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# A NECESSARY CONDITION FOR TWIN BOUNDARY LAYER BEHAVIOR

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**1. Introduction.** In this note we give a simple condition that a constant $k$ must satisfy if it is to be the limit as $\varepsilon \to 0^+$ within $(a, b)$ of a solution $y = y(t, \varepsilon)$ of

$$\varepsilon y'' = p(t, y)\, y'^2 + q(t, y)\, y', \quad a < t < b, \tag{1.1}$$

$$y(a, \varepsilon) = A, \quad y(b, \varepsilon) = B, \quad A \neq B; \quad A, B \neq k. \tag{1.2}$$

More specifically, if the problem (1.1), (1.2) has a solution $y = y(t, \varepsilon)$ which exhibits boundary layer behavior at both $t = a$ and $t = b$ and which satisfies $\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = k$, $a < t < b$, then $k$ satisfies a certain equation involving only $A$, $B$ and the functions $p$ and $q$. A similar necessary condition has recently been given by the author and S. V. PARTER [2] for the related quasilinear problem

$$\text{(QL)} \qquad \begin{aligned} &\varepsilon y'' = f(t, y)\, y', \quad a < t < b, \\ &y(a, \varepsilon) = A, \quad y(b, \varepsilon) = B. \end{aligned}$$

However the presence of the quadratic term in (1.1) requires us to modify the technique used in [2].

**2. A'Priori Estimates.** It follows directly from the form of (1.1) that any solution $y = y(t, \varepsilon)$ of (1.1), (1.2) is strictly increasing (if $A < B$) and strictly decreasing (if $A > B$); indeed, $(B - A)\, y'(t, \varepsilon) > 0$, $a \leqq t \leqq b$. Similarly any solution $y(t, \varepsilon)$ lies between $\min\{A, B\}$ and $\max\{A, B\}$ for $a \leqq t \leqq b$. This can be proved either by means of the maximum principle (cf. [1]) or by means of Nagumo's estimates [3], [4]. Finally suppose that any solution $y(t, \varepsilon)$ of (1.1), (1.2) is such that $\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = k$, $a < t < b$, for a constant $k$ strictly between $A$ and $B$. Then VISHIK and LIUSTERNIK [6] (cf. also [5; Chap. 2]) have given the following estimates for

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$y'(a, \varepsilon)$ and $y'(b, \varepsilon)$: for $A < B$, $y'(a, \varepsilon) = \exp[\varepsilon^{-1} \int_k^A p(a, s)\,\mathrm{d}s]$ and $y'(b, \varepsilon) = [\varepsilon^{-1} \int_k^B p(b, s)\,\mathrm{d}s]$, while for $A > B$,

$$y'(a, \varepsilon) = -\exp\left[\varepsilon^{-1} \int_k^A p(a, s)\,\mathrm{d}s\right] \quad \text{and} \quad y'(b, \varepsilon) = -\exp\left[\varepsilon^{-1} \int_k^B p(b, s)\,\mathrm{d}s\right].$$

Note that in both cases $y'(\tau, \varepsilon) = O(\exp[C\varepsilon^{-1}])$, $\tau = a$ or $b$, for a positive constant $C$ which is equal to the length of the initial boundary layer jump [6], [5; Chap. 2].

**3. A Necessary Condition.** Suppose for definiteness that $A < B$ in the following theorem.

**Theorem.** *Let the functions $p$ and $q$ be of class $C^{(1)}$ on $[a, b] \times \mathbb{R}^1$ and let (1.1), (1.2) have a solution $y = y(t, \varepsilon)$ which satisfies $\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = k$, $a < t < b$, for a constant $k$ in $(A, B)$. Then $k$ is a solution of*

$$(*) \qquad \int_k^B p(b, s)\,\mathrm{d}s - \int_k^A p(a, s)\,\mathrm{d}s = \eta(b, B) - \eta(a, A) + \int_a^b r(t, k)\,\mathrm{d}t\,,$$

*where $\eta = \eta(t, y) = \int^y p(t, s)\,\mathrm{d}s$ and*

$$r(t, y) = q(t, y) - \tilde{q}(t, y)\,,$$

*for*

$$\tilde{q}(t, y) = \frac{\partial}{\partial t}(\eta(t, y))\,.$$

Proof. Since $A < B$, $y'(t, \varepsilon) > 0$, $a \leqq t \leqq b$, and consequently (1.1) is equivalent to $\varepsilon y''/y' = p(t, y)\,y' + q(t, y)$, i.e.,

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}(\ln y') = \frac{\mathrm{d}}{\mathrm{d}t}(\eta(t, y)) + r(t, y)\,,$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\eta(t, y)) = p(t, y)\,y' + \tilde{q}(t, y)\,.$$

Integrating both sides of this equation from $t = a$ to $t = b$ and using (1.2) we obtain

$$\varepsilon \ln(y'(b, \varepsilon)) - \varepsilon \ln(y'(a, \varepsilon)) = \eta(b, B) - \eta(a, A) + \int_a^b r(t, y(t, \varepsilon))\,\mathrm{d}t\,,$$

i.e.,

$$(\sim) \qquad \int_k^B p(b, s)\,\mathrm{d}s - \int_k^A p(a, s)\,\mathrm{d}s = \eta'b, B) - \eta(a, A) + \int_a^b r(t, y(t, \varepsilon))\,\mathrm{d}t$$

by virtue of the estimates of Vishik and Liusternik given in Section 2. Finally since $\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = k$, $a < t < b$, it follows from the Dominated Convergence Theorem that $\lim_{\varepsilon \to 0^+} \int_a^b r(t, y(t, \varepsilon))\, dt = \int_a^b r(t, k)\, dt$. The estimate $(*)$ now results from letting $\varepsilon \to 0^+$ in $(\sim)$.

