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Bohdan Zelinka

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# SUBARBORIANS

BOHDAN ZELINKA, Liberec

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Let a finite tree $T$ be given. We shall define the subarboric function $\sigma$ as a mapping of the vertex set $V(T)$ of $T$ into the the set of all positive integers such that for $x \in \in V(T)$ the value $\sigma(x)$ denotes the number of subtrees of $T$ which contain the vertex $x$. (A graph consisting only of one vertex is also considered a tree.)

If we need to specify to which tree this function is related, we write $\sigma(x, T)$ instead of $\sigma(x)$.

The vertex $s$ of $T$ in which $\sigma$ attains its maximum is called a *subarborian of T.*

First we shall prove a general theorem which can be applied not only to the subarboric function, but also to other functions on a tree.

**Theorem 1.** *Let $T$ be a finite tree, let $f$ be a mapping of the vertex set $V(T)$ of $T$ into the set of all real numbers. Suppose that for any three vertices $u, v, w$ of $T$ such that $v$ is adjacent to both $u$ and $w$ we have*

$$f(v) > \min (f(u), f(w)) .$$

*Then the number of vertices of $T$ in which $f$ attains its maximum is equal to one or to two. If there are two such vertices, they are joined by an edge. The minimum of $f$ is attained only in terminal vertices of $T$.*

Proof. Let $a$ be a vertex of $T$ in which $f$ attains its maximum. Let $u, v$ be two vertices of $T$ joined by an edge and both distinct from $a$. Then either $u$ lies between $a$ and $v$, or $v$ lies between $a$ and $u$. Without loss of generality suppose that $u$ lies between $a$ and $v$. We shall prove that $f(u) > f(v)$. We shall proceed by induction with respect to the distance $d(a, u)$ between $a$ and $u$; evidently $d(a, v) = d(a, u) + 1$. Let $d(a, u) = 1$. Then $u$ is adjacent to $a$ and $v$ and

$$f(u) > \min (f(a), f(v)) = f(v) ,$$

because $f(a)$ is the maximum of $f$ on the whole $V(T)$. Now let $k \geqq 2$; suppose that the assertion holds for $d(a, u) = k - 1$. Let $d(a, u) = k$. By $w$ denote the vertex of the path between $a$ and $u$ adjacent to $u$. Then $d(a, w) = k - 1$ and by the induction assumption $f(w) > f(u)$. We have

$$f(u) > \min(f(w), f(v)) .$$

This minimum cannot be equal to $f(w)$, because $f(w) > f(u)$. It is equal to $f(v)$ and $f(u) > f(v)$.

As $f(u) \leqq f(a)$, we have $f(v) < f(a)$ for each $v$ such that $d(a, v) \geqq 2$. Therefore $f(x) = f(a)$ can hold only in the case $d(a, x) \leqq 1$, i.e. for $x = a$ or for $x$ adjacent to $a$. As $a$ is an arbitrary vertex in which $f$ attains its maximum, we see that all such vertices induce a clique in $T$. As $T$ is a tree, such a clique has at most two vertices and thus the maximum of $f$ is attained in at most two vertices; if these vertices are two, they are joined by an edge.

From the inequality in the assumption of this theorem it is immediately seen that the minimum of $f$ cannot be attained in a vertex $v$ of $T$ which is adjacent to two distinct vertices $u$ and $w$. As $T$ is finite, this minimum must be attained somewhere and therefore it is attained in some terminal vertices of $T$.

This theorem can be dualized.

**Theorem 1'.** *Let $T$ be a finite tree, let $g$ be a mapping of the vertex set $V(T)$ of $T$ into the set of all real numbers. Suppose that for any three vertices $u, v, w$ of $T$ such that $v$ is adjacent to both $u$ and $w$ we have*

$$g(v) < \max(g(u), g(w)) .$$

*Then the number of vertices of $T$ in which $g$ attains its minimum is equal to one or two. If there are two such vertices, they are joined by an edge. The maximum of $g$ is attained only in terminal vertices of $T$.*

Now we shall prove some lemmas.

