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ON 2-FACTORS IN SQUARES OF GRAPHS

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Let G be a graph in the sense of [2] or [5]. We denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. If $u \in V(G)$, then we denote by $\deg u$ or $\deg_G u$ the degree of u in G. A vertex of degree 0 is called isolated. We denote

$V'(G) = \{v \in V(G);\ \deg v \neq 1\}$;
$V^*(G) = \{v \in V(G)$; there exists exactly one vertex w of degree one such that $vw \in E(G)\}$;
$V''(G) = V'(G) \cup V^*(G)$;
$N'(w) = \{v \in V'(G);\ vw \in E(G)\}$, for every $w \in V(G)$;
$N'(W) = \bigcup_{w \in W} N'(w)$, for every $W \subseteq V(G)$.

Finally, for every $w \in V^*(G)$, we denote by $\bar{w}$ the vertex of degree one which is adjacent to w.

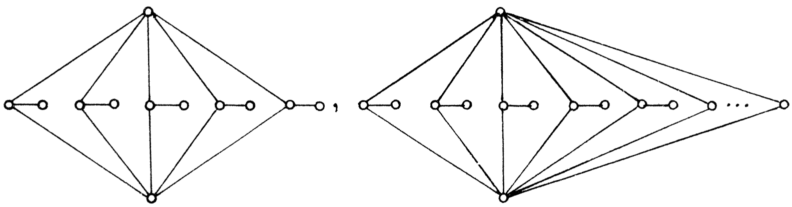

Fig. 1.

We say that a spanning subgraph F of G is an n-factor of G (where n is a positive integer) if F is a regular graph of degree n.

By the square G^2 of a graph G we mean the graph with $V(G^2) = V(G)$ and

$$E(G^2) = \{uv;\ u, v \in V(G) \text{ such that } 1 \leqq d(u, v) \leqq 2\},$$

where $d(w, w')$ denotes the distance between vertices w and w' in G.

Obviously, if a graph G has a 1-factor, then $|V(G)|$ is even. CHARTRAND, POLIMENI and STEWART [3], and SUMNER [8] proved that if G is a connected graph of even order, then G^2 has a 1-factor.

It is easy to see that the squares of none of the connected graphs in Figs 1 or 2 have a 2-factor. A necessary and sufficient condition for the square of a connected

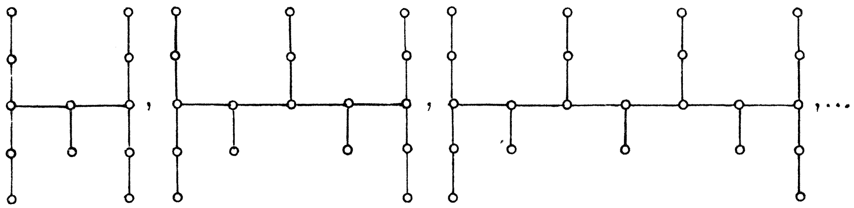

Fig. 2.

graph to have a 2-factor was published in [1]. Unfortunately, the assertion of sufficiency of that condition is false: every connected graph in Figs 1 and 2 can serve as a counter example. In the present paper another condition will be given.

Obviously, if a graph G has a 2-factor. then G contains no isolated vertex. The following theorem represents the main result of this paper:

Theorem. *Let G be a graph with no isolated vertex. Then G^2 has a 2-factor if and only if*

(1) $$|W| \leqq 2|N'(W)| \quad \textit{for every} \quad W \subseteq V^*(G).$$

To obtain the proof of this theorem we shall prove four lemmas.

Lemma 1. *Let G be a graph with no isolated vertex. If G^2 has a 2-factor, then* (1) *holds.*

Proof. Assume that G^2 has a 2-factor, say F, and that (1) does not hold. Then there exists $W \subseteq V^*(G)$ such that $|W| > 2|N'(W)|$. We have that

$$2|W| = \sum_{w \in W} \deg_F w \leqq |W| + 2|N'(W)| < 2|W|,$$

which is a contradiction. Hence the lemma follows.

