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EXTREMAL PROBLEMS FOR FUNCTIONS OF POSITIVE REAL PART WITH A FIXED COEFFICIENT AND APPLICATIONS

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1. INTRODUCTION

Let $\mathbf{B}$ be the class of functions $w(z)$ regular in $\Delta = \{z; |z| < 1\}$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ in Δ. We denote by $\mathbf{P}(A, B)$, $-1 \leqq B < A \leqq 1$, the class of functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ defined by

$$p(z) = \frac{1 + A\,w(z)}{1 + B\,w(z)}, \quad z \in \Delta,$$

for some $w(z) \in \mathbf{B}$. This class, introduced by JANOWSKI [4], is a generalisation of the classical result (see NEHARI [7, p. 169]) that any regular function $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ such that $\operatorname{Re}\{p(z)\} > 0$ in Δ can be written in the form

$$p(z) = \frac{1 + w(z)}{1 - w(z)}, \quad w(z) \in \mathbf{B}.$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots \in \mathbf{P}(A, B)$ and put $\theta = \arg p_1$. Then $p(e^{-i\theta} z) = 1 + |p_1|\, z + \ldots \in \mathbf{P}(A, B)$. Hence there is no loss of generality in limiting our study to functions in $\mathbf{P}(A, B)$ with a non-negative real first coefficient. Also, it is known that $|p_1| \leqq A - B$ (see LIBERA and LIVINGSTON [5]). From these observations, we define the following subclass of $\mathbf{P}(A, B)$:

$$\mathbf{P}_b(A, B) = \{p(z) \in \mathbf{P}(A, B);\ p'(0) = b(A - B),\ 0 \leqq b \leqq 1\}.$$

In this paper, we shall be concerned with the extremal problem

$$\min_{|z|=r<1} \operatorname{Re}\{\alpha\, p(z) + \beta z\, p'(z)/p(z)\}, \quad \alpha \geqq 0, \quad \beta \geqq 0 \tag{1.1}$$

over $\mathbf{P}_b(A, B)$. Two special cases of this problem, namely,

$$\min_{|z|=r<1} \operatorname{Re}\{p(z) + z\, p'(z)/p(z)\} \quad \text{and} \quad \min_{|z|=r<1} \operatorname{Re}\{z\, p'(z)/p(z)\},$$

where $p(z)$ varies in $\mathbf{P}(A, B)$, were considered by Janowski [4]. However, Janowski solved these problems making use of a result due to ROBERTSON which relies on variational techniques, while our approach to (1.1) is classical and based on Dieudonné's lemma (see DUREN [1, p. 25]). The results by Janowski [4] correspond to the cases $\alpha = \beta = b = 1$ and $\alpha = 0$, $\beta = b = 1$, respectively, of the solution to (1.1) (see Theorem 2.1).

For some applications of (1.1), we shall consider two subclasses of univalent functions with fixed second coefficient associated with $\mathbf{P}_b(A, B)$, namely,

$$\mathbf{S}_b^*(A, B) = \{f(z) = z + b(A - B)\,z^2 + \ldots;\ z\,f'(z)/f(z) \in \mathbf{P}_b(A, B),\ z \in \Delta\}\,,$$
$$\mathbf{P}_b'(A, B) = \{f(z) = z + (\tfrac{1}{2}b)\,(A - B)\,z^2 + \ldots;\ f'(z) \in \mathbf{P}_b(A, B),\ z \in \Delta\}\,.$$

By special choices of A, B, these classes reduce to well-known subclasses of univalent functions; for example,

$$\mathbf{S}_b^*(1 - 2\alpha, -1) = \{f(z) = z + 2bz^2 + \ldots;\ \operatorname{Re}\{z\,f'(z)/f(z)\} > \alpha,\ 0 \leqq \alpha < 1,\ z \in \Delta\}\,,$$
$$\mathbf{P}_b'(1 - 2\alpha, -1) = \{f(z) = z + bz + \ldots;\ \operatorname{Re}\{f'(z)\} > \alpha,\ 0 \leqq \alpha < 1,\ z \in \Delta\}\,.$$

