

Eduardo Marques de Sá  
On the diagonals of integral matrices

*Czechoslovak Mathematical Journal*, Vol. 30 (1980), No. 2, 207–212

Persistent URL: <http://dml.cz/dmlcz/101672>

## Terms of use:

© Institute of Mathematics AS CR, 1980

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON THE DIAGONALS OF INTEGRAL MATRICES

E. MARQUES DE SÁ, Coimbra

(Received December 22, 1977, in revised form May 30, 1979)

Throughout this note we deal mainly with matrices over a principal ideal domain,  $R$ . For the general concepts and results on matrices over  $R$ , we refer to [6]. The elements of  $R$  will be denoted by greek letters, and the symbols " $\alpha <: \beta$ " and " $\gamma =: \delta$ " will be used to mean that  $\alpha$  divides  $\beta$  and  $\gamma$  is associated  $\delta$ , respectively. Therefore,  $\alpha =: \beta$  iff  $\alpha <: \beta <: \alpha$ . We let  $M$  be a multiplicatively closed subset of  $R$ , which constitutes a complete system of representatives of the classes of associated elements of  $R$ . The *invariant factors* of a matrix and the *gcd* (*greatest common divisor*) of elements of  $R$ , will be taken from  $M$ .

Let  $A$  be an  $n \times m$  matrix of rank  $r$ . Let  $\alpha_1, \dots, \alpha_r$  be the invariant factors of  $A$ , ordered so that  $\alpha_1 <: \dots <: \alpha_r$ . It is well known that  $A$  is equivalent to the  $n \times m$  matrix

$$\begin{bmatrix} \text{diag}(\alpha_1, \dots, \alpha_r), & 0 \\ & 0, & 0 \end{bmatrix}$$

where the 0's, if they exist, are zero blocks of the appropriate sizes. For convenience, we shall eventually extend the sequence  $\alpha_1 <: \dots <: \alpha_r$ , with a finite or an infinite tail of zeros, so that new sequences as  $\alpha_1 <: \dots <: \alpha_n$  or  $\alpha_1 <: \alpha_2 <: \alpha_3 <: \dots$  are obtained, with  $\alpha_i = 0$  for  $i > r$ . The additional zeros are also considered as invariant factors.

As the first invariant factor,  $\alpha_1$ , is the gcd of the elements of  $A$ , the following easy proposition characterizes the diagonals of the matrices equivalent to  $A$ .

**Proposition.** *Let  $A$  be an  $n$ -square matrix over  $R$ ,  $n \geq 2$ . Let  $\delta_1, \dots, \delta_n$  be elements of  $R$ , that are multiple of  $\alpha_1$ . Then,  $A$  is equivalent to a matrix with diagonal  $(\delta_1, \dots, \delta_n)$ .*

*Proof.* Let  $\alpha_1 <: \alpha_2 <: \dots$  be the invariant factors of  $A$ . If  $n = 2$ , it is easily seen that

$$\begin{bmatrix} \delta_1 & \alpha_1 \\ \delta_1(\delta_2/\alpha_1) - \alpha_2 & \delta_2 \end{bmatrix}$$

is equivalent to  $A$ . Therefore, the proposition is proved for  $n = 2$ . To prove the proposition for a general  $n \geq 3$ , we proceed by induction, assuming that it holds for  $n - 1$ . Firstly, we apply the case  $n = 2$  to prove that  $A$  is equivalent to

$$C = \begin{bmatrix} \delta_1 & \alpha_1 \\ \delta_1 - \alpha_2 & \alpha_1 \end{bmatrix} \dot{+} \text{diag}(\alpha_3, \dots, \alpha_n),$$

Then, we take  $(n - 1)$ -square unimodular matrices,  $P$  and  $Q$ , such that the diagonal of  $P \text{diag}(\alpha_1, \alpha_3, \dots, \alpha_n) Q$  is  $(\delta_2, \dots, \delta_n)$ . The matrix  $(1 \dot{+} P) C (1 \dot{+} Q)$  is equivalent to  $A$  and has the prescribed diagonal.  $\square$

Next, we characterize the diagonals of triangular matrices, equivalent to a given square matrix  $A$ .

