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DECOMPOSITIONS OF HOMOMORPHISMS

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Some sufficient conditions for direct decompositions of homomorphisms were investigated for suitable algebras in [5] and [6] and for lattices and the so called \textit{weakly associative lattices} in [7]. The aim of this paper is to derive new conditions for direct decompositions of homomorphisms on universal algebras by using the well-known conditions for direct decompositions of congruences derived by G. A. Fraser and A. Horn in [2] and the results on the Unique Factorization Property derived by G. Birkhoff (see [1], [3], [4]).

Let $\prod_{i=1}^{n} A_i$ be the direct product of given algebras $A_1, \ldots, A_n$ of the same type and let $pr_i$ be the projection of $\prod_{i=1}^{n} A_i$ onto the $i$-th factor $A_i$. If $\sigma$ is a permutation of the subscript set $\{1, \ldots, n\}$, denote by $\kappa(\sigma)$ the isomorphism of $\prod_{i=1}^{n} A_i$ onto $\prod_{i=1}^{n} A_{\sigma(i)}$ given by the rule

$$\kappa(\sigma)(a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$

i.e., $\kappa(\sigma)$ permutes only the direct factors, which is not essential for the direct product representation.

Let $A_i, B_i$ (for $i = 1, \ldots, n$) be algebras of the same type and let $h_i$ be a homomorphism of $A_i$ into $B_i$. The mapping $h = \prod_{i=1}^{n} h_i$ of $A = \prod_{i=1}^{n} A_i$ into $B = \prod_{i=1}^{n} B_i$ introduced by

$$pr_i h(a) = h_i(pr_i a)$$

for every $a \in A$ and each $i = 1, \ldots, n$ is called the \textit{direct product of homomorphisms} $h_i$ (see e.g. G. Grätzer, Universal Algebra, p. 127).

With respect to the above remark, let us introduce conversely; a homomorphism $h$ of $A$ into $B$ is said to be \textit{direct by decomposable}, provided $h \cdot \kappa(\sigma) = \prod_{i=1}^{n} h_i$ for
a suitable permutation \( \sigma \) of \( \{1, \ldots, n\} \), where \( h_i \) is a homomorphism of \( A_i \) into \( B_{\sigma(i)} \), for all \( i = 1, \ldots, n \), and \( \kappa(\sigma) \) is an isomorphism of \( \prod_{i=1}^{n} B_i \) onto \( \prod_{i=1}^{n} B_{\sigma(i)} \).

**Lemma.** Let \( B \) and \( A_1, \ldots, A_n \) be algebras of the same type and let \( h \) be a homomorphism of \( A = \prod_{i=1}^{n} A_i \) onto \( B \) inducing a congruence \( \Theta_h \) on \( A \). If \( \Theta_h = \prod_{i=1}^{n} \Theta_i \), where \( \Theta_i \) is a congruence on \( A_i \), then there exist algebras \( B_1^*, \ldots, B_n^* \) and homomorphisms \( h_i \) of \( A_i \) onto \( B_i^* \) such that \( B \cong \prod_{i=1}^{n} B_i^* \) (by the isomorphism \( f \)) and

\[
h \cdot f = \prod_{i=1}^{n} h_i.
\]

**Proof.** \( B = h(\prod_{i=1}^{n} A_i) \cong \prod_{i=1}^{n} A_i/\Theta_h = \prod_{i=1}^{n} A_i/\prod_{i=1}^{n} \Theta_i = \prod_{i=1}^{n} (A_i/\Theta_i) \) (see e.g. [1], p. 140). Denote \( B_i^* = A_i/\Theta_i \), hence \( B \cong \prod_{i=1}^{n} B_i^* \) and clearly \( h \cdot f = \prod_{i=1}^{n} h_i \), where \( h_i : A_i \to A_i/\Theta_i \) is a canonical homomorphism. Q.E.D.