We shall investigate how the second coefficient in the series expansion of functions in $\mathbf{S}_b^*(A, B)$ and $\mathbf{P}_b'(A, B)$ affects certain properties such as distortion, covering and convexity of these functions. This type of problems was first studied by GRONWALL [3] on univalent and convex functions. FINKELSTEIN [2] obtained distortion theorems for $\mathbf{S}_b^*(1, -1)$. These results were generalised to $\mathbf{S}_b^*(1 - 2\alpha, -1)$ by TEPPER [8], who also derived the radius of convexity of $\mathbf{S}_b^*(1, -1)$. The radius of convexity of $\mathbf{S}_b^*(1 - 2\alpha, -1)$ was found by MCCARTY [6]. The latter author also obtained corresponding results for $\mathbf{P}_b'(1 - 2\alpha, -1)$. Our results for $\mathbf{S}_b^*(A, B)$ and $\mathbf{P}_b'(A, B)$ will naturally cover all these as special cases.

2. THE FUNCTIONAL $\operatorname{Re}\{\alpha\,p(z) + \beta z\,p'(z)/p(z)\}$, $\alpha \geqq 0$, $\beta \geqq 0$, OVER $\mathbf{P}_b(A, B)$

For $p(z) \in \mathbf{P}_b(A, B)$, we may write

$$p(z) = \frac{1 + A\,w(z)}{1 + B\,w(z)}\,, \quad z \in \Delta\,, \tag{2.1}$$

for some $w(z) \in \mathbf{B}$ so that

$$w(z) = \frac{1 - p(z)}{B\,p(z) - A} = bz + \ldots = z\,\psi(z)\,,$$

where $\psi(z)$ is regular and $|\psi(z)| \leqq 1$ in Δ with $\psi(0) = b$. Now, since $0 \leqq b \leqq 1$, we have

$$\frac{\psi(z) - b}{1 - b\,\psi(z)} \prec z\,, \quad z \in \Delta\,.$$

where $f(z) \prec g(z)$ means "$f(z)$ is subordinate to $g(z)$".

Hence

$$\psi(z) \prec \frac{z+b}{1+bz}, \quad z \in \Delta,$$

which yields

$$(2.2) \qquad \operatorname{Re}\{\psi(z)\} \geqq \frac{b-|z|}{1-b|z|}, \quad |\psi(z)| \leqq \frac{|z|+b}{1+b|z|}, \quad |w(z)| \leqq |z| \frac{|z|+b}{1+b|z|}.$$

We next put $D = (r+b)/(1+br)$, $0 < r < 1$, and define

$$H_r(z) = \frac{1+ADz}{1+BDz}, \quad z \in \Delta;$$

then it is clear that

$$(2.3) \qquad p(z) \prec H_r(z), \quad |z| \leqq r.$$

And so, $p(z)$ maps $|z| \leqq r$ into the disc

$$(2.4) \qquad |p(z) - a_b| \leqq d_b,$$

where

$$(2.5) \qquad a_b = \frac{1-ABC^2}{1-B^2C^2}, \quad d_b = \frac{(A-B)\,C}{1-B^2C^2}, \quad C = r\frac{r+b}{1+br}.$$

It follows immediately from (2.4) and (2.5) that if $p(z) \in \mathbf{P}_b(A, B)$, then on $|z| = r < 1$,

$$(2.6) \qquad \frac{1-AC}{1-BC} \leqq \operatorname{Re}\{p(z)\} \leqq |p(z)| \leqq \frac{1+AC}{1+BC}.$$

The first inequality is sharp for the function

$$p(z) = \frac{1+b(A-1)\,z - Az^2}{1+b(B-1)\,z - Bz^2} \quad \text{at} \quad z = -r$$

while the third inequality is sharp for the function

$$p(z) = \frac{1+b(1+A)\,z + Az^2}{1+b(1+B)\,z + Bz^2} \quad \text{at} \quad z = r.$$