Let G be a graph with no isolated vertex, and let D be a digraph (we shall denote by $V(D)$ and $A(D)$ the set of its vertices and the set of its arcs, respectively). We shall say that D is *suitable* for G, if the following conditions are fulfilled:

(i) $V(D) = V''(G)$;

(ii) if $(u, v) \in A(D)$, then $uv \in E(G)$;

(iii) if $v \in V^*(G)$, then outdeg $v = 1$;
(iv) if $v \in V''(G) - V^*(G)$, then outdeg $v = 0$;
(v) if $v \in V'(G)$, then indeg $v \leqq 2$;
(vi) if $v \in V''(G) - V'(G)$, then indeg $v = 0$

(the symbols indeg v and outdeg v denote the indegree and outdegree of v in D).

Lemma 2. *Let G be a graph with no isolated vertex. If* (1) *holds, then there exists a suitable digraph for G.*

Proof. Assume that (1) holds. Let G^{I} and G^{II} be disjoint copies of G. If $U \subseteq V(G)$ (or $u \in V(G)$), then we denote by U^{I} and U^{II} (or u^{I} and u^{II}) the corresponding copy of U(or u) in G^{I} and G^{II}, respectively. From (1) it follows that

$$|W| \leqq |(N'(W))^{\mathrm{I}} \cup (N'(W)))^{\mathrm{II}}| \quad \text{for every} \quad W \subseteq V^*(G).$$

According to P. HALL's Theorem [4] (see, for example, Theorem 12.3 in [2]), the collection of sets

$$(N'(w))^{\mathrm{I}} \cup (N'(w))^{\mathrm{II}}; \quad w \in V^*(G)$$

has a system of distinct representatives. This means that there exists a mapping f from $V^*(G)$ into $(V'(G))^{\mathrm{I}} \cup (V'(G))^{\mathrm{II}}$ such that

(a) if $u, v \in V^*(G)$ and $u \neq v$, then $f(u) \neq f(v)$;
(b) if $w \in V^*(G)$, then $f(w) \in (N'(W))^{\mathrm{I}} \cup (N'(W))^{\mathrm{II}}$.

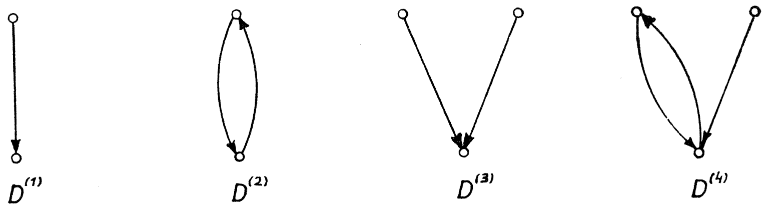

Fig. 3.

We denote by g the mapping from $V^*(G)$ into $V'(G)$ defined as follows: if $u \in V^*(G)$ then $f(u) \in \{g(u)^{\mathrm{I}}, g(u)^{\mathrm{II}}\}$. Finally, we denote by D the digraph with $V(D) = V''(G)$, and

$$A(D) = \{(u, g(u));\ u \in V^*(G)\}.$$

Clearly, D is suitable for G, which completes the proof of the lemma.

Let G be a graph with no isolated vertex. We say that a digraph D is *very suitable* for G if D is suitable for G and every nontrivial weak component of D is isomorphic to one of the weakly connected digraphs $D^{(1)}, \ldots, D^{(4)}$ in Fig. 3.

Lemma 3. *Let G be a graph with no isolated vertex. If there exists a suitable digraph for G, then there exists a very suitable digraph for G.*

Proof. If D is a digraph, then we denote by $i(D)$ the maximum number of vertices of a weak component of D, and by $j(D)$ the number of weak components C of D such that $|V(C)| = i(D)$. Let D_1 and D_2 be digraphs such that $V(D_1) = V(D_2)$; we shall write $D_1 > D_2$ if either (a) $i(D_1) = i(D_2)$ and $j(D_1) > j(D_2)$ or (b) $i(D_1) > i(D_2)$.

