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_Czechoslovak Mathematical Journal_, Vol. 30 (1980), No. 1, 71–79

Persistent URL: [http://dml.cz/dmlcz/101656](http://dml.cz/dmlcz/101656)

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A RELATIONSHIP BETWEEN A HANKEL MATRIX 
OF MARKOV PARAMETERS 
AND THE ASSOCIATED MATRIX POLYNOMIAL 
WITH SOME APPLICATIONS 

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(Received January 19, 1978) 

1. INTRODUCTION 

Let 

\begin{equation} 
f(x) = a_{n+1}x^n - a_n x^{n-1} - \ldots - a_2x - a_1 
\end{equation} 

and 

\begin{equation} 
g(x) = b_{m+1}x^m - b_m x^{m-1} - \ldots - b_2x - b_1 
\end{equation} 

be two polynomials of degree \( n \) and \( m \) with real coefficients. For the sake of simplicity, let us assume that \( a_{n+1} = 1 \) and \( m \leq n \). 

The quantities \( s_i, \ i = -1, 0, 1, 2, \ldots, \) defined, by 

\[
g(x) = \frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \ldots 
\]

are called Markov parameters associated with the rational function 

\[
R(x) = \frac{g(x)}{f(x)} 
\]

and the matrices 

\[
H_{kk} = \begin{bmatrix} 
s_0 & s_1 & \ldots & s_{k-1} 
s_1 & s_2 & \ldots & s_k 
\vdots  
s_{k-1} & s_k & \ldots & s_{2k-2} 
\end{bmatrix} 
\]
(k = 1, 2, ..., n) are called **Hankel matrices of Markov parameters**. There exist simple recursive relations for generating the coefficients of a Hankel matrix of Markov parameters. For example, in case n = m, these relations are given by [8, vol. II, pp. 214]

\begin{equation}
\begin{align*}
    s_{-1} &= b_{n+1}, \\
    s_0 &= a_n s_{-1} = -b_n, \\
    s_n &= a_n s_{n-2} \ldots - a_1 s_{-1} = -b_1, \\
    s_t &= a_n s_{t-1} \ldots - a_1 s_{t-n} = 0,
\end{align*}
\end{equation}

(t = n, n + 1, n + 2, ...). In this paper, we establish an interesting relationship between the Hankel matrix of Markov parameters $H_n$ and the matrix polynomial $g(A)$, where $A$ is the companion matrix of $f(x)$. As an immediate application of this result, we demonstrate the equivalence of the well-known Markov stability criterion [8, vol. II, pp. 235–236] and a recent formulation of the Liénard-Chipart criterion of stability by Barnett [1]. By the use of this result, we also show that a criterion of aperiodicity recently obtained by the author [4] is equivalent to the one given by Barnett in [1]. We indicate several other possible applications.

### 2. LEMMAS

We establish a few lemmas in this section which will be used later.

**Lemma 1.** Let $H_n$ be the Hankel matrix of Markov parameters associated with the polynomials $f(x)$ and $g(x)$ and let $A$ be the companion matrix of $f(x)$. Then

\begin{equation}
    AH_n = H_n A^T
\end{equation}

**Proof.** Let

\begin{equation}
    A = \begin{bmatrix}
        0 & 1 & 0 & 0 & \ldots & 0 \\
        0 & 0 & 1 & 0 & \ldots & 0 \\
        \vdots & \vdots & \ddots & \vdots & & \vdots \\
        0 & 0 & 0 & \ldots & 0 & 1 \\
        a_1 & a_2 & a_3 & a_4 & \ldots & a_{n-1} & a_n
    \end{bmatrix}
\end{equation}

and let $H_n = (s_{i+j})$. Then

\begin{equation}
    AH_n = (s_{i+j+1})
\end{equation}

is symmetric; as is $H_n$.

This proves the lemma.

As an immediate Corollary of Lemma 1, we obtain the following:
Corollary 1. Let \( h_1, h_2, \ldots, h_n \) be the \( n \) successive columns of \( H_{nn} \). Then

\[
(6) \quad h_{i+1} = Ah_i, \quad i = 1, 2, \ldots, n - 1.
\]

Lemma 2. Let \( g(x) \) and \( A \) be the same as defined in (2) and (5) and let \( g_1, g_2, \ldots, g_n \) be the successive \( n \) columns of \( g(A) \). Then

\[
(7) \quad g_{n-1} = (A - a_nI) g_n,
\]

\[
(8) \quad g_{n-i} = Ag_{n-i+1} - a_{n-i+1} g_n, \quad i = 2, 3, \ldots, n - 1.
\]

The above result is a special case of a result recently obtained by the author [5]. For the sake of completeness, however, we give here a short derivation of the lemma.