Also, putting $E(b) = a_b - d_b = (1-AC)/(1-BC)$, $F(b) = a_b + d_b = (1+AC)/(1+BC)$, C being as given by (2.5), we have

$$\frac{\mathrm{d}C}{\mathrm{d}b} = \frac{r(1-r^2)}{(1+br)^2} > 0, \quad \frac{\mathrm{d}E}{\mathrm{d}b} = -\frac{A-B}{(1-BC)^2}\cdot\frac{\mathrm{d}C}{\mathrm{d}b} < 0,$$

$$\frac{\mathrm{d}F}{\mathrm{d}b} = \frac{A-B}{(1+BC)^2}\cdot\frac{\mathrm{d}C}{\mathrm{d}b} > 0.$$

Thus for a fixed r in $(0, 1)$,

$$(2.7) \qquad a_b - d_b \geqq a_1 - d_1, \quad a_b + d_b \geqq a_0 + d_0.$$

We now prove

2.1. Theorem. *If* $p(z) \in \mathbf{P}_b(A, B)$, $\alpha \geqq 0$, $\beta \geqq 0$, *then on* $|z| = r < 1$,

$$\operatorname{Re}\left\{\alpha\, p(z) + \beta\, \frac{z\, p'(z)}{p(z)}\right\} \geqq$$

$$\geqq \begin{cases} \beta\, \dfrac{A + B}{A - B} + \dfrac{1}{(A - B)(1 - r^2)} \cdot \\ \qquad \cdot \left[L_1 \cdot \dfrac{1 - BC}{1 - AC} + K_1 \cdot \dfrac{1 - AC}{1 - BC} - 2\beta(1 - ABr^2)\right], & R_1 \leqq R_2', \\ \beta\, \dfrac{A + B}{A - B} + \dfrac{2}{(A - B)(1 - r^2)}\left[(L_1 K_1)^{1/2} - \beta(1 - ABr^2)\right], & R_2' \leqq R_1, \end{cases}$$

where $R_1 = (L_1/K_1)^{1/2}$, $R_2' = (1 - AC)/(1 - BC)$, $L_1 = \beta(1 - A)(1 + Ar^2)$, $K_1 = \alpha(A - B)(1 - r^2) + \beta(1 - B)(1 + Br^2)$, $C = r(r + b)/(1 + br)$. *The result is sharp.*

Proof. From the representation formula (2.1) we may write

$$\alpha\, p(z) + \beta\, \frac{z\, p'(z)}{p(z)} = \alpha\, \frac{1 + A\, w(z)}{1 + B\, w(z)} + \beta\, \frac{(A - B)\, z\, w'(z)}{[1 + A\, w(z)]\,[1 + B\, w(z)]}\,.$$

Applying Dieudonné's lemma to the second term of the right-hand side, we find

$$\operatorname{Re}\left\{\alpha\, p(z) + \beta\, \frac{z\, p'(z)}{p(z)}\right\} \geqq \beta\, \frac{A + B}{A - B} + \tag{2.8}$$

$$+ \frac{1}{A - B} \cdot \operatorname{Re}\left\{[\alpha(A - B) - \beta B]\, p(z) - \frac{\beta A}{p(z)}\right\} - \beta\, \frac{r^2 |B\, p(z) - A|^2 - |1 - p(z)|^2}{(A - B)(1 - r^2)\, |p(z)|}\,.$$

In view of (2.4), we put $p(z) = a_b + u + iv$, $|p(z)| = R$, then

$$\begin{aligned} & r^2 |B\, p(z) - A|^2 - |1 - p(z)|^2 = \\ &= -(1 - B^2 r^2)\, R^2 + 2(1 - ABr^2)(a_b + u) - (1 - A^2 r^2) = \\ &= -(1 - B^2 r^2)\, R^2 + 2a_1(1 - B^2 r^2)(a_b + u) - (1 - B^2 r^2)(a_1^2 - d_1^2)\,. \end{aligned}$$

