_{i=1}^{n} h_i \,.$$

Hence $\mathscr{C}$ has $\mathscr{M}$-directly decomposable homomorphisms. Q.E.D.

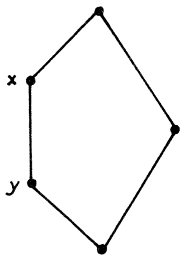

Fig. 1.

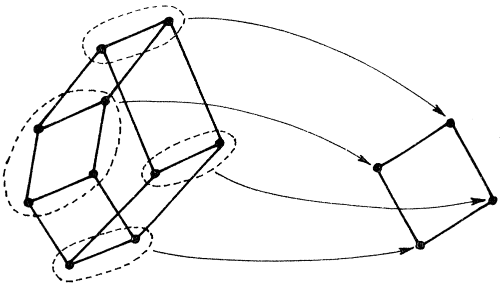

Fig. 2.

Remark. If a subclass $\mathscr{M}$ of $\mathscr{C}$ is not closed under homomorphic images, the implication (a) $\Rightarrow$ (b) of Theorem 1 need not be satisfied, even if $\mathscr{M}$ is a subclass of directly irreducible algebras. It can be demonstrated by the following

Example. Let $\mathscr{M} = \{N_5, I_2\}$, where N_5 is the non-modular pentagon (Fig. 1) and I_2 the two-element lattice, and let $\mathscr{C}$ be a class formed by five lattices: a one-element lattice E and N_5, I_2, $A = N_5 \times I_2$, $B = I_2 \times I_2$. Let h be a homomorphism of A onto B visualized by arrows in Fig. 2. Clearly $\mathscr{C}$ has $\mathscr{M}$-UFP and $\mathscr{M}$-directly decomposable congruences as follows directly from Corollary 1 in [2] (namely the congruence Θ induced by h satisfies $\Theta = \Theta_1 \times \Theta_2$, where Θ_1 on N_5 collapses onto x, y – see Fig. 1, and $\Theta_2 = I_2 \times I_2$), however h is not directly decomposable because h_1 induced by Θ_1 maps N_5 onto the whole B contrary to h_2 induced by Θ_2 which collapses I_2 onto the one-element lattice E.

Theorem 2. *Let $\mathscr{C}$ be a class of algebras of the same type containing a one-element algebra E, let $\mathscr{M}$ be an arbitrary subclass of $\mathscr{C}$ closed under homomorphic images such that $E \in \mathscr{M}$. If $\mathscr{C}$ has $\mathscr{M}$-directly decomposable congruences, the following conditions are equivalent*:

(1) *$\mathscr{C}$ has $\mathscr{M}$-directly decomposable homomorphisms,*
(2) *$\mathscr{C}$ has $\mathscr{M}$-*UFP.

Proof. The implication (2) $\Rightarrow$ (1) follows directly from Theorem 1. Prove (1) $\Rightarrow$ $\Rightarrow$ (2). Let $A \in \mathscr{C}$ and $A \cong \prod_{i=1}^{n} A_i \cong \prod_{j=1}^{m} B_j$, where $A_i, B_j \in \mathscr{M}$ and card $A_i > 1$, card $B_j >$ > 1. Without loss of generality, suppose $m \leqq n$. Put $B_j = E$ for $j = m + 1, \ldots, n$. Then $\prod_{i=1}^{n} A_i \cong \prod_{j=1}^{n} B_j$; denote this isomorphism by h. As h is $\mathscr{M}$-directly decomposable, thus, for every $i = 1, \ldots, n$, A_i is isomorphic to $B_{\sigma(i)}$ for a suitable permutation σ of $\{1, \ldots, n\}$. Hence card $B_j > 1$ for $j = 1, \ldots, n$, i.e. $m = n$ and A has $\mathscr{M}$-UFP. Q.E.D.

Necessary and sufficient conditions for the direct decomposition of congruences are derived by G. A. Fraser and A. Horn in [2].

Corollary 1. *Let $\mathscr{C}$ be a class of algebras with distributive congruences and $\mathscr{M}$ an arbitrary class of directly irreducible algebras of $\mathscr{C}$ closed under homomorphic images and containing a trivial algebra E of $\mathscr{C}$. Then*

(a) *$\mathscr{C}$ has $\mathscr{M}$-directly decomposable homomorphisms if and only if $\mathscr{C}$ has $\mathscr{M}$-*UFP.
(b) *If $A \in \mathscr{M}$ implies that A is finite and a trivial algebra is a subalgebra of A, then $\mathscr{C}$ has $\mathscr{M}$-directly decomposable homomorphisms.*

Proof. If $\mathscr{C}$ has distributive congruences, then by Corollary 1 in [2], $\mathscr{C}$ has $\mathscr{M}$-directly decomposable congruences and (a) follows immediately from Theorem 2.

If algebras from $\mathscr{M}$ are finite and contain a trivial algebra of $\mathscr{C}$ as a subalgebra, then by (i) in [4] (p. 285), $\mathscr{C}$ has also $\mathscr{M}$-UFP. By virtue of the just proved (a), also (b) is valid. Q.E.D.

As an application, we can concentrate on the class of lattices. Let $\mathscr{C}$ be an arbitrary non-trivial variety of lattices and $\mathscr{M}$ the class of all chains with least elements. Then:

(i) $\mathscr{C}$ has distributive congruences (see [1], Theorem VI.9),
(ii) $\mathscr{C}$ has $\mathscr{M}$-UFP (see Theorem 2 in [8] for $\oplus = \vee$ and $\omega = \wedge$),
(iii) $\mathscr{M} \subseteq \mathscr{C}$ (since every non-trivial variety of lattices contains a variety of distributive lattices containing $\mathscr{M}$),

(iv) $\mathcal{M}$ is closed under homomorphic images (evident),

(v) $\mathcal{M}$ contains a one-element lattice (evident),

(vi) $\mathcal{M}$ contains only directly irreducible lattices (see [1]).

Hence, by Corollary 1 (a), we obtain

Corollary 2. *Let $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ be chains with least elements and $A = \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$. Then every homomorphism h of the lattice A onto B is directly decomposable.*

An analogous result can be proved if a variety of weakly associative lattices or the class of tournaments is considered instead of $\mathcal{C}$ or $\mathcal{M}$, respectively. The result obtained is contained in [7].

Let $\mathcal{C}$ be a class of relatively complemented finite modular lattices and $\mathcal{M}$ a subclass of all directly irreducible lattices of $\mathcal{C}$. Then clearly the previous conditions (i), (iii), (v) and (vi) are satisfied. By Theorem 7 in [1] (p. 89), $\mathcal{C}$ contains only modular geometric lattices (Definition in [1], p. 80) and by Corollary on p. 93 in [1], all lattices in $\mathcal{M}$ are simple. Hence, $\mathcal{M}$ is closed under homomorphic images, thus also (iv) is satisfied. Since lattices of $\mathcal{C}$ are relative by complemented, $\mathcal{C}$ has permutable congruences ([1], p. 163, Ex. 7). As lattices in $\mathcal{C}$ are finite, they have congruence lattices with finite lengths and, by Corollary 1 on p. 169 in [1]. $\mathcal{C}$ has $\mathcal{M}$-UFP. Hence (ii) is also true and by Corollary 2(a), we have

Corollary 3. *Let $\mathcal{C}$ be a class of relatively complemented finite modular lattices, let $A, B \in \mathcal{C}$ and $A = \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$, where A_i, B_i are directly irreducible lattices of $\mathcal{C}$. Then every homomorphism h of A onto B is directly decomposable.*

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