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PERTURBATIONS OF VARIATIONAL INEQUALITIES AND RATE OF CONVERGENCE OF SOLUTIONS

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INTRODUCTION

Let $H$ be a Hilbert space with an inner product $(\cdot, \cdot)$ and with the corresponding norm $\| \cdot \|$. We shall consider two closed convex sets $K_1, K_2$ in $H$ and two (in general nonlinear) operators $A_1, A_2 : H \to H$. We shall study the connection between solutions $u_1, u_2$ of the following two variational inequalities:

(I) $u \in K_n$,

(II) $(A_n u, v - u) \geq (f_n, v - u)$ for all $v \in K_n$

($n = 1, 2$), where $f_1, f_2 \in H$ are given. More precisely, we shall estimate the value $\|u_1 - u_2\|$ in terms of $\|f_1 - f_2\|$, the "distance between the sets $K_1, K_2$" and the "distance between the operators $A_1, A_2$" (see Section 2, Theorem 2.1). Further, we can consider a sequence $\{K_n\}$ of closed convex sets, a sequence $\{A_n\}$ of operators and a sequence $\{f_n\}$ of right-hand sides converging in a certain sense to a closed convex set $K_0$, to an operator $A_0$ and to $f_0 \in H$, respectively. Convergence of the sequence of solutions of the corresponding variational inequalities (I), (II) to a solution of the variational inequality (I), (II) (without an estimate of the rate of convergence) has been proved under various assumptions in a number of papers (see for example U. Mosco [3], [4]). As a consequence of the above mentioned Theorem 2.1, we obtain under certain special assumptions an estimate for the rate of convergence of solutions in terms of the rate of convergence of $K_n, A_n, f_n$ (see Remark 2.6). Concrete examples are given in Section 3.

1. NOTATION, GENERAL REMARKS

If $K$ is a closed convex set in the Hilbert space $H$, then we shall denote by $P_K$ the projection onto $K$, i.e., $P_K u$ for an arbitrary $u \in H$ is the unique element of $K$ satisfying the condition

$$\|u - P_K u\| = \inf_{v \in K} \|u - v\|$$

(see [1]).
Remark 1.1. It is well-known (and easy to see) that $P_Ku$ is the unique element of $K$ satisfying the condition
\[ (u - P_Ku, v - P_Ku) \leq 0 \quad \text{for all} \quad v \in K \]
(see [1]).

Remark 1.2. The projection onto a closed convex set is a Lipschitzian mapping:
\[ \|P_Ku - P_Kv\| \leq \|u - v\| \quad \text{for all} \quad u, v \in H. \]

Remark 1.3. Let $\gamma$ be an arbitrary positive number. Then $u \in H$ is a solution of the variational inequality
\begin{align*}
& (I) \quad u \in K, \\
& (II) \quad (Au, v - u) \geq (f, v - u) \quad \text{for all} \quad v \in K
\end{align*}
if and only if
\[ u = P_K(u - \gamma(Au - f)) \]
(see [1]). Indeed, it follows from Remark 1.1 that (1.2) is equivalent to (I) and
\[ (II') \quad (u - \gamma(Au - f) - u, v - u) \leq 0 \quad \text{for all} \quad v \in K, \]
which is equivalent to (II).

Lemma 1.1. (see [1]). Let $A : H \to H$ be an operator satisfying the assumptions
\begin{align*}
& (1.3) \quad (Au - Av, u - v) \geq M\|u - v\|^2 \quad \text{for all} \quad u, v \in H, \\
& (1.4) \quad \|Au - Av\| \leq L\|u - v\| \quad \text{for all} \quad u, v \in H
\end{align*}
where $M \leq L$ are positive constants. Let $f \in H$ and $\gamma \in (0, 2M/L^2)$. Then the operator $T$ defined by
\[ Tu = P_K(u - \gamma(Au - f)) \]
is a contraction. Namely, we have
\[ \|Tu - Tv\| \leq L'\|u - v\| \quad \text{for all} \quad u, v \in H, \]
where $L' = \sqrt{(1 - 2\gamma M + \gamma^2 L^2)} \in (0, 1)$.