Thus, denoting the right-hand side of (2.8) by $S(u, v)$, we get

$$S(u, v) = \beta\, \frac{A + B}{A - B} + \frac{1}{A - B}\left\{[\alpha(A - B) - \beta B](a_b + u) - \frac{\beta A(a_b + u)}{R^2} + \right.$$

$$\left. + \beta\, \frac{1 - B^2 r^2}{1 - r^2}\left[R - 2a_1\, \frac{a_b + u}{R} + \frac{a_1^2 - d_1^2}{R}\right]\right\} =$$

$$= \beta \frac{A+B}{A-B} + \frac{1}{A-B}\left\{\left[\alpha(A-B) - \beta B - \frac{\beta A}{R^2}\right](a_b + u) + \right.$$

$$\left. + \beta \frac{1-B^2r^2}{1-r^2} \cdot \frac{1}{R}\left[(a_b + u - a_1)^2 + v^2 - d_1^2\right]\right\}.$$

This gives

$$\frac{\partial S}{\partial v} = \frac{\beta}{A-B} \cdot \frac{v}{R^4}\, T(u, v) \tag{2.9}$$

where

$$T(u, v) = 2A(a_b + u) + \frac{1-B^2r^2}{1-r^2}\{R^3 - R[a_1^2 - 2(a_b + u)\, a_1 - d_1^2]\} =$$

$$= 2(a_b + u)\left(A + \frac{1-B^2r^2}{1-r^2} \cdot a_1 R\right) + \frac{1-B^2r^2}{1-r^2}\left[R^3 - R(a_1^2 - d_1^2)\right].$$

Since $R \geqq a_b - d_b \geqq a_1 - d_1$ as seen from (2.7), it follows that

$$A + \frac{1-B^2r^2}{1-r^2} \cdot a_1 R \geqq A + (a_1 - d_1)^2 = \tag{2.10}$$

$$= \frac{(1+B)(1-Ar)^2 + (A-B)(1-ABr^2)}{(1-Br)^2} > 0.$$

Consequently,

$$T(u, v) \geqq 2(a_1 - d_1)\left(A + \frac{1-B^2r^2}{1-r^2} \cdot a_1 R\right) + \frac{1-B^2r^2}{1-r^2}\left[R^3 - R(a_1^2 - d_1^2)\right].$$

Denote the right-hand side by $G(R)$, then

$$\frac{\mathrm{d}G}{\mathrm{d}R} = \frac{1-B^2r^2}{1-r^2}\left[(a_1 - d_1)^2 + 3R^2\right] > 0.$$

Thus, by (2.10)

$$G(R) \geqq G(a_1 - d_1) = 2(a_1 - d_1)\left[A + \frac{1-B^2r^2}{1-r^2}(a_1 - d_1)^2\right] > 0.$$

Hence $T(u, v) > 0$, and in view of (2.9), we see that minimum of $S(u, v)$ on the disc $|p(z) - a_b| \leqq d_b$ is attained when $v = 0$ and $u \in [-d_b, d_b]$. Setting $v = 0$, we get

$$S(u, 0) = \beta \frac{A+B}{A-B} + \frac{1}{A-B}\left\{\beta \frac{(1-A)(1+Ar^2)}{1-r^2} \cdot \frac{1}{a_b + u} + \right.$$

$$\left. + \frac{\alpha(A-B)(1-r^2) + \beta(1-B)(1+Br^2)}{1-r^2}(a_b + u) - 2\beta \frac{1-ABr^2}{1-r^2}\right\}$$

which yields

$$\frac{dS(u, 0)}{du} = \frac{1}{(A - B)(1 - r^2)} \left[- \frac{L_1}{(a_b + u)^2} + K_1 \right].$$

It is clear that the absolute minimum of $S(u, 0)$ occurs at the point $u_0 = (L_1/K_1)^{1/2} - a_b$ if u_0 lies in $[-d_b, d_b]$, its value being

$$S(u_0, 0) = \beta \frac{A + B}{A - B} + \frac{2}{(A - B)(1 - r^2)} [(L_1 K_1)^{1/2} - \beta(1 - ABr^2)] .$$

Now, from the conditions $-1 \leqq B < A \leqq 1$, $\alpha \geqq 0$, $\beta \geqq 0$, $r < 1$, it is clear that

$$(a_b + u_0)^2 \leqq \frac{(1 - A)(1 + Ar^2)}{(1 - B)(1 + Br^2)} < \frac{1 + Ar^2}{1 + Br^2} .$$