**Lemma 1.** *Let $T$ be a finite tree, let $x$ be its terminal vertex, let $y$ be the vertex of $T$ adjacent to $x$. Let $T'$ be the tree obtained from $T$ by deleting the vertex $x$ and the edge $xy$. Then*

$$\sigma(x, T) = \sigma(y, T') + 1 .$$

Proof. There exists exactly one subtree of $T$ containing $x$ and not containing $y$, namely the one-vertex tree consisting of $x$. Each other subtree of $T$ containing $x$ is obtained from a subtree of $T'$ containing $y$ by adding the vertex $x$ and the edge $xy$. On the other hand, each subtree of $T'$ containing $y$ can be obtained from a subtree of $T$ containing $x$ and $y$ by deleting the vertex $x$ and the edge $xy$. There is a one-to-one correspondence between subtrees of $T'$ containing $y$ and subtrees of $T$ containing $x$ and $y$, thus the numbers of such trees are equal and the assertion is proved.

**Lemma 2.** *Let $T$ be a finite tree, let $x$ be a vertex of $T$. Let $T_1$, $T_2$ be two subtrees of $T$ such that their intersection consists of the unique vertex $x$ and their union is $T$. Then*

$$\sigma(x, T) = \sigma(x, T_1)\, \sigma(x, T_2)\,.$$

Proof. Each subtree of $T$ containing $x$ can be uniquely decomposed into two trees containing $x$, one being a subtree of $T_1$, the other being a subtree of $T_2$ (some of them may consist only of the vertex $x$). Therefore there is a one-to-one correspondence between the subtrees of $T$ containing $x$ and the pairs of trees containing $x$, one being a subtree of $T_1$, the other being a subtree of $T_2$. This implies the assertion.

Now we can prove the main theorem.

**Theorem 2.** *Every finite tree has either exactly one, or exactly two subarborians. If it has two subarborians, then they are joined by an edge. The minimum of the subarboric function is attained only in terminal vertices of the tree.*

Proof. According to Theorem 1 it is sufficient to prove

$$\sigma(v) > \min\,(\sigma(u), \sigma(w))$$

for each triple of vertices $u, v, w$ such that $v$ is adjacent to $u$ and $w$. Let $T_1$ be the subtree of $T$ whose vertex set consists of the vertices $u, v$ and all vertices which are separated from $v$ by $u$. Let $T_2$ be the subtree of $T$ whose vertex set consists of $v$ and all vertices of $T$ not belonging to $T_1$. Let $T_1'$ (or $T_2'$) be the subtree of $T$ obtained from $T_1$ (or $T_2$) by deleting (or adding, respectively) the vertex $v$ and the edge $uv$. By Lemma 2 we have

$$\sigma(v, T) = \sigma(v, T_1)\, \sigma(v, T_2)\,,$$
$$\sigma(u, T) = \sigma(u, T_1')\, \sigma(u, T_2')\,.$$

Lemma 1 yields

$$\sigma(v, T_1) = \sigma(u, T_1') + 1\,,$$
$$\sigma(v, T_2) = \sigma(u, T_2') - 1\,.$$

Combining this, we obtain

$$\sigma(v, T) = \sigma(u, T_1')\, \sigma(u, T_2') - \sigma(u, T_1') + \sigma(u, T_2') - 1 =$$
$$= \sigma(u, T) - \sigma(u, T_1') + \sigma(u, T_2') - 1\,.$$

If $\sigma(v, T) > \sigma(u, T)$, the assertion is true. Suppose that $\sigma(v, T) \leqq \sigma(u, T)$; this implies $\sigma(u, T_2') \leqq \sigma(u, T_1') + 1$ and $\sigma(v, T_2) + 1 \leqq \sigma(v, T_1)$. Let $S_1$ be the subtree of $T$ whose vertex set consists of the vertices $v$, $w$ and all vertices which are separated from $v$ by $w$. Let $S_2$ be the subtree of $T$ whose vertex set consists of $v$ and all vertices of $T$ not belonging to $S_1$. Let $S'_1$ (or $S'_2$) be the subtree of $T$ obtained from $S_1$ (or $S_2$) by deleting (or adding, respectively) the vertex $v$ and the edge $vw$. Analogously we obtain

$$\sigma(v, T) = \sigma(v, S_1)\,\sigma(v, S_2)\,,$$
$$\sigma(w, T) = \sigma(w, S'_1)\,\sigma(w, S'_2)\,,$$
$$\sigma(v, S_1) = \sigma(w, S'_1) + 1\,,$$
$$\sigma(v, S_2) = \sigma(w, S'_2) - 1\,,$$
$$\sigma(v, T) = \sigma(w, S'_1)\,\sigma(w, S'_2) - \sigma(w, S'_1) + \sigma(w, S'_2) - 1 =$$
$$= \sigma(w, T) - \sigma(w, S'_1) + \sigma(w, S'_2) - 1\,.$$