An algebra is said to have the *Unique Factorization Property over the class \( \mathcal{M} \)* (briefly \( \mathcal{M} \)-UFP), provided \( A \) is isomorphic with a direct product of algebras from \( \mathcal{M} \) and this representation is unique up to isomorphism, i.e. provided the condition

\[
A \cong \prod_{i \in I} A_i \cong \prod_{j \in J} B_j, \quad I = \{1, \ldots, n\}, \quad J = \{1, \ldots, m\}
\]

(where \( \text{card } A_i > 1 \), \( \text{card } B_j > 1 \) for every subscript \( i, j \)) with \( A_i, B_j \in \mathcal{M} \) for \( i \in I \), \( j \in J \) always implies the existence of a bijection \( \sigma \) of \( I \) onto \( J \) with \( A_i \cong B_{\sigma(i)} \) for all \( i \in I \). If \( \mathcal{M} \) is the class of directly irreducible algebras (of the same type as \( A \)), \( A \) is said briefly to have the *Unique Factorization Property*. (For this definition, see e.g. [3] and [8]).

The concepts involved can be relativized by the following

**Definition.** Let \( \mathcal{C} \) be a class of algebras of the same type and \( \mathcal{M} \) its subclass. We say that:

(i) \( \mathcal{C} \) has \( \mathcal{M} \)-directly decomposable congruences provided for every congruence \( \Theta \) on each \( A \in \mathcal{C} \) with \( A = \prod_{i=1}^{n} A_i \) for \( A_1, \ldots, A_n \in \mathcal{M} \) there exists \( \Theta_i \) on \( A_i \) such that

\[
\Theta = \prod_{i=1}^{n} \Theta_i.
\]

(ii) \( \mathcal{C} \) has \( \mathcal{M} \)-directly decomposable homomorphisms provided for every \( A, B \in \mathcal{C} \), \( A = \prod_{i=1}^{n} A_i \), \( B = \prod_{i=1}^{n} B_i \) with \( A_i, B_i \in \mathcal{M} \), every homomorphism \( h \) of \( A \) onto \( B \) is directly decomposable.

(iii) \( \mathcal{C} \) has \( \mathcal{M} \)-UFP provided every \( A \in \mathcal{C} \) has \( \mathcal{M} \)-UFP.
Theorem 1. Let $\mathcal{C}$ be a class of algebras of the same type and $\mathcal{M}$ its arbitrary subclass closed under homomorphic images. Then the implication $(a) \implies (b)$ is satisfied, where:

(a) $\mathcal{C}$ has $\mathcal{M}$-directly decomposable congruences and $\mathcal{M}$-UFP;
(b) $\mathcal{C}$ has $\mathcal{M}$-directly decomposable homomorphisms.

Proof. Suppose $A, B \in \mathcal{C}$, $A = \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$ with $A_i, B_i \in \mathcal{M}$ and let $h$ be a homomorphism of $A$ onto $B$. Let $\Theta_h$ be a congruence induced by $h$ on $A$. By $(a)$, $\Theta_h = \prod_{i=1}^{n} \Theta_i$, where $\Theta_i$ is a congruence on $A_i$. By Lemma, there exist $B_i^* \in \mathcal{C}$ ($i = 1, \ldots, n$) and homomorphisms $h_i^*$ of $A_i$ onto $B_i^*$ with $B = \prod_{i=1}^{n} B_i^*$ (an isomorphism $f$) and $h \cdot f = \prod_{i=1}^{n} h_i^*$.