Let D be a suitable digraph for G. First, let $i(D) \leqq 3$. Assume that D is not very suitable. Let $C_1, \ldots, C_n$ be the nontrivial weak components of D which are not isomorphic to any of the digraphs $D^{(1)}, \ldots, D^{(4)}$. Then there exist distinct vertices $u_1, v_1, w_1, \ldots, u_n, v_n, w_n$ such that $V(C_1) = \{u_1, v_1, w_1\}, \ldots, V(C_n) = \{u_n, v_n, w_n\}$ and $u_1v_1, v_1w_1, \ldots, u_nv_n, v_nw_n \in A(D)$. It is clear that $D - v_1w_1 - \ldots - v_nw_n + v_1u_1 + \ldots + v_nu_n$ is very suitable for G.

We now assume that $i(D) \geqq 4$, and that if there exists a suitable digraph D_0 for G such that $D > D_0$, then there exists a very suitable digraph for G. Let C be an arbitrary component of D such that $|V(C)| = i(D)$. Hence, $|V(C)| \geqq 4$. We distinguish two cases:

1. There exist $u, v, w \in V(C)$ such that $(u, v), (v, w) \in A(D)$, indeg $u = 0$, and $C - (v, w)$ is not weakly connected. Then $D - (v, w) + (v, u)$ is suitable for G and $D > D - (v, w) + (v, u)$.

2. For every $u, v, w \in V(C)$ such that $(u, v), (v, w) \in A(D)$ and indeg $u = 0$ it holds that $C - (v, w)$ is weakly connected. Then C contains exactly one directed cycle, say C', and every arc in C is incident with a vertex in C'. Since $|V(C)| \geqq 4$, there exist $u_0, u, u_1, v_0, v, v_1 \in V(C)$ such that $(u_0, u), (u, u_1), (v_0, v)$ and (v, v_1) are distinct arcs in C, (u, u_1) and (v, v_1) belong to C', indeg $u_0 \leqq 1$, and indeg $v_0 \leqq 1$. Then $(u, u_0), (v, v_0) \notin A(D)$, $D - (u, u_1) - (v, v_1) + (u, u_0) + (v, v_0)$ is suitable for G, and $D > D - (u, u_1) - (v, v_1) + (u, u_0) + (v, v_0)$.

From the induction assumption the assertion of the lemma follows.

Lemma 4. *If G is a graph with no isolated vertex such that there exists a very suitable digraph for G, then G^2 has a 2-factor.*

Proof. Assume that the lemma is false. Then there exists a graph G such that the lemma is false for G but it is true for every proper spanning subgraph of G. Since the lemma is false for G, we have that G is a graph with no isolated vertex, there exists a very suitable digraph for G, say a digraph D, and G^2 has no 2-factor. This means that the square of no spanning subgraph of G has a 2-factor. Since for every proper spanning subgraph of G the lemma is true, we have that for every $e \in E(G)$, either $G - e$ contains an isolated vertex or there exists no very suitable digraph for $G - e$.

From the definition of a suitable digraph it follows that every component of G contains at least three vertices.

First, let every component of G be homeomorphic to a star (note that a path is also homeomorphic to a star). From the existence of D it follows that there exists $A \subseteq E(G)$ such that every component of $G - A$ is a tree with at least three vertices which contains no subgraph isomorphic to the subdivision graph $S(K(1, 3))$ of the star $K(1, 3)$. According to a result due to F. NEUMAN [7], every component of $(G - A)^2$ is hamiltonian, and therefore G^2 has a 2-factor, which is a contradiction.

We now assume that there exists a component G_1 of G which is not homeomorphic to a star. We shall prove that there exists $e \in E(G_1)$ such that $G - e$ contains no isolated vertex and there exists a very suitable digraph for $G - e$, which will be a contradiction. We shall distinguish a number of cases:

1. There exists no nontrivial weak component of D whose vertices belong to G_1. Then $V^*(G_1) = \emptyset$.

1.1. G_1 is a tree. Since G_1 is not homeomorphic to a star, we have that there exists $e \in E(G_1)$ such that every component of $G_1 - e$ contains at least three vertices. It is easy to see that there exists a very suitable digraph for $G_1 - e$, and therefore there exists a very suitable digraph for $G - e$.

1.2. G_1 is not a tree. Then there exists $e \in E(G_1)$ such that $G_1 - e$ is connected. Clearly, there exists a very suitable digraph for $G - e$.