**Proof.** Let \( l_i \) be the \( i \)th column of the identity matrix \( I \) of order \( n \).

Then

\[
g_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} = (g(A) (A - a_nI) l_n = (A - a_nI) (g(A) l_n) = (A - a_nI) g_n
\]

(note that \( g(A) \) and \( A \) commute with each other).

In general

\[
g_{n-i} = g(A) l_{n-i} = g(A) (Al_{n-i+1} - a_{n-i+1} l_n) = Ag_{n-i+1} - a_{n-i+1} g_n,
\]

\[
(i = 2, 3, \ldots, n - 1).
\]

3. A RELATIONSHIP BETWEEN \( g(A) \) AND \( H_{nn} \)

**Theorem 1.**

\[
g(A) = H_{nn} \begin{bmatrix} -a_2 & -a_3 & \ldots & -a_n & 1 \\ -a_3 & -a_4 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} = H_{nn} U
\]

**Proof.** Let \( h'_1, h'_2, \ldots, h'_n \) be the columns of \( H_{nn} U \) and \( h_1, h_2, \ldots, h_n \) be those of \( H_{nn} \).

Then

\[
(9) \quad h'_n = h_1
\]

\[
(10) \quad h'_{n-1} = h_2 - a_1 h_1
\]

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By Corollary 1, \( h_2 = Ah_1 \)

So,

\[
(11) \quad h'_{n-1} = Ah_1 - a_nh_1 = (A - a_nI)h_1.
\]

In general,

\[
(12) \quad h'_{n-i} = h_{i+1} - a_nh_i - a_{n-1}h_{i-1} - \ldots - a_{n-i+2}h_2 - a_{n-i+1}h_1 =
\]

\[
= Ah_i - a_nAh_{i-1} - a_{n-1}Ah_{i-2} - \ldots - a_{n-i+2}Ah_1 - a_{n-i+1}h_1 \quad \text{(Using Corollary 1)}
\]

\[
= A(h_i - a_nh_{i-1} - a_{n-1}h_{i-2} - \ldots - a_{n-i+2}h_1) - a_{n-i+1}h_1 =
\]

\[
= Ah'_{n-i+1} - a_{n-i+1}h_1 =
\]

\[
= Ah'_{n-i+1} - a_{n-i+1}h'_i, \quad i = 2, 3, \ldots, n - 1 \quad \text{(since } h_1 = h'_n).\]

Thus, by the results of lemma 2 and from (11) and (12), it follows that the first \((n - 1)\) columns of \( H_{n-1}U \) satisfy the same recursive relations as do those of \( g(A) \).

Also, let \( g_n \) be the last column of \( g(A) \),

\[
g_n = \begin{bmatrix} g_{n1} \\ g_{n2} \\ \vdots \\ g_{nn} \end{bmatrix}
\]

Then in case \( n = m \)

\[
g_{n1} = b_{n+1}a_n - b_n,
\]

\[
g_{n2} = b_{n+1}(a_n^2 + a_{n-1}) - b_n(a_n) - b_{n-1} =
\]

\[
= a_n(b_{n+1}a_n - b_n) + a_{n-1}b_{n+1} - b_{n-1} =
\]

\[
= a_ng_{n1} + a_{n-1}b_{n+1} - b_{n-1}
\]

\[\vdots\]

\[\text{etc.}\]

Bringing the Markov parameters into the picture, we see by means of relations (3) that

\[
g_{n1} = s_0,
\]

\[
g_{n2} = a_ns_0 + a_{n-1}s_{-1} - b_{n-1} = s_1,
\]

\[\vdots\]

\[\text{etc.}\]

This shows that

\[
g_n = (g_{n1}, g_{n2}, \ldots, g_{nm})^T = (s_0, s_1, \ldots, s_n)^T = h_1 = h'_n
\]

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This relation is also valid in case $n < m$ and can be verified similarly. The proof is now complete.