Suppose that $\sigma(v, T) \leqq \sigma(w, T)$; this implies $\sigma(v, S_2) + 1 \leqq \sigma(v, S_1)$. But $S_1$ is a subtree of $T_2$ and thus $\sigma(v, S_1) \leqq \sigma(v, T_2)$; analogously $T_1$ is a subtree of $S_2$ and $\sigma(v, T_1) \leqq \sigma(v, S_2)$. We have

$$\sigma(v, T_2) + 1 \leqq \sigma(v, T_1) \leqq \sigma(v, S_2) \leqq \sigma(v, S_1) - 1 \leqq \sigma(v, T_2) - 1$$

which is a contradiction. Therefore if $\sigma(v, T) \leqq \sigma(u, T)$, then $\sigma(v, T) > \sigma(w, T)$ and the assertion is true.

Evidently both the mentioned cases can occur; a snake with an odd number of vertices has exactly one subarborian, a snake with an even number of vertices has exactly two subarborians.

Therefore a subarborian of a tree is in a certain sense analogous to such concepts as centre and median [1] of a tree.

For studying interrelations among these concepts it is advantageous to consider a special type of a tree called comet. Consider a snake consisting of vertices $u_0, u_1, \ldots$ $\ldots, u_p$ and edges $u_iu_{i+1}$ for $i = 0, 1, \ldots, p - 1$ and a star consisting of vertices $u_0, v_1, \ldots, v_q$ and edges $u_0v_j$ for $j = 1, \ldots, q$. Identifying the vertices denoted by $u_0$ in these two trees we obtain a comet $C(p, q)$.

For our further considerations we choose $p$ and $q$ so that both these numbers are even and $p > 2^q$.

The centres of the comet $C(p, q)$ are the vertices $u_{p/2-1}, u_{p/2}$. We recall the definition of a median. On the vertex set $V(T)$ of $T$ a function $m_0$ is given so that for $x \in V(T)$ the value $m_0(x)$ is equal to the sum of distances between $x$ and all other vertices of $T$ divided by the number of vertices of $T$. A median of $T$ is a vertex of $T$ in which $m_0$ attains its minimum. In the comet $C(p, q)$ we have

$$m_0(u_k) = [\tfrac{1}{2}k(k + 1) + \tfrac{1}{2}(p - k)(p - k + 1) + (k + 1)\,q]/(p + q + 1)$$

for $k = 0, 1, \ldots, p$. For the subarboric function on $C(p, q)$ we have

$$\sigma(u_k) = (k + 2^q)(p - k + 1)\,.$$

Both these expressions can be considered as functions of a real variable $k$. By standard means of the differential calculus we can prove that $m_0(u_k)$ attains its minimum for $k = \frac{1}{2}(p - q)$ and $\sigma(u_k)$ attains its maximum for $k = \frac{1}{2}p - 2^{q-1}$. The numbers $\frac{1}{2}(p - q)$ and $\frac{1}{2}p - 2^{q-1}$ are integers between 0 and $p$ and among the vertices $v_1, \ldots, v_q$ no median and no subarborian can occur. Therefore $T$ has exactly one median $u_{(p-q)/2}$ and exactly one subarborian $u_{p/2-2^q}$. The distances between the subarborian and the centres of $C(p, q)$ are $2^{q-1} - 1$ and $2^{q-1}$, the distance between the subarborian and the median of $C(p, q)$ is $2^{q-1} - \frac{1}{2}q$. All of these distances can be arbitrarily large. We have a theorem.

**Theorem 3.** *In a finite tree $T$ the distance between a subarborian of $T$ and a centre of $T$ or between a subarborian of $T$ and a median of $T$ can be arbitrarily large.*

*Reference*

[1] *Zelinka, B.:* Medians and peripherians of trees. Arch. Math. Brno *4* (1968), 87—95.

*Author's address:* Komenského 2, 460 01 Liberec 1, ČSSR (katedra matematiky VŠST).

## NEWS and NOTICES

### SEVENTY YEARS OF PROFESSOR ZDENĚK PÍRKO

RNDr. Zdeněk Pírko, DrSc., Professor of the Faculty of Electrical Engineering, Czech Technical University at Prague, reaches seventy years of age on December 12, 1979.

Professor Z. Pírko was distinguished for his merits by the National Medal for Development. His main field of interest is Kinematic Geometry.

A more detailed biography of Professor Z. Pírko is published in Časopis pro pěstování matematiky *105* (1980), 216—218.

*Editorial Board*