Thus, together with (2.7), we find

$$(a_b + u_0)^2 < \frac{1 + Ar^2}{1 + Br^2} = a_0 + d_0 \leqq a_b + d_b \leqq (a_b + d_b)^2 .$$

Thus $u_0 < d_b$. However, it is not necessary that $u_0 > -d_b$. For the case $u_0 \leqq -d_b$, that is, if $R_1 \leqq R_2'$, the absolute minimum of $S(u, 0)$ occurs at the end-point $u = -d_b$, the value of which is

$$S(-d_b, 0) = \beta \frac{A + B}{A - B} + \frac{1}{(A - B)(1 - r^2)} \cdot$$

$$\cdot \left[L_1 \cdot \frac{1 - BC}{1 - AC} + K_1 \cdot \frac{1 - AC}{1 - BC} - 2\beta(1 - ABr^2) \right].$$

The result is sharp for the function

$$p(z) = \frac{1 + b(A - 1) z - Az^2}{1 + b(B - 1) z - Bz^2}$$

at the point $z = -r$ for $R_1 \leqq R_2'$ and at the point $z = re^{i\theta}$ for $R_2' \leqq R_1$, where θ is determined from the equation

$$\operatorname{Re} \left\{ \frac{1 + b(A - 1) re^{i\theta} - Ar^2 e^{2i\theta}}{1 + b(B - 1) re^{i\theta} - Br^2 e^{2i\theta}} \right\} = R_1 .$$

3. TWO SUBCLASSES OF UNIVALENT FUNCTIONS WITH FIXED SECOND COEFFICIENT

We first establish certain distortion properties for the class $S_b^*(A, B)$. These refine several results obtained previously by Janowski [4] on the class $S^*(A, B)$.

3.1. Theorem. *Let* $f(z) \in S_b^*(A, B)$; *then on* $|z| = r < 1$,

$$r\,G(r) \leqq |f(z)| \leqq r\,H(r)$$

$$\frac{1 + b(1 - A)\,r - Ar^2}{1 + b(1 - B)\,r - Br^2} \cdot G(r) \leqq |f'(z)| \leqq \frac{1 + b(1 + A)\,r + Ar^2}{1 + b(1 + B)\,r + Br^2} \cdot H(r)$$

where

$$H(r) = \begin{cases} \exp\{H_1(r; A, B)\}, \ \text{for } B < 0 \text{ or } \{B > 0 \text{ and } b^2 \geqq 4B/(1 + B)^2\}, \\ \exp\{H_2(r; A, B)\}, \ \text{for } B > 0 \text{ and } b^2 \leqq 4B/(1 + B)^2, \\ \exp\left\{A\left[\dfrac{r}{b} + \left(1 - \dfrac{1}{b^2}\right)\log(1 + br)\right]\right\}, \ \text{for } B = 0 \text{ and } b \neq 0, \\ \exp\{\tfrac{1}{2}Ar^2\}, \ \text{for } B = 0 \text{ and } b = 0; \end{cases}$$

$$G(r) = \begin{cases} \exp\{H_1(r; -A, -B)\}, \ \text{for } B > 0 \text{ or } \{B < 0 \text{ and } b^2 \geqq -4B/(1 - B)^2\}, \\ \exp\{H_2(r; -A, -B)\}, \ \text{for } B < 0 \text{ and } b^2 \leqq -4B/(1 - B)^2, \\ \exp\left\{-A\left[\dfrac{r}{b} + \left(1 - \dfrac{1}{b^2}\right)\log(1 + br)\right]\right\}, \ \text{for } B = 0 \text{ and } b \neq 0, \\ \exp\{-\tfrac{1}{2}Ar^2\}, \ \text{for } B = 0 \text{ and } b = 0; \end{cases}$$

$$H_1(r; A, B) = \frac{A - B}{2B}\log(1 + b(1 + B)\,r + Br^2) +$$

$$+ \frac{(A - B)(1 - B)\,b}{4B^2 r\sqrt{-c_1}}\log\left|\frac{b(1 + B) + 2Br(1 + \sqrt{-c_1})}{b(1 + B) + 2Br(1 - \sqrt{-c_1})} \cdot \frac{b(1 + B) - 2Br\sqrt{-c_1}}{b(1 + B) + 2Br\sqrt{-c_1}}\right|,$$

