Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 30 (1980), No. 1, 71–79

Persistent URL: `http://dml.cz/dmlcz/101656`

Czechoslovak Mathematical Journal, 30 (105) 1980, Praha

A RELATIONSHIP BETWEEN A HANKEL MATRIX OF MARKOV PARAMETERS AND THE ASSOCIATED MATRIX POLYNOMIAL WITH SOME APPLICATIONS

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(Received January 19, 1978)

1. INTRODUCTION

Let

(1) $$f(x) = a_{n+1}x^n - a_n x^{n-1} \dots - a_2 x - a_1$$

and

(2) $$g(x) = b_{m+1}x^m - b_m x^{m-1} \dots - b_2 x - b_1$$

be two polynomials of degree n and m with real coefficients. For the sake of simplicity, let us assume that $a_{n+1} = 1$ and $m \leqq n$.

The quantities s_i, $i = -1, 0, 1, 2, \dots$, defined, by

$$\frac{g(x)}{f(x)} = s_{-1} + \frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \dots$$

are called *Markov parameters associated with the rational function*

$$R(x) = \frac{g(x)}{f(x)}$$

and the matrices

$$H_{kk} = \begin{bmatrix} s_0 & s_1 \dots s_{k-1} \\ s_1 & s_2 \dots s_k \\ \vdots & \\ s_{k-1} & s_k \dots s_{2k-2} \end{bmatrix}$$

($k = 1, 2, \ldots, n$) are called *Hankel matrices of Markov parameters*. There exist simple recursive relations for generating the coefficients of a Hankel matrix of Markov parameters. For example, in case $n = m$, these relations are given by [8. vol. II, pp. 214]

$$
\begin{aligned}
&s_{-1} = b_{n+1} ,\\
&s_0 - a_n s_{-1} = -b_n ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&s_{n-1} - a_n s_{n-2} \ldots - a_1 s_{-1} = -b_1 ,\\
&s_t - a_n s_{t-1} \ldots - a_1 s_{t-n} = 0 ,
\end{aligned} \tag{3}
$$

($t = n,\ n + 1,\ n + 2, \ldots$). In this paper, we establish an interesting relationship between the Hankel matrix of Markov parametres H_{nn} and the matrix polynomial $g(A)$, where A is the companion matrix of $f(x)$. As an immediate application of this result, we demonstrate the equivalence of the well-known Markov stability criterion [8, vol. II, pp. 235–236] and a recent formulation of the Liénard-Chipart criterion of stability by Barnett [1]. By the use of this result, we also show that a criterion of aperiodicity recently obtained by the author [4] is equivalent to the one given by Barnett in [1]. We indicate several other possible applications.

2. LEMMAS

We establish a few lemmas in this section which will be used later.

Lemma 1. *Let H_{nn} be the Hankel matrix of Markov parameters associated with the polynomials $f(x)$ and $g(x)$ and let A be the companion matrix of $f(x)$. Then*

$$AH_{nn} = H_{nn}A^T \tag{4}$$

Proof. Let

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0\ldots0 & 0\\
0 & 0 & 1 & 0\ldots0 & 0\\
. & . & . & \ldots\ldots & .\\
. & . & . & \ldots\ldots & .\\
0 & 0 & 0 & 0\ldots0 & 1\\
a_1 & a_2 & a_3 & a_4\ldots a_{n-1} & a_n
\end{bmatrix} \tag{5}
$$

and let $H_{nn} = (s_{i+j})$. Then

$$AH_{nn} = (s_{i+j+1})$$

is symmetric; as is H_{nn}.

This proves the lemma.

As an immediate Corollary of Lemma 1, we obtain the following:

Corollary 1. *Let $h_1, h_2, \ldots, h_n$ be the n successive columns of H_{nn}. Then*

(6) $$h_{i+1} = Ah_i\,, \quad i = 1, 2, \ldots, n - 1\,.$$

Lemma 2. *Let $g(x)$ and A be the same as defined in (2) and (5) and let $g_1, g_2, \ldots, g_n$ be the successive n columns of $g(A)$. Then*

(7) $$g_{n-1} = (A - a_n I)\, g_n\,,$$

(8) $$g_{n-i} = Ag_{n-i+1} - a_{n-i+1} g_n\,, \quad i = 2, 3, \ldots, n - 1\,.$$

The above result is a special case of a result recently obtained by the author [5]. For the sake of completeness, however, we give here a short derivation of the lemma.