**Theorem.** *Let  $A$  be an  $n$ -square matrix over  $R$ , with invariant factors  $\alpha_1 < \dots < \alpha_n$ . Let  $\delta_1, \dots, \delta_n$  be  $n$  elements of  $R$ . Then, there exists a triangular matrix, equivalent to  $A$ , with diagonal  $(\delta_1, \dots, \delta_n)$ , if and only if the following relations hold:*

- (1)  $\alpha_1 \dots \alpha_k < \text{gcd} \{ \delta_{i_1} \dots \delta_{i_k} : 1 \leq i_1 < \dots < i_k \leq n \}, \quad k = 1, \dots, n - 1,$
- (2)  $\alpha_1 \dots \alpha_n =: \delta_1 \dots \delta_n.$

To prove our theorem we need the following basic result, that has been proved in [7] and [8], and that can be viewed as a consequence of a theorem in [1].

- (3) *Let  $A$  and  $B$  be matrices over  $R$ , of dimensions  $(n + p) \times (m + q)$  and  $n \times m$ , respectively ( $p, q \geq 0$ ). Let  $\alpha_1 < \alpha_2 < \dots$  and  $\beta_1 < \beta_2 < \dots$  be their respective sequences of invariant factors. Then,  $A$  is equivalent to a matrix having  $B$  as a submatrix, if and only if the following relations hold:*
- (4)  $\alpha_i < \beta_i < \alpha_{i+p+q}, \quad i = 1, 2, \dots$

Remark. R. C. THOMPSON [8] has already observed the striking analogy between the relations (4), and the interlacing inequalities for eigenvalues of hermitian matrices and for singular values of arbitrary complex matrices.

Also remarkable is the resemblance between our Theorem and a result by H. WEYL [9] and A. HORN [4], that can be stated as follows: *Let  $A$  be an  $n$ -square complex matrix, with singular values  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$ . Let  $\delta_1, \dots, \delta_n$  be complex numbers. Then, there exists a triangular matrix, unitarily equivalent to  $A$ , with diagonal  $(\delta_1, \dots, \delta_n)$ , if and only if the following relations hold:*

$$\alpha_1 \dots \alpha_k \geq \sup \{ |\delta_{i_1} \dots \delta_{i_k}| : 1 \leq i_1 < \dots < i_k \leq n \}, \quad k = 1, \dots, n - 1,$$

$$\alpha_1 \dots \alpha_n = |\delta_1 \dots \delta_n|.$$

Though not so impressive, the analogy between our relations (1)–(2) and the so called Hardy-Littlewood-Polya inequalities

$$(5) \quad a_1 \leq \dots \leq a_n,$$

$$(6) \quad a_1 + \dots + a_k \leq \min \{d_{i_1} + \dots + d_{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}, \\ k = 1, \dots, n - 1,$$

$$(7) \quad a_1 + \dots + a_n = d_1 + \dots + d_n,$$

is worth noting as well. These inequalities characterize the diagonals,  $(d_1, \dots, d_n)$ , of the hermitian matrices with prescribed eigenvalues  $a_1 \leq \dots \leq a_n$  [3]. The claimed analogy is easily seen in the case when  $A$  is nonsingular. For, given a prime  $\pi \in R$ , let us define the integers  $a_i$  and  $d_i$  as being the multiplicites of  $\pi$  in the prime factorization of  $\alpha_i$  and  $\delta_i$ , respectively. Then, (5)–(7) follow from  $\alpha_1 < \dots < \alpha_n$ , (1) and (2).

As a matter of fact, we shall need in the sequel the following lemma, involving both the interlacing inequalities for eigenvalues of hermitian matrices and the Hardy-Littlewood-Polya inequalities.

**Lemma.** *Let  $n \geq 2$ ,  $a_1, \dots, a_n$  and  $d_1, \dots, d_n$  be integers such that (5), (6) and (7) are satisfied. Then the following system of inequalities:*

$$(8) \quad x_1 + \dots + x_k \leq \min \{d_{i_1} + \dots + d_{i_k} : 1 \leq i_1 < \dots < i_k \leq n - 1\}, \\ k = 1, \dots, n - 2,$$

$$(9) \quad x_1 + \dots + x_{n-1} = d_1 + \dots + d_{n-1},$$

$$(10) \quad a_1 \leq x_1 \leq a_2 \leq x_2 \leq \dots \leq x_{n-1} \leq a_n,$$

has an integral solution,  $x_1, \dots, x_{n-1}$ .