Remark 1.4: It follows from Lemma 1.1, Remark 1.3 and from the well-known Banach contraction principle that under the assumptions (1.3), (1.4) the problem (I), (II) has precisely one solution and this solution can be obtained by the usual iterative method as a fixed point of the operator $T$.

For the sake of completeness, we present

Proof of Lemma 1.1. Using (1.1), (1.3), (1.4) we obtain
\[ \|Tu - Tv\|^2 = \|P_K(u - \gamma(Au - f)) - P_K(v - \gamma(Av - f))\|^2 \leq \]
\[ \leq \|u - \gamma(Au - f - v + \gamma(Av - f))\|^2 = \]

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\[ = (u - v, u - v) - 2\gamma(Au - Av, u - v) + \gamma^2(Av - Au, Av - Au) \leq \]
\[ \leq (1 - 2\gamma M + \gamma^2 L^2) \|u - v\|^2. \]

It is \(1 - 2\gamma M + \gamma^2 L^2 \in (0, 1)\) for \(\gamma \in (0, 2M/L^2)\).

2. PERTURBATION OF THE VARIATIONAL INEQUALITY.
RATE OF CONVERGENCE OF THE APPROXIMATIVE SOLUTION

In this section, we shall establish an estimate of the norm of the difference of solutions \(u_1, u_2\) of the problems (I), (II), \(i = 1, 2\). To this end, let us first define the expressions which characterize the “distance” between two closed convex sets and between two operators.

Let \(K_1, K_2\) be closed convex nonempty sets in \(H\). For each \(r > 0\) such that \(\{x \in K_i; \|x\| \leq r\} \neq \emptyset\) \((i = 1, 2)\), we define

\[ S(r; K_1, K_2) = \sup_{\|v\| \leq r} \inf_{\|u\| \leq r} \|u - v\|, \]

\[ \sigma(r; K_1, K_2) = \max (S(r; K_1, K_2), S(r; K_2, K_1)). \]

For each \(r > 0\), we set

\[ \varrho(r; K_1, K_2) = \sup_{u \in H} \inf_{\|u\| \leq r} \|P_{K_1}u - P_{K_2}u\|. \]

(If no misunderstanding can occur we shall not specify the convex sets writing briefly \(\sigma(r; K_1, K_2) = \sigma(r)\) e.t.c..)

Remark 2.1. The expression \(\sigma(r)\) is the so-called local gap (or opening) of the sets \(K_1, K_2\) (see [4]). Given a sequence of convex sets \(\{K_n\}_{n=1}^{\infty}\) we can define the convergence \(K_n \to K\) by means of the conditions

\[ (CK) \quad \lim_{n \to \infty} \varrho(r; K, K_n) = 0 \quad \forall r > 0 \]

or

\[ (CK') \quad \exists r_0 \geq 0 \quad \forall r > r_0 \lim_{n \to \infty} \sigma(r; K, K_n) = 0 \]

which are equivalent (see Remark 2.2 and Lemma 2.1). The condition \((CK')\) ensures that \(K_n\) tend to \(K\) in the following sense:

\( (M1) \) to each \(u \in K\) there exist \(u_n \in K_n, n = 1, 2, \ldots, \) such that \(u_n \to u\);

\( (M2) \) if \(u_n \in K_{l_n}\) where \(l_n\) is an increasing sequence of indices and \(u_n \to u\), then \(u \in K\).

The conditions \((M1), (M2)\) were used by U. Mosco [3] in the proof of convergence of the corresponding solutions (without estimates for the rate of convergence).