4. APPLICATIONS

(a) EQUIVALENCE BETWEEN TWO CRITERIA OF STABILITY

Let $f(x)$ be the same as defined in (1) and represent it in the form

$$f(x) = h(x^2) + x \gamma(x^2).$$

This representation gives rise to two polynomials $h(u)$ and $\gamma(u)$ defined as follows:

$$h(u) = -a_1 - a_3 u - a_5 u^2 - \ldots,$$
$$\gamma(u) = -a_2 - a_4 u - a_6 u^3 - \ldots.$$ 

Assume that $h(u)$ and $\gamma(u)$ are relatively prime and generate $s_{-1}, s_0, s_1 \ldots$ by

$$\frac{\gamma(u)}{h(u)} = s_{-1} + \frac{s_0}{u} + \frac{s_1}{u^2} + \ldots.$$

The following theorem gives a criterion of stability of $f(x)$ ($f(x)$ is said to be stable if all the roots of $f(x)$ have negative real parts).

**Theorem 2.** (Markov Criterion of Stability [8. vol. II, pp. 235–236]). $f(x)$ is stable if and only if the following system of determinantal inequalities hold:

$$s_0 > 0, \begin{vmatrix} s_0 & s_1 & \cdots & s_{m-1} \\ s_1 & s_2 & \cdots & s_m \\ \vdots & \vdots & \ddots & \vdots \\ s_{m-1} & s_m & \cdots & s_{2m-2} \end{vmatrix} > 0,$$

$$s_1 < 0, \begin{vmatrix} s_1 & s_2 & \cdots & s_m \\ s_2 & s_3 & \cdots & s_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m} & s_{m+1} & \cdots & s_{2m-1} \end{vmatrix} > 0,$$

where $n = 2m$ or $2m + 1$ according as $n$ is even or odd. If $n$ is odd, in addition to the above inequalities, $s_{-1}$ is needed to be positive.

Assume now all the coefficients of $h(u)$, namely $a_1, a_3, a_5 \ldots$ etc are negative (there is no loss of generality in this assumption, because, the necessity condition of stability demands that all the coefficients of $f(x)$ be negative).
The condition that $s_{-1} > 0$ in case $n$ is odd, is trivially satisfied in this case. For, when $n$ is odd, $s_{-1} = -1/a_n > 0$. Furthermore, under this assumption, we show that the second set of inequalities is redundant. To do this, first we give a matrix formulation of theorem 2.

Let $H$ be the companion matrix of the form (5) of $h(u)$ when $n$ is even and of $-(1/a_n) h(u)$ when $n$ is odd. Let $H_{mm} = (s_{i+j})$ be the associated Hankel matrix of Markov parameters. Then,

$$HH_{mm} = (s_{i+j+1}).$$

The first set of inequalities, therefore, implies that $H_{mm}$ is positive definite and the second set implies that $HH_{mm}$ is negative definite.

This later condition is redundant. For since $H$ is nonderogatory, positive definite, $HH_{mm}$ implies that all the roots of $h(u)$ are real and distinct. Moreover, since all the coefficients of $h(u)$ are negative, $h(u) > 0$ for all $u \geq 0$. This implies that the roots of $h(u)$ are all negative as well.

$$HH_{mm} = H_{mm}H^T$$

is therefore, negative definite. The above discussion allows us to reformulate Theorem 2 in Liénard-Chipart style as follows:

**Theorem 2'**. $f(x)$ is stable if and only if

$$a_1 < 0, \ a_3 < 0, \ a_5 < 0, \ ...$$

and $H_{mm}$ is positive definite.

In [1], Barnett presented a new formulation of the classical Liénard-Chipart stability criterion using certain matrix polynomials. In the following Theorem we present his results with some modifications*).

**Theorem 3.** Let $R_k$ denote the minor of the first $k$ rows and the last $k$ columns of $\gamma(H)$ and define

$$(14) \quad t_k = (-1)^k \frac{(k - 1)}{2}$$

then, $f(x)$ is stable if and only if $a_1 < 0, a_3 < 0, a_5 < 0, \ ...$ and $t_k R_k > 0$, $k = 1, 2, \ldots, m$.

We now prove:

**Theorem 4.** Theorem 3 and Theorem 2' are equivalent.

*) In case $n$ is odd; Barnett gave his results using a different matrix polynomial $h(R)$, where $R$ is the companion matrix of $\gamma(u)$. However, as stated in Theorem 3, both the cases can be handled using the same matrix polynomial $\gamma(H)$.  

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Proof. Consider two cases.