*Proof.* From (5)–(7), and from [3], it follows that there exists an  $n \times n$  hermitian matrix  $H$ , having  $a_1, \dots, a_n$  as eigenvalues and  $(d_1, \dots, d_n)$  as main diagonal. Applying the Cauchy interlacing inequalities [5], and applying again [3], we deduce that the eigenvalues,  $\hat{x}_1, \dots, \hat{x}_{n-1}$ , of the hermitian matrix obtained from  $H$  by deleting the last row and column, constitute a solution of (8)–(10).

It remains to prove that one of the solutions of (8)–(10), is integral. For, let us rewrite the system (8)–(10) in the form:

$$(11) \quad \begin{bmatrix} -\infty \\ -\infty \\ \vdots \\ -\infty \\ e_{n-1} \end{bmatrix} \leq \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} \leq \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_{n-1} \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} \leq \begin{bmatrix} a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{bmatrix},$$

where  $e_k = \min \{d_{i_1} + \dots + d_{i_k} : 1 \leq i_1 < \dots < i_k \leq n - 1\}$ ,  $k = 1, \dots, n - 1$ .

Using the same notation as in [2, p. 224], (11) is a system of the type:

$$b \leq Ax \leq b', \quad c \leq x \leq c',$$

where  $b, b', c$  and  $c'$  are vectors, whose entries are integers ( $\pm \infty$  allowed), and  $A$  is the triangular matrix of 0's and 1's, appearing in (11). Then, [2, Theorem 2] asserts that every vertex of the polyhedron

$$Q(b, b', c, c') = \{x : b \leq Ax \leq b', \quad c \leq x \leq c'\}$$

has all integral coordinates, if and only if  $A$  has the *unimodular property* (i.e., every minor determinant of  $A$  equals 0, +1 or -1). It is easy to prove that our triangular matrix  $A$  has the unimodular property (apply, for instance, Theorem 5 of [2], with  $V_2 = \emptyset$ ). The proof of the Lemma is now complete.  $\square$

**Proof of the Theorem.** It is a well known fact that the invariant factors of  $A$  can be given by:

$$\alpha_1 \dots \alpha_k = \text{gcd} \{ \text{minors of } A, \text{ of order } k \}, \quad k = 1, \dots, n.$$

If the  $\alpha_k$ 's are the invariant factors of a triangular matrix  $T$ , with diagonal  $(\delta_1, \dots, \delta_n)$ , then conditions (1)–(2) must hold. In fact,  $\delta_{i_1} \dots \delta_{i_k}$  is the determinant of a  $k \times k$  principal submatrix of  $T$ , for  $1 \leq k \leq n$ ,  $1 \leq i_1 < \dots < i_k \leq n$ . Thus, the “only if” part of the Theorem is proved.

The converse will be easier to prove, if we show first that the ordering of the  $\delta$ 's is irrelevant. For, let  $T$  be a lower triangular matrix, with diagonal  $(\delta_1, \dots, \delta_n)$ , and let  $s$  be an integer,  $1 \leq s < n$ . Consider the following block decomposition of  $T$ :

$$T = \begin{bmatrix} T_1 & 0 & 0 \\ F & T_2 & 0 \\ G & H & T_3 \end{bmatrix}, \quad \text{with} \quad T_2 = \begin{bmatrix} \delta_s & 0 \\ \gamma & \delta_{s+1} \end{bmatrix},$$

where the diagonal blocks are square. As

$$T_2' = \begin{bmatrix} \delta_{s+1} & 0 \\ \gamma & \delta_s \end{bmatrix}$$

is equivalent to  $T_2$ , there exist  $2 \times 2$  unimodular matrices,  $U$  and  $V$ , such that  $UT_2V = T_2'$ . Therefore,  $T$  is equivalent to the triangular matrix

$$T' = \begin{bmatrix} T_1 & 0 & 0 \\ UF & T_2' & 0 \\ G & HV & T_3 \end{bmatrix}$$

the diagonal of which is  $(\delta_1, \dots, \delta_{s-1}, \delta_{s+1}, \delta_s, \delta_{s+2}, \dots, \delta_n)$ . As any permutation,  $\sigma$ , is the product of transpositions, then  $T$  equivalent to a triangular matrix with diagonal  $(\delta_{\sigma(1)}, \dots, \delta_{\sigma(n)})$ .