Since $\mathcal{M}$ is closed under homomorphic images and $B_i^* = h_i^*(A_i)$, also $B_i^* \in \mathcal{M}$ for $i = 1, \ldots, n$. As $\mathcal{C}$ has $\mathcal{M}$-UFP, there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $B_{\sigma(i)} \cong B_i^*$. Denote by $g_i$ this isomorphism $B_{\sigma(i)}^* \cong B_i$ and put $h_i = h_i^* \cdot g_i$. Then clearly

$$h = h \cdot f \cdot \prod_{i=1}^{n} g_i \cdot \prod_{i=1}^{n} h_i^* \cdot \prod_{i=1}^{n} g_i \cdot \prod_{i=1}^{n} (h_i^* \cdot g_i) = \prod_{i=1}^{n} h_i.$$

Hence $\mathcal{C}$ has $\mathcal{M}$-directly decomposable homomorphisms. Q.E.D.

Remark. If a subclass $\mathcal{M}$ of $\mathcal{C}$ is not closed under homomorphic images, the implication $(a) \implies (b)$ of Theorem 1 need not be satisfied, even if $\mathcal{M}$ is a subclass of directly irreducible algebras. It can be demonstrated by the following example.

Example. Let $\mathcal{M} = \{N_5, I_2\}$, where $N_5$ is the non-modular pentagon (Fig. 1) and $I_2$ the two-element lattice, and let $\mathcal{C}$ be a class formed by five lattices: a one-element lattice $E$ and $N_5, I_2$, $A = N_5 \times I_2$, $B = I_2 \times I_2$. Let $h$ be a homomorphism of $A$ onto $B$ visualized by arrows in Fig. 2. Clearly $\mathcal{C}$ has $\mathcal{M}$-UFP and $\mathcal{M}$-directly
decomposable congruences as follows directly from Corollary 1 in [2] (namely the congruence \( \Theta \) induced by \( h \) satisfies \( \Theta = \Theta_1 \times \Theta_2 \), where \( \Theta_1 \) on \( N_5 \) collapses onto \( x, y \) — see Fig. 1, and \( \Theta_2 = I_2 \times I_2 \)), however \( h \) is not directly decomposable because \( h_1 \) induced by \( \Theta_1 \) maps \( N_5 \) onto the whole \( B \) contrary to \( h_2 \) induced by \( \Theta_2 \) which collapses \( I_2 \) onto the one-element lattice \( E \).

**Theorem 2.** Let \( \mathcal{C} \) be a class of algebras of the same type containing a one-element algebra \( E \), let \( \mathcal{M} \) be an arbitrary subclass of \( \mathcal{C} \) closed under homomorphic images such that \( E \in \mathcal{M} \). If \( \mathcal{C} \) has \( \mathcal{M} \)-directly decomposable congruences, the following conditions are equivalent:

1. \( \mathcal{C} \) has \( \mathcal{M} \)-directly decomposable homomorphisms,
2. \( \mathcal{C} \) has \( \mathcal{M} \)-UFP.

**Proof.** The implication (2) \( \Rightarrow \) (1) follows directly from Theorem 1. Prove (1) \( \Rightarrow \) (2). Let \( A \in \mathcal{C} \) and \( A \cong \prod_{i=1}^{n} A_i \cong \prod_{j=1}^{m} B_j \), where \( A_i, B_j \in \mathcal{M} \) and card \( A_i \) > 1, card \( B_j \) > 1. Without loss of generality, suppose \( m \leq n \). Put \( B_j = E \) for \( j = m + 1, \ldots, n \).

Then \( \prod_{i=1}^{n} A_i \cong \prod_{j=1}^{n} B_j \), denote this isomorphism by \( h \). As \( h \) is \( \mathcal{M} \)-directly decomposable, thus, for every \( i = 1, \ldots, n \), \( A_i \) is isomorphic to \( B_{\sigma(i)} \) for a suitable permutation \( \sigma \) of \( \{1, \ldots, n\} \). Hence card \( B_j \) > 1 for \( j = 1, \ldots, n \), i.e. \( m = n \) and \( A \) has \( \mathcal{M} \)-UFP.

Q.E.D.

Necessary and sufficient conditions for the direct decomposition of congruences are derived by G. A. Fraser and A. Horn in [2].