Case 1. $n$ is even. By Theorem 1,

\[
\gamma(H) = H_{mm} \begin{bmatrix} -a_3 & -a_5 & \ldots & -a_{n-1} & 1 \\ -a_5 & -a_7 & \ldots & 1 & 0 \\ \vdots \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Case 2. $n$ is odd. Let $H'_{mm}$ denote the Hankel matrix of Markov parameters associated with $-(1/a_n) h(u)$ and $\gamma(u)$. Then by Theorem 1,

\[
\gamma(H) = H'_{mm} \begin{bmatrix} a_3 & a_5 & \ldots & a_{n-2} & 1 \\ a_5 & a_7 & \ldots & 1 & 0 \\ \vdots \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}.
\]

Again, it is easy to check that

\[H'_{mm} = -a_n H_{mm}.\]

Therefore,

\[
\gamma(H) = H_{mm} \begin{bmatrix} -a_3 & -a_5 & \ldots & -a_{n-1} & -a_n \\ -a_5 & -a_7 & \ldots & -a_n & 0 \\ \vdots \\ -a_n & 0 & \ldots & 0 & 0 \end{bmatrix}.
\]

Applying now the Cauchy-Binet Theorem [8, vol. I, pp. 9–12] to (15) and (16), we see that Theorem 3 and Theorem 2' are equivalent.

(b) EQUIVALENCE BETWEEN TWO CRITERIA OF APERIODICITY

A polynomial $f(x)$ with real coefficients is said to be aperiodic if all its roots are distinct and negative real. The concept of aperiodicity is an important concept in Mathematical Control Theory [1].

In [1], Barnett gave a criterion of aperiodicity using the matrix polynomial $f'(A)$, where $f'(x)$ is the derivative of $f(x)$.

Theorem 5. $f(x)$ is periodic if and only if all $a_i < 0$ and $t_k F_k > 0$, $k = 1, 2, \ldots, n$;
where $F_k$ is the minor of the first $k$ rows and last $k$ columns of $f'(A)$ and $t_k$ is the same as defined in (14).

Recently the author [4], [6] has shown.

**Theorem 6.** $f(x)$ is aperiodic if and only if all $a_i < 0$ and the Hankel matrix of Markov parameters associated with $f(x)$ and $f'(x)$ is positive definite.

In view of Theorem 1, Theorem 5 and Theorem 6 are easily seen to be equivalent.

**Remark.** In [4], the author gave the criterion of aperiodicity using Hankel matrix of Newton sums. However later in [6], it has been shown that the Hankel matrix of Newton sums is just the Hankel matrix of Markov parameters associated with $f(x)$ and $f'(x)$.

5. DISCUSSIONS

We have established here a relationship between the Hankel matrix of Markov parameters $H_n$ associated with two polynomials $f(x)$ and $g(x)$ and the matrix polynomial $g(A)$, where $A$ is the companion matrix of $f(x)$. As an immediate application of this result, we have demonstrated the equivalence of the well-known Markov criterion of stability (modified in Liénard-Chipart style) and a recent result of Barnett on the classical stability criterion of Liénard and Chipart. By the use of this result we have also shown that a recently obtained criterion of aperiodicity of the author is equivalent to the one obtained by Barnett earlier. It is to be noted also that there exist a few results involving $g(A)$ on the root separation of polynomials and other related problems. For example, Barnett [2] and later (independently) the author [3] have shown how $g(A)$ may be employed to obtain information on the location of roots a polynomial in a given half plane and inside the unit circle. It is also well-known that polynomials $f(x)$ and $g(x)$ are relatively prime if and only if $g(A)$ is nonsingular. The rank of $g(A)$ even determines the degree of the greatest common divisor of $f(x)$ and $g(x)$. These results and a few others have been nicely summarized in a recent survey of Barnett [2].

The matrix polynomial $g(A)$ is again related to the classical Bézout matrix associated with Bézoutian defined by $f(x)$ and $g(x)$, and there exists a great variety of classical results involving Bézoutian. For more details, the readers may again refer to the survey of Barnett [2] (see also [7]).

In view of the relationship between $H_n$ and $g(A)$ established in this paper, all the results involving $g(A)$ (and therefore those involving the Bézoutian as well) can now be given new interpretations in terms of $H_n$. One can be used as a complete alternative to the other. Computationally, the use of $H_n$ is attractive in the sense that there exist simple recursive relations for generating the elements of a Hankel matrix of Markov parameters.
References


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