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Josef Nedoma

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# COMPLETE EXTENSION OF A CONVEX FUNCTION

JOSEF NEDOMA, Praha

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When studying the properties of subsets of the so-called convex manifolds a special attention is paid to a class of convex functions the definition domain of which is "as large as possible". A certain type of such functions is called complete here (see Def. 2). In this paper the problem of extending a convex function to a complete convex function is discussed. In order to explain the motivation of our approach we mention briefly the problem of the analytic expression of a relatively convex surface.

We shall deal with the real linear space $R^n$ (or $R^{n+1}$). The closure, the boundary and the convex hull of a set $A$ are denoted by $\operatorname{cl} A$, $\operatorname{bd} A$ and $[A]$, respectively. Further, ray $A$ is the set of all positive multiples of the elements of $A$ and $A^*$ is the polar cone of $A$. (We shall consider polar cones both in $R^n$ and in $R^{n+1}$). The normal cone of a set $A$ at a point $x \in A$ is denoted by $N(A)(x)$, i.e. $N(A)(x) = \{v \mid A \subset x + \{v\}^*, v \neq 0\}$. A closed halfspace $H$ is called a supporting halfspace of $A$ if $A \subset H$ and $A \cap \operatorname{bd} H \neq \emptyset$. The term function means a finite function exclusively. The domain of definition of a function $f$ is denoted by $\operatorname{dom} f$ and $\partial f(x)$ means the subdifferential of $f$ at a point $x$. Finally, $N(A) = \bigcup\{N(A)(x) \mid x \in A\}$, $\partial f(A) = \bigcup\{\partial f(x) \mid x \in A\}$ and $\partial f = \partial f(\operatorname{dom} f)$.

## I. RELATIVELY CONVEX SURFACES

A surface $P \subset R^n$ is called *relatively convex with respect to a vector u* (briefly: *u-convex*) if 1°. $u \notin \operatorname{cl} \operatorname{ray}(P - P)$ and 2°. $P + \operatorname{ray}\{u\}$ is a convex set.

Consider a convex function $f : R^n \to R$. It can be easily shown that $P = \{x \mid f(x) = 0\}$ is $u$-convex for each $u \in \operatorname{int}(\partial f(P))^*$ provided $\partial f(P)$ does not contain the zero vector. Indeed: If $x^1, x^2 \in P$ then there exists no $t > 0$ such that $u = t(x^1 - x^2)$ since otherwise $f(x^1) \leqq f(x^2) + \langle v, x^1 - x^2 \rangle < 0$ would hold for $v \in \partial f(x^1)$, which would contradict the assumption $x^1 \in P$. Consequently $u \notin \operatorname{ray}(P - P)$ and therefore 1° is fulfilled because $\operatorname{int}(\partial f(P))^*$ is open. Further, $x + \operatorname{ray}\{u\} \subset M^- = \{x \mid f(x) < 0\}$ for every $x \in P$ and conversely $(z - \operatorname{ray}\{u\}) \cap P \neq \emptyset$ for every $z \in M^-$ so that $P + \operatorname{ray}\{u\} = M^-$ which is convex.

The latter argument, however, cannot be used in the case that $f$ is defined in a convex region $G \neq R^n$. From this point of view it is interesting to study the problem of extending a given convex function $f$ to the whole space in such a manner that $(\partial f)^*$ is kept. If such an extension is possible then the mentioned surface stands for a part of a relatively convex one.

This problem is even more important in the case of manifolds the dimension of which is less than $n - 1$.

## II. COMPLETE CONVEX FUNCTIONS

**Definition 1.** A function $f$ is called *open* if $F = \{(x, \mu) \mid \mu > f(x),\ x \in \operatorname{dom} f\}$ is an open set in $R^{n+1}$.

**Lemma 1.** *An open function is continuous.*

Proof. Let $Y \subset R$ be an arbitrary open interval. Then $Z = (R^n \times Y) \cap F$ is an open set and therefore its orthogonal projection $X = f^{-1}(Y)$ into $R^n$ is also open. □

**Lemma 2.** *A continuous function $f$ is open if and only if* $\operatorname{dom} f$ *is open.*

Proof. Let $\operatorname{dom} f$ be open and choose $(\bar{x}, \bar{\mu}) \in F$. Then there exist $\varepsilon > 0$ and a neighbourhood $O(\bar{x}) \subset \operatorname{dom} f$ such that $f(x) < \bar{\mu} - \varepsilon\ \forall x \in O(\bar{x})$. Hence $\Omega = \{(x, \mu) \mid \mu > \bar{\mu} - \varepsilon,\ x \in O(\bar{x})\}$ is an open set satisfying $(\bar{x}, \bar{\mu}) \in \Omega \subset F$. Thus $F$ in open. The "only if" part of the lemma holds trivially. □