Proof. Let l_i be the its column of the identity matrix I of order n.
Then

$$g_{n-1} = (g_1, g_2, \ldots, g_{n-1}, g_n) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} =$$

$$= g(A)(A - a_n I)\, l_n = (A - a_n I)(g(A)\, l_n) = (A - a_n I)\, g_n$$

(note that $g(A)$ and A commute with each other).
In general

$$g_{n-i} = g(A)\, l_{n-i} = g(A)(Al_{n-i+1} - a_{n-i+1} l_n) =$$
$$= A\, g(A)\, l_{n-i+1} - a_{n-i+1}\, g(A)\, l_n = Ag_{n-i+1} - a_{n-i+1} g_n\,,$$
$$(i = 2, 3, \ldots, n - 1)\,.$$

3. A RELATIONSHIP BETWEEN $g(A)$ AND H_{nn}

Theorem 1.

$$g(A) = H_{nn} \begin{bmatrix} -a_2 & -a_3 & \ldots & -a_n & 1 \\ -a_3 & -a_4 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = H_{nn} U$$

Proof. Let $h'_1, h'_2, \ldots, h'_n$ be the columns of $H_{nn}U$ and $h_1, h_2, \ldots, h_n$ be those of H_{nn}. Then

(9) $$h'_n = h_1$$

(10) $$h'_{n-1} = h_2 - a_n h_1$$

By Corollary 1, $h_2 = Ah_1$
So,

(11) $$h'_{n-1} = Ah_1 - a_n h_1 = (A - a_n I)\, h_1\,.$$

In general,

(12) $$\begin{aligned} h'_{n-i} &= h_{i+1} - a_n h_i - a_{n-1} h_{i-1} \ldots - a_{n-i+2} h_2 - a_{n-i+1} h_1 = \\ &= Ah_i - a_n Ah_{i-1} - a_{n-1} Ah_{i-2} \ldots - a_{n-i+2} Ah_1 - \\ &- a_{n-i+1} h_1 \quad \text{(Using Corollary 1)} \\ &= A(h_i - a_n h_{i-1} - a_{n-1} h_{i-2} \ldots - a_{n-i+2} h_i) - a_{n-i+1} h_1 = \\ &= Ah'_{n-i+1} - a_{n-i+1} h_1 = \\ &= Ah'_{n-i+1} - a_{n-i+1} h'_n\,, \quad i = 2, 3, \ldots, n-1 \quad (\text{since } h_1 = h'_n)\,. \end{aligned}$$

Thus, by the results of lemma 2 and from (11) and (12), it follows that the first $(n - 1)$ columns of $H_{nn}U$ satisfy the same recursive relations as do those of $g(A)$.

Also, let g_n be the last column of $g(A)$,

$$g_n = \begin{bmatrix} g_{n1} \\ g_{n2} \\ \vdots \\ g_{nn} \end{bmatrix}.$$

Then in case $n = m$

$$g_{n1} = b_{n+1} a_n - b_n\,,$$

$$\begin{aligned} g_{n2} &= b_{n+1}(a_n^2 + a_{n-1}) - b_n(a_n) - b_{n-1} = \\ &= a_n(b_{n+1} a_n - b_n) + a_{n-1} b_{n+1} - b_{n-1} = \\ &= a_n g_{n1} + a_{n-1} b_{n+1} - b_{n-1} \end{aligned}$$

....

etc.

Bringing the Markov parameters into the picture, we see by means of relations (3) that

$$g_{n1} = s_0\,,$$

$$g_{n2} = a_n s_0 + a_{n-1} s_{-1} - b_{n-1} = s_1\,,$$

....

etc.

This shows that

$$g_n = (g_{n1}, g_{n2}, \ldots, g_{nn})^T = (s_0, s_1, \ldots, s_n)^T = h_1 = h'_n$$

This relation is also valid in case $n < m$ and can be verified similarly.

The proof is now complete.