*) By \(\to\) and \(\rightharpoonup\) we denote the strong convergence and the weak convergence in \(H\), respectively.
Remark 2.2. We shall establish an estimate of \( \| u_1 - u_2 \| \) in terms of the expression \( \varrho \). However, it is usually difficult to calculate this expression directly, while it is often possible to evaluate the expression \( \sigma \) (cf. also Section 3). The following lemma describes the relation (in general nonlinear) between the expressions \( \varrho, \sigma \) and hence between the conditions (CK), (CK'):

**Lemma 2.1.** Let \( K_1, K_2 \) be closed convex nonempty sets in \( H \), and let us denote \( d_i = \text{dist} (\theta, K_i) \) \((i = 1, 2), d = \max (d_1, d_2).) \) Then

(E) \[ \sigma(r) \leq \varrho(r) \leq \sqrt{(8r + 4d) \sigma(r + d) + \sigma^2(r + d)} \]

for each \( r > d \).

**Proof.** (i) If \( v \in K_1 \), then \( P_{K_i} v = v \) and therefore

\[ \inf_{u \in K_2} \| u - v \| = \| P_{K_i} v - v \| = \| P_{K_i} v - P_{K_2} v \|. \]

Thus we have

\[ S(r; K_1, K_2) = \sup_{v \in K_1} \inf_{u \in K_2} \| u - v \| = \sup_{\| v \| \leq r} \| P_{K_2} v - P_{K_1} v \| \leq \sup_{\| v \| \leq r} \| P_{K_2} v - P_{K_1} v \| = \varrho(r) \]

for an arbitrary \( r > d \); analogously for \( S(r; K_2, K_1) \) and the first inequality of (E) is proved.

(ii) Secondly, let \( u \in H \) be an arbitrary point, \( \| u \| \leq r \) and let us denote \( u_1 = P_{K_1} u, u_2 = P_{K_2} u \). We have

(2.1) \[ \| u_i \| \leq \| P_{K_i} (\theta) \| + \| P_{K_i} (u) - P_{K_i} (\theta) \| \leq d + r \], \( i = 1, 2 \)

in virtue of Remark 1.2. This together with the definition of \( \sigma \) implies that

(2.2) \[ \text{dist} (u_2, K_1) \leq \sigma(r + d). \]

It follows from Remark 1.1 that the set \( K_1 \) lies in the half-space \( H_1 = \{ w; (u - u_1, w - u_1) \leq 0 \} \) and (2.2) yields that

\[ u_2 \in H_2 = H_1 + \frac{u - u_1}{\| u - u_1 \|} \sigma, \]

where we write \( \sigma \) instead of \( \sigma(r + d) \) (see Fig. 2.1). It follows from the definition of \( \sigma \) that \( B(\varrho, \sigma) \cap K_2 \neq \emptyset \), where \( B(z, k) \) denotes the closed ball with the center \( z \) and with the radius \( k \). Thus \( u_2 \in B(u, R + \sigma) \), where \( R = \| u - u_1 \| \). Hence we have

\[ \varrho(r) \leq \sup \{ \| w - u_1 \|; w \in B(u, R + \sigma) \cap H_2 \} \overset{\text{def}}{=} q. \]

Easy calculation by methods of the plane geometry yields \( q = \sqrt{(4R \sigma + \sigma^2)} \). We have \( R = \| u - u_1 \| \leq \| u \| + \| u_1 \| \leq 2r + d \) and this implies (E).

\[ \star \) By \( \theta \) we denote the origin in \( H \).
Remark 2.3. It is easy to see from the proof of Lemma 2.1 that the following more precise estimate holds for each \( r > 0 \):

\[
\varrho(r) \leq \sqrt{(8r + 4d_2) \sigma(r + d_1) + \sigma^2(r + d_1)}.
\]

Particularly, if one of the sets \( K_i \) contains the origin, then \( \varrho(r) \) is estimated in terms of \( \sigma(r) \) instead of \( \sigma(r + d) \).