**Corollary 2.1.** *A convex function $f$ is open if and only if* $\operatorname{dom} f$ *is open.*

**Definition 2.** A function is called *complete* if it is both open and closed.

**Theorem 1.** *A convex function is complete if and only if it increases infinitely near the boundary of its definition domain.*

Proof. a) If the condition is satisfied then $\operatorname{bd}(\operatorname{dom} f) \cap \operatorname{dom} f = \emptyset$ which means that $\operatorname{dom} f$ is open and therefore $f$ is open according to Lemma 2. On the other hand for every $(\bar{x}, \bar{\mu}) \in \operatorname{bd}(\operatorname{epi} f)$ we have $\bar{\mu} < +\infty$ and hence $(\bar{x}, \bar{\mu}) \in \operatorname{graph} f \subset \operatorname{epi} f$. Thus $f$ is closed and therefore complete.

b) Suppose that there exist a point $x^0 \in \operatorname{bd} \operatorname{dom} f$ and a number $\alpha$ such that $\inf \{f(x) \mid x \in O(x^0) \cap \operatorname{dom} f\} < \alpha$ for any neighbourhood $O(x^0)$ of $x^0$. Then there is a sequence $x^k$ such that $x^k \in \operatorname{dom} f$, $x^k \to x^0$ and $f(x^k) \leqq \alpha$. Hence $x^0 \in \operatorname{dom}(\operatorname{cl} f)$ which means that either $x^0 \in \operatorname{dom} f$ (then $f$ cannot be open) or $f \neq \operatorname{cl} f$ ($f$ is not closed). □

Note 1. A convex function defined in the whole space is complete.

Let $f$ be a convex function and consider the sets $K_f = N(\operatorname{epi} f) = \{(tv, -t) \mid v \in \partial f, t > 0\}$ and

$$A_f = \operatorname{cl}(\operatorname{epi} f + K_f^*)\,.$$

Evidently $e^{n+1} = (0^n, 1) \in K_f^*$ and thus $A_f + \operatorname{ray} e^{n+1} = A_f$. It means that $A_f$ stands for the epigraph of a finite function.

**Definition 3.** Let $f$ be an open convex function. Then a function $f^{\text{up}}$ defined by

$$\operatorname{epi} f^{\text{up}} = \operatorname{cl}(\operatorname{epi} f + K_f^*) \tag{1}$$

will be called *an upper extension of $f$.*

Let us denote by $Z_f$ the intersection of all supporting halfspaces of $\operatorname{epi} f$.

**Definition 4.** Let $f$ be an open convex function. Then the function $f^{\text{low}}$ defined by

$$\operatorname{epi} f^{\text{low}} = Z_f \tag{2}$$

will be called *a lower extension of $f$.*

Note 2. $f^{\text{low}}$ is the supremum of all linear functions $h$ such that $h \leqq f$ and $h(x) = f(x)$ for an $x \in \operatorname{dom} f$.

**Theorem 2.** *Let $f$ be an open convex function. Then*

1°. *$f^{\text{up}}, f^{\text{low}}$ are closed convex functions;*
2°. $\operatorname{dom} f \subset \operatorname{dom} f^{\text{low}} = \operatorname{dom} f^{\text{up}}$;
3°. $f^{\text{low}} \leqq f^{\text{up}}$;
4°. $f^{\text{low}}(x) = f^{\text{up}}(x) = f(x)\ \forall x \in \operatorname{dom} f$;
5°. $\partial f^{\text{low}} \subset \operatorname{cl}[\partial f]$, $\partial f^{\text{up}} \subset \operatorname{cl}[\partial f]$.

Proof. 1° follows immediately from (1), (2). To verify 2° take notice first of all that

$$\operatorname{int} \operatorname{dom} f^{\text{low}} \subset \operatorname{dom} f^{\text{up}}\,. \tag{3}$$

Indeed, $x^0 \in \operatorname{bd}(\operatorname{dom} f^{\text{up}})$ implies that $(w, 0) \in \operatorname{cl} K_f$ for any $w \in N(\operatorname{dom} f^{\text{up}})(x^0)$. Therefore there exist supporting halfspaces $H_k$ of $\operatorname{epi} f$ such that their normals converge to $(w, 0)$. It means that there exists an $\alpha$ such that $H_0 = \{x \mid \langle w, x \rangle \leqq \alpha\} \subset R^n$ is a supporting halfspace of both $\operatorname{cl} \operatorname{dom} f$ and $\operatorname{cl} \operatorname{dom} f^{\text{low}}$. Since $\operatorname{epi} f \subset f^{\text{up}}$ and consequently $\operatorname{dom} f \subset \operatorname{dom} f^{\text{up}}$, we have $x^0 \notin \operatorname{int} H_0$ which proves (3). Further, $K_f^* = (K_{f^{\text{low}}})^*$ yields $A_f \subset Z_f + K_f^* = Z_f$ which together with (3) proves 2° and also 3°.