$$H_2(r; A, B) = \frac{A - B}{2B}\log(1 + b(1 + B)\,r + Br^2) -$$

$$- \frac{(A - B)(1 - B)\,b}{2B^2 r\sqrt{c_1}}\left[\tan^{-1}\left(\frac{2Br + b(1 + B)}{2Br\sqrt{c_1}}\right) - \tan^{-1}\left(\frac{b(1 + B)}{2Br\sqrt{c_1}}\right)\right],$$

$$c_1 = \frac{1}{Br^2} - \left[\frac{b(1 + B)}{2Br}\right]^2.$$

Proof. The structural formula for the class $S_b^*(A, B)$ is

$$f(z) = z\exp\int_0^z \frac{p(\xi) - 1}{\xi}\,d\xi, \quad p(z) \in P_b(A, B).$$

Hence

$$\left|\frac{f(z)}{z}\right| = \exp \operatorname{Re}\left\{\int_0^z \frac{p(\xi) - 1}{\xi} \mathrm{d}\xi\right\}.$$

Substituting ξ by zt in the integral we get

$$\left|\frac{f(z)}{z}\right| = \exp \int_0^1 \operatorname{Re}\left\{\frac{p(zt) - 1}{t}\right\} \mathrm{d}t . \tag{3.1}$$

An application of (2.6) yields, on $|zt| = rt$,

$$\operatorname{Re}\left\{\frac{p(zt) - 1}{t}\right\} \geqq -(A - B)\frac{br + r^2t}{1 + b(1 - B)\,rt - Br^2t^2}.$$

Replacing this bound into (3.1) and carrying out the integration will give the lower bound for $|f(z)|$. The upper bound may be obtained similarly. From the definition of $\mathbf{S}_b^*(A, B)$ we have

$$|f'(z)| = \left|\frac{f(z)}{z}\right| |p(z)|, \quad p(z) \in \mathbf{P}_b(A, B), \quad z \in \Delta . \tag{3.2}$$

Hence making use of the bounds derived above for $|f(z)|$ together with inequalities (2.6), we obtain the corresponding bounds for $|f'(z)|$.

The lower bounds for $|f(z)|$ and $|f'(z)|$ are sharp for the function

$$f(z) = z \exp \int_0^z \frac{(A - B)(b - \xi)}{1 + b(B - 1)\,\xi - B\xi^2} \mathrm{d}\xi ,$$

while their upper bounds are attained for the function

$$f(z) = z \exp \int_0^z \frac{(A - B)(b + \xi)}{1 + b(1 + B)\,\xi + B\xi^2} \mathrm{d}\xi .$$

3.2. Remark. For an application of the above theorem, let us consider the function $g(z) = 1/z + b_1z + b_2z^2 + \ldots$ which maps the unit disc onto a domain whose complement is starlike with respect to the origin. Then the function $f(z)$ defined by $f(z) = 1/g(z)$, $z \in \Delta$, is starlike in Δ and has the series expansion

$$f(z) = z + a_3z^3 + a_4z^4 + \ldots$$

Hence Theorem 3.1 with $A = 1$, $B = -1$, $b = 0$ gives

$$\frac{1}{r} - r \leqq |g(z)| = \frac{1}{|f(z)|} \leqq \frac{1}{r} + r , \quad |z| = r .$$

Equalities occur for the function $g(z) = 1/z + \varepsilon z$, $|\varepsilon| = 1$.

3.3. Theorem. *The radius of convexity of* $\mathbf{S}_b^*(A, B)$ *is given by the smallest root in* $(0, 1]$ *of*

(i) $$A^2r^4 + b(2A^2 - 3A + B)\,r^3 + [b^2(1 - A)^2 - 4A + 2B]\,r^2 + b(2 + B - 3A)\,r + 1 = 0\,, \quad \textit{for} \quad R_1 \leqq R_2'\,,$$

(ii) $$(4A^2 - 5A + B)\,r^4 - 2(2A^2 - 3A + 2 - B)\,r^2 + 4 - 5A + B = 0\,, \quad \textit{for} \quad R_2' \leqq R_1\,,$$

where R_1, R_2' *are as given in Theorem* 2.1 *with* $\alpha = \beta = 1$.