...,  $x_{n-1}(\pi)$ , such that

$$(15) \quad x_1(\pi) + \dots + x_k(\pi) \leq \min \{d_{i_1}(\pi) + \dots + d_{i_k}(\pi) : 1 \leq i_1 < \dots < i_k \leq n-1\},$$

$$k = 1, \dots, n-2,$$

$$(16) \quad x_1(\pi) + \dots + x_{n-1}(\pi) = d_1(\pi) + \dots + d_{n-1}(\pi),$$

$$(17) \quad a_1(\pi) \leq x_1(\pi) \leq a_2(\pi) \leq \dots \leq x_{n-1}(\pi) \leq a_n(\pi).$$

If we denote the powers  $\pi^{a_i(\pi)}$ ,  $\pi^{d_i(\pi)}$  and  $\pi^{x_j(\pi)}$ , respectively by  $\alpha_i(\pi)$ ,  $\delta_i(\pi)$  and  $\chi_j(\pi)$ , for  $1 \leq i \leq n$ ,  $1 \leq j < n$ , then (15)–(17) can be written as

$$(18) \quad \chi_1(\pi) \dots \chi_k(\pi) <: \gcd \{ \delta_{i_1}(\pi) \dots \delta_{i_k}(\pi) : 1 \leq i_1 < \dots < i_k \leq n-1 \},$$

$$k = 1, \dots, n-2,$$

$$(19) \quad \chi_1(\pi) \dots \chi_{n-1}(\pi) = \delta_1(\pi) \dots \delta_{n-1}(\pi),$$

$$(20) \quad \alpha_1(\pi) <: \chi_1(\pi) <: \alpha_2(\pi) <: \dots <: \chi_{n-1}(\pi) <: \alpha_n(\pi).$$

If we define  $\chi_i$  by  $\chi_i = \prod_{\pi \in \mathcal{P}} \chi_i(\pi)$ ,  $i = 1, \dots, n-1$ , it is a simple exercise to prove that (12)–(14) follow from (18)–(20).

**Acknowledgement.** We are indebted to Professores DAVID CARLSON, GRACIANO DE OLIVEIRA and MARIA EMÍLIA MIRANDA, for useful remarks about this note.

#### References

- [1] *David Carlson*: Inequalities for the Degrees of Elementary Divisors of Modules, *Lin. Algebra and Appl.* 5 (1972), 293–298.
- [2] *A. J. Hoffman* and *J. B. Kruskal*: Integral Boundary Points of Convex Polyhedra, *Linear Inequalities and Related Systems*, Princeton, New Jersey, 1956.
- [3] *A. Horn*: Doubly Stochastic Matrices and the Diagonal of a Rotation Matrix, *Amer. J. of Math.* LXXVI (1954), 620–630, Theorem 5.
- [4] *A. Horn*: On the Eigenvalues of a Matrix with Prescribed Singular Values, *Proc. Amer. Math. Soc.* 5 (1954), 4–7.
- [5] *Ky Fan* and *G. Pall*: Imbedding Conditions for Hermitian and Normal Matrices, *Can. J. of Math.* 9 (1957), 298–304.
- [6] *Morris Newman*: *Integral Matrices*, Academic Press, New York, 1972.
- [7] *E. Marques de Sá*: Imbedding Conditions for  $\lambda$ -Matrices, *Lin. Algebra and Appl.* 24 (1979), 33–50.
- [8] *R. C. Thompson*: Interlacing Inequalities for Invariant Factors, *Lin. Algebra and Appl.* 24 (1979), 1–32.
- [9] *H. Weyl*: Inequalities Between the Two Kinds of Eigenvalues of a Linear Transformation, *Proc. Nat. Acad. Sci. U.S.A.* 35 (1949), 401–411.

*Author's address*: Instituto de Matemática, Universidade de Coimbra, Portugal.