4. APPLICATIONS

(a) EQUIVALENCE BETWEEN TWO CRITERIA OF STABILITY

Let $f(x)$ be the same as defined in (1) and represent it in the form

$$f(x) = h(x^2) + x\,\gamma(x^2).$$

This representation gives rise to two polynomials $h(u)$ and $\gamma(u)$ defined as follows:

$$h(u) = -a_1 - a_3 u - a_5 u^2 - \dots,$$

$$\gamma(u) = -a_2 - a_4 u - a_6 u^3 - \dots.$$

Assume that $h(u)$ and $\gamma(u)$ are relatively prime and generate $s_{-1}, s_0, s_1 \dots$ by

$$\frac{\gamma(u)}{h(u)} = s_{-1} + \frac{s_0}{u} + \frac{s_1}{u^2} + \dots.$$

The following theorem gives a criterion of stability of $f(x)$ ($f(x)$ is said to be *stable* if all the roots of $f(x)$ have negative real parts).

Theorem 2. (Markov Criterion of Stability [8. vol. II, pp. 235–236]). *$f(x)$ is stable if and only if the following system of determinantal inequalities hold*:

$$s_0 > 0, \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} > 0, \dots, \begin{vmatrix} s_0 & s_1 & \dots & s_{m-1} \\ s_1 & s_2 & \dots & s_m \\ \vdots & & & \\ s_{m-1} & s_m & \dots & s_{2m-2} \end{vmatrix} > 0,$$

$$s_1 < 0, \quad \begin{vmatrix} s_1 & s_2 \\ s_2 & s_3 \end{vmatrix} > 0, \dots, (-1)^m \begin{vmatrix} s_1 & s_2 & \dots & s_m \\ s_2 & s_3 & \dots & s_{m+1} \\ \vdots & & & \\ s_m & s_{m+1} & \dots & s_{2m-1} \end{vmatrix} > 0,$$

where $n = 2m$ or $2m + 1$ according as n is even or odd. If n is odd, in addition to the above inequilities, s_{-1} is needed to be positive.

Assume now all the coefficients of $h(u)$, namely $a_1, a_3, a_5 \dots$ etc are negative (there is no loss of generality in this assumption, be cause, the necessity condition of stability demands that all the coefficients of $f(x)$ be negative).

The condition that $s_{-1} > 0$ in case n is odd, is trivially satisfied in this case. For, when n is odd, $s_{-1} = -1/a_n > 0$. Fur theremore, under this assumption, we show that the second set of inequalities is redundant. To do this, first we give a matrix formulation of theorem 2.

Let H be the companion matrix of the form (5) of $h(u)$ when n is even and of $-(1/a_n)\, h(u)$ when n is odd. Let $H_{mm} = (s_{i+j})$ be the associated Hankel matrix of Markov parameters. Then,

$$HH_{mm} = (s_{i+j+1}) .$$

The first set of inequalities, therefore, implies that H_{mm} is positive definite and the second set implies that HH_{mm} is negative definite.

This later condition is redundant. For since H is nonderogatory, positive definite. ness of H_{mm} implies that all the roots of $h(u)$ are real and distinct. Moreover, since all the coefficients of $h(u)$ are negative, $h(u) > 0$ for all $u \geqq 0$. This implies that the roots of $h(u)$ are all negative as well.

$$HH_{mm} = H_{mm}H^T$$

is therefore, negative definite. The above discussion allows us to reformulate Theorem 2 in Liénard-Chipart style as follows:

Theorem 2′. *$f(x)$ is stable if and only if*

$$a_1 < 0\,, \quad a_3 < 0\,, \quad a_5 < 0\,, \ldots$$

and H_{mm} is positive definite.

In [1], Barnett presented a new formulation of the classical Liénard-Chipart stability criterion using certain matrix polynomials. In the following Theorem we present his results with some modifications*).

Theorem 3. *Let R_k denote the minor of the first k rows and the last k columns of $\gamma(H)$ and define*

$$t_k = (-1)^k \,.\, \frac{(k-1)}{2} \tag{14}$$

then, $f(x)$ is stable if and only if $a_1 < 0$, $a_3 < 0$, $a_5 < 0, \ldots$ and $t_kR_k > 0$, $k = 1, 2, \ldots, m$.

We now prove:

Theorem 4. *Theorem 3 and Theorem 2′ are equivalent.*

*) In case n is odd; Barnett gave his results using a different matrix polynomial $h(R)$, where R is the companion matrix of $\gamma(u)$. However, as stated in Theorem 3, both the cases can be handled using the same matrix polynomial $\gamma(H)$.

Proof. Consider two cases.