**Corollary 1.** Let \( \mathcal{C} \) be a class of algebras with distributive congruences and \( \mathcal{M} \) an arbitrary class of directly irreducible algebras of \( \mathcal{C} \) closed under homomorphic images and containing a trivial algebra \( E \) of \( \mathcal{C} \). Then

(a) \( \mathcal{C} \) has \( \mathcal{M} \)-directly decomposable homomorphisms if and only if \( \mathcal{C} \) has \( \mathcal{M} \)-UFP.
(b) If \( A \in \mathcal{M} \) implies that \( A \) is finite and a trivial algebra is a subalgebra of \( A \), then \( \mathcal{C} \) has \( \mathcal{M} \)-directly decomposable homomorphisms.

**Proof.** If \( \mathcal{C} \) has distributive congruences, then by Corollary 1 in [2], \( \mathcal{C} \) has \( \mathcal{M} \)-directly decomposable congruences and (a) follows immediately from Theorem 2.

If algebras from \( \mathcal{M} \) are finite and contain a trivial algebra of \( \mathcal{C} \) as a subalgebra, then by (i) in [4] (p. 285), \( \mathcal{C} \) has also \( \mathcal{M} \)-UFP. By virtue of the just proved (a), also (b) is valid. Q.E.D.

As an application, we can concentrate on the class of lattices. Let \( \mathcal{C} \) be an arbitrary non-trivial variety of lattices and \( \mathcal{M} \) the class of all chains with least elements. Then:

(i) \( \mathcal{C} \) has distributive congruences (see [1], Theorem VI.9),
(ii) \( \mathcal{C} \) has \( \mathcal{M} \)-UFP (see Theorem 2 in [8] for \( \oplus = \lor \) and \( \omega = \land \)),
(iii) \( \mathcal{M} \subseteq \mathcal{C} \) (since every non-trivial variety of lattices contains a variety of distributive lattices containing \( \mathcal{M} \)),

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(iv) $\mathcal{M}$ is closed under homomorphic images (evident),
(v) $\mathcal{M}$ contains a one-element lattice (evident),
(vi) $\mathcal{M}$ contains only directly irreducible lattices (see [1]).
Hence, by Corollary 1 (a), we obtain

**Corollary 2.** Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be chains with least elements and $A =$
$= \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$. Then every homomorphism $h$ of the lattice $A$ onto $B$ is directly decomposable.

An analogous result can be proved if a variety of weakly associative lattices or the class of tournaments is considered instead of $\mathcal{C}$ or $\mathcal{M}$, respectively. The result obtained is contained in [7].

Let $\mathcal{C}$ be a class of relatively complemented finite modular lattices and $\mathcal{M}$ a subclass of all directly irreducible lattices of $\mathcal{C}$. Then clearly the previous conditions (i), (iii), (v) and (vi) are satisfied. By Theorem 7 in [1] (p. 89), $\mathcal{C}$ contains only modular geometric lattices (Definition in [1], p. 80) and by Corollary on p. 93 in [1], all lattices in $\mathcal{M}$ are simple. Hence, $\mathcal{M}$ is closed under homomorphic images, thus also (iv) is satisfied. Since lattices of $\mathcal{C}$ are relative by complemented, $\mathcal{C}$ has permutable congruences ([1], p. 163, Ex. 7). As lattices in $\mathcal{C}$ are finite, they have congruence lattices with finite lengths and, by Corollary 1 on p. 169 in [1], $\mathcal{C}$ has $\mathcal{M}$-UFP. Hence (ii) is also true and by Corollary 2(a), we have

**Corollary 3.** Let $\mathcal{C}$ be a class of relatively complemented finite modular lattices, let $A, B \in \mathcal{C}$ and $A =$
$= \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$, where $A_i, B_i$ are directly irreducible lattices
of $\mathcal{C}$. Then every homomorphism $h$ of $A$ onto $B$ is directly decomposable.

References


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