Proof. For $f(z) \in \mathbf{S}_b^*(A, B)$, we may write

$$1 + \frac{z\,f''(z)}{f'(z)} = p(z) + \frac{z\,p'(z)}{p(z)}\,,$$

for some $p(z) \in \mathbf{P}_b(A, B)$. Thus an application of Theorem 2.1 with $\alpha = \beta = 1$ yields immediately the equations giving the radius of convexity of $\mathbf{S}_b^*(A, B)$. The result is sharp for the function $f_0(z)$ determined from $z\,f_0'(z)/f_0(z) = p(z)$, where $p(z)$ is extremal for Theorem 2.1.

Theorem 3 of McCarty [6] corresponds to the case $A = 1 - 2\alpha$, $B = -1$. We note that the two bounds in Theorem 2.1 are attained by the same function at two different points. Thus the function $f_0(z)$ defined above serves as an extremal function for both cases of Theorem 3.3. The second extremal function given by McCarty [6, Theorem 3], in fact, does not belong to the class.

In [5], Libera and Livingston found the radius of convexity for functions $f(z)$ satisfying

$$\left|\frac{z\,f'(z)}{f(z)} - \alpha\right| < \alpha\,, \quad z \in \Delta$$

for $\alpha \geqq 1$. The complete result which includes the range $\frac{1}{2} < \alpha < 1$ may be obtained by putting $A = 1$, $B = 1/\alpha - 1$, $b = 1$ in Theorem 3.3 above.

We next consider the class $\mathbf{P}_b'(A, B)$.

3.4. Theorem. *Let* $f(z) \in \mathbf{P}_b'(A, B)$; *then on* $|z| = r < 1$,

$$\frac{1 + b(1 - A)\,r - Ar^2}{1 + b(1 - B)\,r - Br^2} \leqq \operatorname{Re}\{f'(z)\} \leqq |f'(z)| \leqq \frac{1 + b(1 + A)\,r + Ar^2}{1 + b(1 + B)\,r + Br^2}\,;$$

$$|f(z)| \leqq \begin{cases} G_1(r; A, B), \ \textit{for}\ B < 0 \ \textit{or}\ \{B > 0 \ \textit{and}\ b^2 \geqq 4B/(1 + B)^2\}\,, \\ G_2(r; A, B), \ \textit{for}\ B > 0 \ \textit{and}\ b^2 \leqq 4B/(1 + B)^2\,, \\ \dfrac{Ar^2}{2b} + \left(1 + A - \dfrac{A}{b^2}\right) r + \dfrac{A(1 - b^2)}{b^3} \log(1 + br), \ \textit{for}\ B = 0,\ b \neq 0\,, \\ r + Ar^3/3, \ \textit{for}\ B = 0,\ b = 0\,; \end{cases}$$

$$
|f(z)| \geqq \begin{cases} G_1(r; -A, -B), \; for \; B > 0 \; or \; \{B < 0 \; and \; b^2 \geqq -4B/(1-B)^2\}, \\ G_2(r; -A, -B), \; for \; B < 0 \; and \; b^2 \leqq -4B/(1-B)^2, \\ -\dfrac{Ar^2}{2b} + \left(1 - A + \dfrac{A}{b^2}\right) r - \dfrac{A(1-b^2)}{b^3} \log(1+br), \; for \; B = 0, \; b \neq 0, \\ r - Ar^3/3, \; for \; B = 0, \; b = 0; \end{cases}
$$

where

$$
G_1(r; A, B) = \frac{Ar}{B} - \frac{b(A-B)}{2B^2} \log(1 + b(1+B)r + Br^2) + \frac{A-B}{2B^2} \cdot
$$

$$
\cdot \left[1 - \frac{b^2(1+B)}{2B}\right] \frac{1}{\sqrt{-c_2}} \log \left| \frac{2Br + b(1+B) + 2B\sqrt{-c_2}}{2Br + b(1+B) - 2B\sqrt{-c_2}} \cdot \frac{b(1+B) - 2B\sqrt{-c_2}}{b(1+B) + 2B\sqrt{-c_2}} \right|,
$$