2. There exists a nontrivial weak component of D whose vertices belong to G_1. Since G_1 is not homeomorphic to a star, Fig. 3 implies that there exist adjacent vertices u and v of G_1 such that (a) u belongs to a nontrivial weak component of D, say D_1, (b) (u, v), $(v, u) \notin A(D)$, (c) every component of $G_1 - uv$ contains at least three vertices. Clearly, $\deg v \geqq 2$.

2.1. $\deg v > 2$. If $\deg u > 2$, then D is very suitable for $G - uv$. Let $\deg u = 2$. Then D_1 is isomorphic to $D^{(1)}$, $\text{indeg}\, u = 1$, and $\text{outdeg}\, u = 0$. Clearly, the vertex of D_1 different from u, say u_1, belongs to $V^*(G)$. This means that $D - (u_1, u)$ is a very suitable digraph for $G - uv$.

2.2. $\deg v = 2$. Let w denote the vertex different from u and adjacent to v. Since every component of $G_1 - uv$ contains at least three vertices, we have that $w \in V'(G)$. Hence, $v \notin V^*(G)$.

2.2.1. v belongs to a nontrivial weak component of D, say D_2. Since $(u, v) \notin A(D)$, D_2 is isomorphic to $D^{(1)}$. Hence, $(w, v) \in A(D)$. If $\deg u > 2$, then $D - (w, v)$ is very suitable for $G - uv$. Let $\deg u = 2$. Then $D - (u_1, u) - (w, v)$ is very suitable for $G - uv$, where u_1 is the same as in Case 2.1.

2.2.2. v belongs to no nontrivial weak component of D. From the fact that $w \in V'(G)$ it follows that every component of $G_1 - vw$ contains at least three vertices. If $u \in V^*(G)$, then there exists a vertex u' such that $(u, u') \in A(D)$. If $u \notin V^*(G)$, then $\text{outdeg}\, u = 0$ and there exists a vertex u'' such that $(u'', u) \in A(D)$.

2.2.2.1. $\deg w > 2$. Then either $D - (u, u')$ (if $u \in V^*(G)$) or $D + (u, u'')$ (if $u \notin V^*(G)$) is very suitable for $G - vw$.

2.2.2.2. $\deg w = 2$. Let x denote the vertex different from v and adjacent. to w. Since every component of $G - vw$ contains at least three vertices, we have that $x \in V'(G)$. Hence, $w \notin V^*(G)$.

2.2.2.2.1. w belongs to a nontrivial weak component of D, say D_3. Since (v, w), $(w, v) \notin A(D)$, we have that D_3 is isomorphic to $D^{(1)}$. It is easy to see that either $D - (u, u') - (x, w)$ or $D + (u, u'') - (x, w)$ is very suitable for $G - vw$.

2.2.2.2.2. w belongs to no nontrivial weak component of D.

2.2.2.2.2.1. x belongs to a nontrivial weak component of D. If $x \in V^*(G)$, then there exists a vertex x' such that $(x, x') \in A(D)$. If $x \notin V^*(G)$, then $\text{outdeg}\, x = 0$ and there exists a vertex x'' such that $(x'', x) \in A(D)$. If $u = x$, then either D or $D - (x, x')$ is very suitable for $G - vw$. If $u \neq x$, then one of the following digraphs is very suitable for $G - vw$: $D - (u, u') - (x, x')$, $D - (u, u') + (x, x'')$, $D + (u, u'') - - (x, x')$, $D + (u, u'') + (x, x'')$.

2.2.2.2.2.2. x belongs to no nontrivial component of D. If $\deg u > 2$, then $D + + (w, x)$ is very suitable for $G - uv$. If $\deg u = 2$, then $D + (w, x) - (u', u)$ is very suitable for $G - uv$, where u' is the same as in Case 1.

Hence the lemma follows.

Thus the proof of the theorem is complete.

Corollary 1 (A. HOBBS [6]). *If G is a nontrivial connected graph with no vertex of degree one, then G^2 has a 2-factor.*

Since for every nontrivial graph G, the total graph of G si isomorphic to the square of the subdivision graph of G, we have the following corollary, which was stated in [1]:

Corollary 2. *Let G be a nontrivial connected graph. Then the total graph of G has a 2-factor if and only if every vertex of G is adjacent to at most two vertices of degree one.*

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