If $A > B$ then a similar argument shows that $k$ is also a solution of $(*)$, i.e.,

$$\int_k^B p(b, s)\, ds - \int_k^A p(a, s)\, ds = \eta(b, B) - \eta(a, A) + \int_a^b r(t, k)\, dt .$$

**4. Two Examples.** Consider first

$$\text{(E1)} \quad \varepsilon y'' = yy'^2 - yy' , \quad 0 < t < 1 , \quad y(0, \varepsilon) = A , \quad y(1, \varepsilon) = B , \quad A \neq B .$$

Suppose that (E1) has a solution $y(t, \varepsilon)$ satisfying $\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = k$, $a < t < b$, for $k$ between $A$ and $B$. Here $p(t, y) = y$ and $q(t, y) = -y$ and so $\eta(t, y) = \frac{1}{2}y^2$ and $r(t, y) = -y$. A short computation shows that $(*)$ reduces to $k = 0$.

Consider next

$$\text{(E2)} \qquad \varepsilon w'' = ww'^2 - w , \quad 0 < t < 1 , \quad w(0, \varepsilon) = A , \quad w(1, \varepsilon) = B .$$

Our theory does not apply directly to (E2); however, the change of dependent variable $y = w - t$ converts it into

$$\text{(E3)} \qquad \varepsilon y'' = (y + t)\, y'^2 + 2(y + t)\, y' , \quad 0 < t < 1 , \quad y(0, \varepsilon) = A ,$$
$$y(1, \varepsilon) = B - 1$$

which is of the form (1.1), (1.2). Suppose now that for $A < B - 1$ (E3) has a solution $y(t, \varepsilon)$ such that $\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = k$, $0 < t < 1$, for a constant $k$ in $(A, B - 1)$. Since $p(t, y) = y + t$ and $q(t, y) = 2(y + t)$ we set

$$\eta(t, y) = \tfrac{1}{2}y^2 + ty$$

and

$$r(t, y) = q(t, y) - \frac{\partial}{\partial t}(\eta(t, y)) = y + t .$$

It follows directly that $(*)$ reduces to $k = -\frac{1}{2}$; moreover, one can show that for $A < -\frac{1}{2} < B - 1$ (E3) has a solution $y = y(t, \varepsilon)$ for which $\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = -\frac{1}{2}$, $0 < t < 1$. In terms of (E2) this means that for $A < -\frac{1}{2}$ and $B > \frac{1}{2}$ there is a solution $w = w(t, \varepsilon)$ which satisfies $\lim_{\varepsilon \to 0^+} w(t, \varepsilon) = t - \frac{1}{2}$, $0 < t < 1$.

**5. Concluding Remark.** For the quasilinear problem (QL) (with $A \neq B$) the analog of $(*)$ is $\int_a^b f(t, k)\,dt = 0$, where $\lim\limits_{\varepsilon \to 0^+} y(t, \varepsilon) = k$, $a < t < b$. This follows by noting that

$$\int_a^b f(t, k)\,dt = \lim_{\varepsilon \to 0^+} \int_a^b f(t, y(t, \varepsilon))\,dt = \lim_{\varepsilon \to 0^+} \{\varepsilon \ln |y'(b, \varepsilon)| - \varepsilon \ln |y'(a, \varepsilon)|\} = 0$$

since $y'(\tau, \varepsilon) = O(\varepsilon^{-1})$, $\tau = a$ or $b$ (cf. [6]), and as a result, $\lim\limits_{\varepsilon \to 0^+} \varepsilon \ln |y'(\tau, \varepsilon)| = $ $= \lim\limits_{\varepsilon \to 0^+} \varepsilon |\ln \varepsilon| = 0$.

A complete discussion can be found in [2].

*References*


[1] *F. W. Dorr, S. V. Parter* and *L. F. Shampine:* Application of the Maximum Principle to Singular Perturbation Problems, SIAM Rev. *15* (1973), 44—88.

[2] *F. A. Howes* and *S. V. Parter:* A Model Nonlinear Problem Having a Continuous Locus of Singular Points, Studies in Appl. Math. *58* (1978), 249—262.

[3] *L. K. Jackson:* Subfunctions and Second Order Ordinary Differential Inequalities, Adv. in Math. *2* (1968), 307—363.

[4] *M. Nagumo:* Über die Differentialgleichung $y'' = f(x, y, y')$, Proc. Phys. Math. Soc. Japan *19* (1937), 861—866.

[5] *A. B. Vasil'eva:* Asymptotic Behavior of Solutions to Certain Problems Involving Nonlinear Differential Equations Containing a Small Parameter Multiplying the Highest Derivatives, Russian Math. Surveys *18* (1963), 13—84.

[6] *M. I. Vishik* and *L. A. Liusternik:* Initial Jump for Nonlinear Differential Equations Containing a Small Parameter, Sov. Math. Dokl. *1* (1960), 749—752.



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