Remark 2.4. Let us discuss the case of a sequence \( \{K_n\}_{n=1}^{\infty} \). It is easy to see that if \((CK')\) is valid, then \( \text{dist}(K_i, \emptyset) \leq D \) for \( i = 1, 2, \ldots \) and for some \( D \). Thus, for each \( r > 0 \) we have \( \varrho(r, K, K_n) \leq 2(r + D) < \infty \) and hence there exists \( C(r) \) such that

\[
\varrho(r; K, K_n) \leq C(r) \left\{ \sigma(r + D; K, K_n) \right\}^\alpha, \quad \alpha = \frac{1}{2}.
\]
A simple example illustrated by Fig. 2.2a, b shows that this estimate is not true with \( \alpha > \frac{1}{2} \).

Now, let us consider operators \( A_1, A_2 : H \to H \). The following assumptions will be used:

(M) \( (A_n u - A_n v, u - v) \geq M \| u - v \|^2 \) for all \( u, v \in H \), \( n = 1, 2 \), where \( M > 0 \) (monotonicity);

(B) \( B(r) = \sup_{n=1,2, \| u \| \leq r} \| A_n u \| \) is a finite number for each \( r > 0 \) (boundedness);

(L) \( \| A_n u - A_n v \| \leq L \| u - v \| \) for all \( u, v \in H \), \( n = 1, 2 \), where \( L > 0 \) (Lipschitz property).

For each \( r > 0 \), let us denote

\[
a(r) = a(r; A_1, A_2) = \sup_{\| u \| \leq r} \| A_1 u - A_2 u \| .
\]

Remark 2.5. If \( \{ A_n \} \) is a sequence of operators, then the following convergence condition can be considered:

(CA) \( \lim_{n \to \infty} a(r; A, A_n) = 0 \) for each \( r > 0 \).

This condition is stronger than the assumptions about the convergence of operators studied by U. Mosco [3].

Theorem 2.1. Let \( K_1, K_2 \) be closed convex sets in \( H \) and let \( A_1, A_2 : H \to H \) be operators satisfying the conditions (M), (B), (L). Let us suppose that \( f_1, f_2 \in H \). Let us denote by \( u_n, n = 1, 2 \) the unique solutions of \( (I_n), (II_n) \). Let us choose \( \gamma \in (0, 2M/|L|^2) \). Then

\[
(2.3) \quad \| u_n \| \leq U, \quad n = 1, 2 ;
\]

\[
(2.4) \quad \| u_1 - u_2 \| \leq \frac{1}{1 - L'} \left[ \| f_1 \| + \| f_2 \| \right] + \gamma \left( \| f_1 \| + \| f_2 \| \right) + \gamma a(U),
\]

where

\[
L' = \sqrt{(1 - 2\gamma M + \gamma^2 L^2)} \in (0, 1),
\]

\[
U = \frac{1}{M} \left[ F + B(d) \right] + d,
\]

\[
F = \max_{n=1,2} (\| f_1 \|, \| f_2 \|), \quad d = \max_{n=1,2} \left( \text{dist} (K_n, \Theta) \right).
\]

Proof. Choose \( v_n \in K_n \) such that \( \| v_n \| \leq d, n = 1, 2 \). The conditions (M), (II), (B) imply that

\[
M \| u_n - v_n \|^2 \leq (A_n u_n - A_n v_n, u_n - v_n) \leq (f_n - A_n v_n, u_n - v_n) \leq \left[ F + B(d) \right] \| u_n - v_n \|
\]

*) The existence and unicity of the solution of (I), (II) is well-known under more general assumptions (for example, see [2]). In our special case it follows directly from Remark 1.4.

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which yields (2.3). With respect to Remark 1.3, we have

\[(2.5) \quad \|u_1 - u_2\| \leq \|P_{K_1}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_1}(u_2 + \gamma(f_2 - A_2u_2))\| \leq \\
\leq \|P_{K_1}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_1}(u_1 + \gamma(f_1 - A_1u_1))\| + \\
+ \|P_{K_1}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_1}(u_2 + \gamma(f_1 - A_1u_2))\| + \\
+ \|P_{K_1}(u_2 + \gamma(f_1 - A_1u_2)) - P_{K_1}(u_2 + \gamma(f_2 - A_2u_2))\|.
\]