According to Note 2, $f(x) \leqq f^{\text{low}}(x)\ \forall x \in \operatorname{dom} f$. This together with 2° yields 4°

since epi $f \subset$ epi $f^{up}$. Finally, we have $N(A_f) \subset K_f^{**}$, $N(Z_f) \subset K_f^{**}$. Since $K_f^{**} = \text{cl}[K_f]$, the relations (1), (2) yield 5°. □

Of course, the functions $f^{low}$, $f^{up}$ are not identical in general. See the following

Example. Consider $f : G \to R$ where $G$ is the positive orthant of $R^2$ and

$$f = \max \begin{Bmatrix} 0 \\ x_1 - x_2 - 1 \\ -x_1 \end{Bmatrix}, \quad x \in G .$$

Evidently, $f$ is open and convex. Then

$$f^{up} = \max \begin{Bmatrix} 0 \\ x_1 - x_2 - 1 \\ -x_1 \\ -\frac{1}{2}x_2 \end{Bmatrix}, \quad x \in R^2$$

while

$$f^{low} = \max \begin{Bmatrix} 0 \\ x_1 - x_2 - 1 \\ -x_1 \end{Bmatrix}, \quad x \in R^2 .$$

Both the extensions are defined in the whole space and therefore they are complete.

**Theorem 3.** *Let $f$, $g$ be open convex functions such that*

1°. $g(x) = f(x)$ $\forall x \in \text{dom} f$;

2°. $\partial g \subset \text{cl}[\partial f]$.

*Then* $f^{low} \leqq g \leqq f^{up}$.

Proof. a) To verify $f^{low} \leqq g$ suppose that there exists an $y \in R^n$ such that $g(y) < f^{low}(y)$. Then there is a supporting halfspace $H$ of epi $f$ at a point $(x, f(x)) \in$ graph $f$, such that $(y, g(y)) \notin H$. It means that $g$ is not convex which contradicts the hypothesis.

b) Since epi $f \subset$ epi $g$ and $K_f^* \subset K_g^*$ due to 2°, we have epi $f^{up} \subset$ epi $g^{up}$ or $g^{up} \leqq f^{up}$. It proves $g \leqq f^{up}$ because $g^{up}(x) = g(x)$ $\forall x \in \text{dom}\, g$ according to Theorem 2.

□

**Theorem 4.** *If $f$ is a complete convex function then*

$$f^{up} = f^{low} = f .$$

Proof. Every closed convex set $A$ can be expressed as the intersection of its supporting halfspaces, i.e. $A = A + (N(A))^*$. Applying this to $A = \text{epi} f$ we obtain the statement of the theorem.

**Lemma 3.** *An open convex function $f$ satisfies the Lipschitz condition if and only if $\partial f$ is bounded.*

Proof. a) We have $\langle u, x^1 - x^2 \rangle \leqq f(x^1) - f(x^2) \leqq -\langle v, x^2 - x^1 \rangle \ \forall x^1, x^2 \in \in \operatorname{dom} f$, $u \in \partial f(x^2)$, $v \in \partial f(x^1)$. If $\partial f$ is bounded then the set of norms of all subgradients of $f$ possesses a finite supremum $\beta$ which can be taken as the Lipschitz constant:

$$|f(x^2) - f(x^1)| \leqq \beta |x^2 - x^1| \ \forall x^1, x^2 \in \operatorname{dom} f. \tag{4}$$

b) For any $x^1 \in \operatorname{dom} f$ and $v \in \partial f(x^1)$ there exists a $t > 0$ such that $x^2 = x^1 + tv \in \operatorname{dom} f$. Then

$$f(x^2) - f(x^1) \geqq \langle v, x^2 - x^1 \rangle = |v| \, |x^2 - x^1| .$$

Consequently: if $\partial f$ is not bounded then there is no $\beta$ satisfying (4). □

**Theorem 5.** *Let $f$ be an open convex Lipschitz-type function. Then $f^{\mathrm{up}}, f^{\mathrm{low}}$ are complete convex functions.*

Proof. $\partial f$ is bounded by Lemma 3 and thus for an arbitrary $w \in R^n$ there exists $t > 0$ such that

$$\langle (w, t), (v, -1) \rangle = \langle w, v \rangle - t \leqq 0 \ \forall v \in \partial f. \tag{5}$$

Every vector $(w, t)$ satisfying (5) belongs to $K_f^*$. Since $\operatorname{dom} f^{\mathrm{up}}$ stands for the orthogonal projection of $\operatorname{epi} f^{\mathrm{up}} = \operatorname{cl}(\operatorname{epi} f + K_f^*)$ into $R^n$, we have $\operatorname{dom} f^{\mathrm{up}} = R^n$. The same holds for $\operatorname{dom} f^{\mathrm{low}}$ according to Theorem 2. □

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*Author's address:* 111 73 Praha 1, Politických vězňů 7, ČSSR. (Ekonomicko-matematická laboratoř EÚ ČSAV).