Case 1. n is even. By Theorem 1,

(15)
$$\gamma(H) = H_{mm} \begin{bmatrix} -a_3 & -a_5 & \dots & -a_{n-1} & 1 \\ -a_5 & -a_7 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & & 0 & 0 \end{bmatrix}.$$

Case 2. n is odd. Let H'_{mm} denote the Hankel matrix of Markov parameters associated with $-(1/a_n)\,h(u)$ and $\gamma(u)$. Then by Theorem 1,

$$\gamma(H) = H'_{mm} \begin{bmatrix} \frac{a_3}{a_n} & \frac{a_5}{a_n} & \dots & \frac{a_{n-2}}{a_n} & 1 \\ \frac{a_5}{a_n} & \frac{a_7}{a_n} & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Again, it is easy to check that

$$H'_{mm} = -a_n H_{mm}\,.$$

Therefore,

(16)
$$\gamma(H) = H_{mm} \begin{bmatrix} -a_3 & -a_5 & \dots & -a_{n-2} & -a_n \\ -a_5 & -a_7 & \dots & -a_n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_n & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Applying now the Cauchy-Binet Theorem [8. vol. I, pp. 9–12] to (15) and (16), we see that Theorem 3 and Theorem 2′ are equivalent.

(b) EQUIVALENCE BETWEEN TWO CRITERIA OF APERIODICITY

A polynomial $f(x)$ with real coefficients is said to be aperiodic if all its roots are distinct and negative real. The concept of aperiodicity is an important concept is Mathematical Control Theory [1].

In [1], Barnett gave a criterion of aperiodicity using the matrix polynomial $f'(A)$, where $f'(x)$ is the derivative of $f(x)$.

Theorem 5. *$f(x)$ is periodic if and only if all $a_i < 0$ and $t_k F_k > 0$, $k = 1, 2, \dots, n$;*

where F_k is the minor of the first k rows and last k columns of $f'(A)$ and t_k is the same as defined in (14).

Recently the author [4], [6] has shown.

Theorem 6. *$f(x)$ is aperiodic if and only if all $a_i < 0$ and the Hankel matrix of Markov parameters associated with $f(x)$ and $f'(x)$ is positive definite.*

In view of Theorem 1, Theorem 5 and Theoıem 6 are easily seen to be equivalent.

Remark. In [4], the author gave the criterion of aperiodicity using Hankel matrix of Newton sums. However later in [6], it has been shown that the Hankel matrix of Newton sums is just the Hankel matrix of Markov parameters associated with $f(x)$ and $f'(x)$.

5. DISCUSSIONS

We have established here a relationship between the Hankel matrix of Markov parameters H_{nn} associated with two polynomials $f(x)$ and $g(x)$ and the matrix polynomial $g(A)$, where A is the companion matrix of $f(x)$. As an immediate application of this result, we have demonstrated the equivalence of the well-known Markov criterion of stability (modified in Liénard-Chipart style) and a recent result of Barnett on the classical stability criterion of Liénard and Chipart. By the use of this result we have also shown that a recently obtained criterion of aperiodicity of the author is equivalent to the one obtained by Barnett earlier. It is to be noted also that there exist a few results involving $g(A)$ on the root separation of polynomials and other related problems. For example, Barnett [2] and later (independently) the author [3] have shown how $g(A)$ may be employed to obtain information on the location of roots a polynomial in a given half plane and inside the unit circle. It is also well-known that polynomials $f(x)$ and $g(x)$ are relatively prime if and only if $g(A)$ is nonsingular. The rank of $g(A)$ even determines the degree of the greatest common divisor of $f(x)$ and $g(x)$. These results and a few others have been nicely summarized in a recent survey of Barnett [2].

The matrix polynomial $g(A)$ is again related to the classical Bézout matrix associated with Bézoutian defined by $f(x)$ and $g(x)$, and there exists a great variety of classical results involving Bézoutian. For more details, the readers may again refer to the survey of Barnett [2] (see also [7]).

In view of the relationship between H_{nn} and $g(A)$ established in this paper, all the results involving $g(A)$ (and therefore those involving the Bézoutian as well) can now be given new interpretations in terms of H_{nn}. One can be used as a complete alternative to the other. Computationally, the use of H_{nn} is attractive in the sense that there exist simple recursive relations for generating the elements of a Hankel matrix of Markov parameters.

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