Using (B) and (2.3) we obtain

\[(2.6) \quad \|u_n + \gamma(f_n - A_nu_n)\| \leq U + \gamma F + \gamma B(U), \quad n = 1, 2
\]

and therefore

\[(2.7) \quad \|P_{K_1}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_1}(u_1 + \gamma(f_1 - A_1u_1))\| \leq \\
\leq \varrho(U + \gamma F + \gamma B(U)).
\]

Further, Lemma 1.1 implies that

\[(2.8) \quad \|P_{K_1}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_1}(u_2 + \gamma(f_1 - A_1u_2))\| \leq L\|u_1 - u_2\|.
\]

Remark 1.2 implies that

\[(2.9) \quad \|P_{K_1}(u_2 + \gamma(f_1 - A_1u_2)) - P_{K_1}(u_2 + \gamma(f_2 - A_2u_2))\| \leq \\
\leq \gamma\|f_1 - f_2\| + a(U).
\]

Putting (2.7) - (2.9) into (2.5) we obtain (2.4).

Remark 2.6. Let us consider closed convex sets \(K, K_n\) in \(H (n = 1, 2, \ldots)\) satisfying the condition (CK). Further, let \(A, A_n : H \to H (n = 1, 2, \ldots)\) be operators satisfying the assumptions (M), (L) (with some positive \(M, L\) independent of \(n\)), (CA) and

\[(\tilde{B}) \quad \tilde{B}(r) = \sup_{\|u\| \leq r} \|A_nu\| \text{ is a finite number for each } r > 0.
\]

Suppose that \(f, f_n \in H, f_n \to f\). Denote by \(u\) and \(u_n\) the unique solutions of the problems (I), (II) and (I\(_n\)), (II\(_n\)), respectively.

Theorem 2.1 ensures that \(u_n \to u\) and it gives an estimate of the rate of this convergence. If we set \(\varrho_a(r) = \varrho(r; K, K_n), a_a(r) = a(r; A, A_n)\), then

\[\|u - u_n\| \leq \frac{1}{1 - L'} \left[\varrho_a(U + \gamma \tilde{B}(U) + \gamma \tilde{F}) + \gamma \|f - f_n\| + \gamma a_a(U)\right],
\]

where \(\gamma \in (0, 2M/L^2)\) is arbitrary,

\[L' = 1 - 2\gamma M + \gamma^2 L^2,
\]

\[\tilde{F} = \sup_{n = 1, 2, \ldots} \|f_n\|, \quad d = \sup_{n = 1, 2, \ldots} \text{dist}(K_n, \theta),
\]

\[U = \frac{1}{M} \left[\tilde{F} + \tilde{B}(d)\right] + d.
\]
Further,

\[ \|u_n\| \leq U . \]

Let us remark that the convergence of solutions without an estimate of its rate is proved in [3] in a more general situation.

3. EXAMPLES

In this section, we shall explain two easy applications of Theorem 2.1. For the sake of simplicity, we shall choose the simplest fixed operator \( A ( = A_1 = A_2 ) \) in Example 3.1 and give the estimate of the difference between the solutions in terms of the distance between the sets \( K_1, K_2 \) only. On the other hand, a simple fixed set \( K ( = K_1 = K_2 ) \) will be considered in Example 3.2, where the estimate of the difference between the solutions in terms of the distance between the operators \( A_1, A_2 \) will be given. It will be clear that both examples can be generalized and combined.

In the whole section, \( \Omega \) is a given domain in \( \mathbb{R}^N \) with a lipschitzian boundary.