$$
G_2(r; A, B) = \frac{Ar}{B} - \frac{b(A-B)}{2B^2} \log(1 + b(1+B)r + Br^2) - \frac{A-B}{B^2} \cdot
$$

$$
\cdot \left[1 - \frac{b^2(1+B)}{2B}\right] \frac{1}{\sqrt{c_2}} \left[\tan^{-1}\left(\frac{2Br + b(1+B)}{2B\sqrt{c_2}}\right) - \tan^{-1}\frac{b(1+B)}{2B\sqrt{c_2}}\right],
$$

$$
c_2 = \frac{1}{B} - \left[\frac{b(1+B)}{2B}\right]^2.
$$

Proof. Since $f'(z) \in \mathbf{P}_b(A, B)$, the bounds for $\operatorname{Re}\{f'(z)\}$ and $|f'(z)|$ follow immediately from (2.6). The bounds for $|f(z)|$ are derived from the fact that

$$
f(z) = \int_0^z f'(\xi)\, \mathrm{d}\xi = \int_0^{|z|} f'(te^{i\theta})\, e^{i\theta}\, \mathrm{d}t.
$$

Thus, on $|z| = r$,

$$
|f(z)| \leqq \int_0^r |f'(te^{i\theta})|\, \mathrm{d}t \leqq \int_0^r \frac{1 + b(1+A)t + At^2}{1 + b(1+B)t + Bt^2}\, \mathrm{d}t,
$$

$$
|f(z)| \geqq \int_0^r \operatorname{Re}\{f'(te^{i\theta})\}\, \mathrm{d}t \geqq \int_0^r \frac{1 + b(1-A)t - At^2}{1 + b(1-B)t - Bt^2}\, \mathrm{d}t.
$$

Carrying out the integration we get the bounds for $|f(z)|$.

The upper bounds for $|f'(z)|$ and $|f(z)|$ are attained for the function

$$
f(z) = \int_0^z \frac{1 + b(1+A)\xi + A\xi^2}{1 + b(1+B)\xi + B\xi^2}\, \mathrm{d}\xi \quad \text{at} \quad z = r,
$$

while the lower bounds for $\operatorname{Re}\{f'(z)\}$ and $|f(z)|$ are attained for the function

$$
f(z) = \int_0^z \frac{1 + b(A-1)\xi - A\xi^2}{1 + b(B-1)\xi - B\xi^2}\, \mathrm{d}\xi \quad \text{at} \quad z = -r.
$$

For $f(z) \in \mathbf{P}'_b(A, B)$, we have

$$1 + \frac{z f''(z)}{f'(z)} = 1 + \frac{z p'(z)}{p(z)}, \quad z \in \Delta$$

for some $p(z) \in \mathbf{P}_b(A, B)$. Thus an application of Theorem 2.1 with $\alpha = 0$, $\beta = 1$ gives

3.5. Theorem. *The radius of convexity of* $\mathbf{P}'_b(A, B)$ *is given by the smallest root in* $(0, 1]$ *of*

(i) $$ABr^4 - 2bA(1 - B) r^3 + [b^2(1 - A)(1 - B) + B - 3A] r^2 + 2b(1 - A) r + 1 = 0, \quad \text{for} \quad R_1 \leqq R'_2,$$

(ii) $$A(1 - B) r^4 + (1 - A)(1 - B) r^2 - (1 - A) = 0, \quad \text{for} \quad R'_2 \leqq R_1,$$

where R_1, R'_2 *are as given in Theorem* 2.1 *with* $\alpha = 0$, $\beta = 1$.

The result is sharp for the function $f_1(z) = \int_0^z p(\xi)\, d\xi$, where $p(z)$ is extremal for Theorem 2.1.

Putting $A = 1 - 2\alpha$, $B = -1$, we obtain Theorem 2 of McCarty [6]. Again here, we remark that the function $f_1(z)$ defined above is extremal for both cases of Theorem 3.5. The second extremal function given by McCarty [6, Theorem 2], in fact, does not belong to the class.

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