Example 3.1. Denote \( H = W_2^3(\Omega) \) (the well-known Sobolev space). Let \( \varphi_1, \varphi_2, \psi_1, \psi_2 \in H \) be given functions satisfying the conditions

\[
\begin{align*}
(3.1) & \quad \psi_1 - \varphi_1 \geq \delta , \\
(3.2) & \quad \|\varphi_2 - \varphi_1\| \leq \varepsilon , \quad \|\psi_2 - \psi_1\| \leq \varepsilon , \\
(3.3) & \quad \varphi_n \leq 0 \leq \psi_n , \quad n = 1, 2 ,
\end{align*}
\]

where \( \varepsilon, \delta \) are constants such that

\[
(3.4) \quad 0 < \varepsilon \leq \frac{\delta}{4} .
\]

(We write \( \nu \leq u \) for the functions \( \nu, u \in H \) if and only \( \nu(x) \leq u(x) \) for almost all \( x \in \Omega \) etc..) The assumption (3.3) is not necessary and it is considered for the sake of simplicity only. This assumption ensures that \( d = 0 \) in Theorem 2.1 and that (E) holds for all \( r > 0 \) in Lemma 2.1. Therefore the estimate of \( \|u_1 - u_2\| \) will be simpler in this case.

Let us consider convex closed sets

\[
(3.5) \quad K_n = \{ u \in H ; \varphi_n \leq u \leq \psi_n \} .
\]

\((n = 1, 2)\) and an operator \( A : H \to H \) defined by

\[
(3.6) \quad (Au, v) = \int_\Omega \left( \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + uv \right) dx \quad \text{for all } u, v \in H .
\]
We shall show that if \( u_n \) is the solution of the problem \((I_n), (II_n)\) \((n = 1, 2)\) with \( K_n \) from (3.5), \( A_1 = A_2 = A \) from (3.6) and with some \( f_1 = f_2 = f \in H \), then
\[
\|u_1 - u_2\| \leq \left[ 24\|f\| \frac{4\varepsilon}{\delta} (3\|f\| + \frac{1}{2}\|\varphi_1 + \psi_1\|) + \right.
\]
\[
\left. + \frac{16\varepsilon^2}{\delta^2} (3\|f\| + \frac{1}{2}\|\varphi_1 + \psi_1\|)^2 \right]^{1/2}.
\]

It is clear the assumptions \((M), (B), (L)\) are fulfilled with
\[
(3.7) \quad M = L = 1, \quad B(r) = r
\]
and we can choose
\[
(3.8) \quad \gamma = 1, \quad L' = 0
\]
in Theorem 2.1. The assumption (3.3) implies
\[
(3.9) \quad d = 0, \quad U = \|f\|.
\]

Now, we want to estimate \( \sigma(r; K_1, K_2) \). Denote \( \xi = \frac{1}{2}(\psi_1 + \varphi_1) \). Let \( u \in K_1 \) be an arbitrary function such that \( \|u\| \leq r \). Set
\[
u_k = k(u - \xi) + \xi = ku + (1 - k) \xi
\]
for each \( k \in (0, 1) \). It follows from (3.1), (3.5) that
\[
u_k \geq ku + (1 - k) \frac{1}{2}(2\varphi_1 + \delta) \geq \varphi_1 + \frac{1 - k}{2} \delta,
\]
\[
u_k \leq ku + (1 - k) \frac{1}{2}(2\psi_1 - \delta) \leq \psi_1 - \frac{1 - k}{2} \delta.
\]

If we set \( k = 1 - 2\varepsilon/\delta \), we obtain
\[
\varphi_1 + \varepsilon \leq u_k \leq \varphi_1 - \varepsilon
\]
and this together with (3.2), (3.5) implies \( u_k \in K_2 \). Further,
\[
\sup_{\|u\| \leq r} \|u - u_k\| = \sup_{\|u\| \leq r} (1 - k) \|u - \xi\| \leq \frac{2\varepsilon}{\delta} (r + \|\xi\|)
\]
and hence
\[
S(r; K_1, K_2) \leq \frac{2\varepsilon}{\delta} (r + \|\xi\|).
\]

On the other hand, let \( u \in K_2, \|u\| \leq r \). It follows from (3.1), (3.2), (3.4) that
\[
u_k \geq ku + (1 - k) \frac{1}{2}(2\varphi_2 + \delta) \geq
\]
\[
\geq k\varphi_2 + (1 - k) \frac{1}{2}(2\varphi_2 - 2\varepsilon + \delta) \geq \varphi_2 + \frac{1 - k}{4} \delta,
\]
\[ u_k \leq ku + (1 - k) \frac{1}{2}(2\psi_1 - \delta) \leq k\psi_2 + (1 - k) \frac{1}{2}(2\psi_2 + 2\varepsilon - \delta) \leq \psi_2 - \frac{1 - k}{4} \delta. \]

It we set \( k = 1 - 4\varepsilon/\delta \), we obtain

\[ \varphi_2 + \varepsilon \leq u_k \leq \psi_2 - \varepsilon \]

and this together with (3.2), (3.5) implies \( u_k \in K_1 \). Hence we have

\[ \sup_{\|u\| \leq r} \|u - u_k\| = \sup_{\|u\| \leq r} (1 - k) \|u - \xi\| \leq \frac{4\varepsilon}{\delta} (r + \|\xi\|) \]

which yields

\[ S(r; K_2, K_1) \leq \frac{4\varepsilon}{\delta} (r + \|\xi\|). \]

On the whole, we have

(3.10) \[ \sigma(r; K_1, K_2) \leq \frac{4\varepsilon}{\delta} (r + \|\xi\|). \]

Using (E) from Lemma 2.1, (3.9) and (3.10), we obtain

(3.11) \[ g(r; K_1, K_2) \leq \left(8r \frac{4\varepsilon}{\delta} (r + \|\xi\|) + \frac{16\varepsilon^2}{\delta^2} (r + \|\xi\|)^2 \right). \]

Putting (3.7), (3.8), (3.9), (3.11) into (2.4), we obtain the estimate announced above.

Example 3.2. Let us denote by \( H = \hat{W}^{1/2}_0(\Omega) \) the subspace of \( W^1_2(\Omega) \) of functions with zero traces on the boundary of \( \Omega \) and introduce the inner product on \( H \) by

\[ (u, v) = \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx \quad \text{for all} \quad u, v \in H. \]

Let \( g_1, g_2 \) be two continuous functions defined on \( (0, \infty) \) which have the first derivative on \( (0, \infty) \), and satisfy the following conditions:

(3.12) \[ M \leq g_n(t) \leq B \quad \text{for all} \quad t \in (0, \infty), \quad n = 1, 2, \]

(3.13) \[ M \leq \frac{d}{dt} (g_n(t)) = g_n(t) + g'_n(t) \leq L, \quad t \in (0, \infty), \quad n = 1, 2, \]

(3.14) \[ |g_1(t) - g_2(t)| \leq \varepsilon \quad \text{for all} \quad t \in (0, \infty), \]

where \( M, B, L \) are positive constants. We shall consider operators \( A_1, A_2; H \to H \) defined by

(3.15) \[ (A_n u, v) = \int_\Omega g_n(|\text{grad } u|) \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx \]

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and a closed convex set

\[(3.16) \quad K = \{u \in H; u \geq 0\}.\]

We shall show that if \(u_n\) is the solution of the problem \((I_n), (II_n)\) \((n = 1, 2)\) with \(A_n\) from \((3.15)\), \(K_1 = K_2 = K\) from \((3.16)\) and with some \(f_1, f_2 \in H\), then

\[\|u_1 - u_2\| \leq \frac{\gamma}{1 - \sqrt{1 - 2\gamma M + \gamma^2 L^2}} \left( \|f_1 - f_2\| + \frac{\varepsilon}{M} \max (\|f_1\|, \|f_2\|) \right)\]

for an arbitrary \(\gamma \in (0, 2M/L^2)\).

First, we shall show that \(A_1, A_2\) satisfy the assumptions of Theorem 2.1. If \(r = [r_1, \ldots, r_N], s = [s_1, \ldots, s_N] \in \mathbb{R}^N\) are arbitrary, then we can write (omitting the index of \(g\) for a moment)

\[\sum_{i=1}^{N} g(|r_i|) r_i - g(|s_i|) s_i \approx F(1) \approx F(0),\]

where

\[F(t) = \sum_{i=1}^{N} g(|s_i + t(r_i - s_i)|) (s_i + t(r_i - s_i)) r_i - s_i.\]

There exist \(\tau \in (0, 1)\) and \(\theta = s + \tau(r - s)\) such that

\[F(1) - F(0) = g(|\theta|) |r - s|^2 + \frac{g'(|\theta|)}{|\theta|} \sum_{i=1}^{N} (r_i - s_i) \theta_i \sum_{j=1}^{N} (r_j - s_j) \theta_j.\]

If \(g'(|\theta|) \geq 0\), then \(F(1) - F(0) \geq g(|\theta|) |r - s|^2\); if, conversely, \(g'(|\theta|) < 0\), we use the Cauchy inequality which yields

\[F(1) - F(0) \geq g(|\theta|) |r - s|^2 + \frac{g'(|\theta|)}{|\theta|} |r - s|^2 |\theta|^2;\]

hence we have in both cases

\[\sum_{i=1}^{N} [g(|r_i|) r_i - g(|s_i|) s_i] (r_i - s_i) \approx M \sum (r_i - s_i)^2.\]

This implies that

\[(A_n u - A_n v, u - v) \geq M \|u - v\|^2,\]

i.e. the condition \((M)\) is fulfilled.

To obtain the condition \((B)\), we conclude from the relations

\[\|A_n u\| = \sup_{\|v\| \leq 1} |(A_n u, v)| = \sup_{\|v\| \leq 1} \left| \int_{\Omega} g_n(|\text{grad } u|) \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right| \leq \]

\[\leq \sup_{\|v\| \leq 1} B \int_{\Omega} \left| \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right| \leq B \sup_{\|u\| \leq 1} \|u\| \|v\|\]

that \((B)\) is fulfilled with

\[(3.17) \quad B(r) = B \cdot r.\]
Now, if \( r, s, t \in \mathbb{R}^N \), then we have (supposing \( g_n(|r|) - g_n(|s|) \geq 0 \))
\[
\sum_{i=1}^N \left[ g_n(|r_i|) r_i - g_n(|s_i|) s_i \right] t_i =
\]
\[
= \sum_{i=1}^N \left\{ \left[ g_n(|r_i|) - g_n(|s_i|) \right] r_i t_i + g_n(|s_i|) (r_i - s_i) t_i \right\} \leq
\]
\[
\leq \left[ g_n(|r_i|) - g_n(|s_i|) \right] |r| |t| + g_n(|s_i|) |r - s| |t| \leq
\]
\[
\leq \left[ g_n(|r|) - g_n(|s|) \right] |s| |t| \leq L |r - s| |t| ;
\]
analogously as above we obtain
\[
\| A_n u - A_n v \| \leq L \| u - v \| ,
\]
i.e. the condition (L) is fulfilled. Further, it is easy to see that
\[
(3.18) \quad a(r; A_1, A_2) = \sup_{\|u\| = 1} \left\| \sum_{i=1}^N \left[ g_1(|\text{grad } u|) \frac{\partial u}{\partial x_i} - g_2(|\text{grad } u|) \frac{\partial u}{\partial x_i} \right] \frac{\partial w}{\partial x_i} \right\| \leq
\]
\[
\leq \varepsilon \sup_{\|w\| = 1} \int_{\Omega} \left\| \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i} \right\| \, dx = r \cdot \varepsilon .
\]

Obviously, we have \( g(r) = 0 \) for all \( r \) because \( K_1 = K_2 \). The assumptions (M), (B), (L) are fulfilled with the constants \( M, L \) from (3.12), (3.13) and putting (3.17), (3.18), into (2.4) we obtain the estimate mentioned above.

Remark 3.1. Evidently, we could consider sequences of sets \( K_n \), of operators \( A_n \) and right hand sides \( f_n \) (\( n = 1, 2, \ldots \)) converging to \( K, A, f \) and we could give an estimate of the rate of convergence in Examples 3.1, 3.2 (cf. Remark 2.6).

References


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