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# POLYNOMIALLY DETERMINED TOLERANCES

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By a *tolerance* $T$ on an algebra $\mathfrak{A} = (A, F)$ we mean a reflexive and symmetric binary relation on $A$ satisfying the Substitution Property with respect to all operations from $F$, i.e. for each $n$-ary $f \in F$ the validity of $\langle a_i, b_i \rangle \in T$ $(i = 1, \ldots, n)$ implies $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in T$. Denote by $LT(\mathfrak{A})$ the set of all tolerances on $\mathfrak{A}$. Evidently, $LT(\mathfrak{A})$ is an algebraic lattice with respect to the set inclusion (see [2]).

The concept of a polynomially determined congruence was introduced in [5] and [6]. The aim of this paper is to generalize this concept for tolerances and to give examples of such algebras.

**Definition 1.** Let $\mathfrak{A} = (A, F)$ be an algebra and $p(x, y)$ a binary polynomial over $F$. A tolerance $T \in LT(\mathfrak{A})$ is called $(p, e)$-*determined* if there exists an element $e \in A$ such that

$$\langle a, b \rangle \in T \quad \text{if and only if} \quad \langle p(a, b), e \rangle \in T .$$

**Remark.** Since every congruence $\theta$ on $\mathfrak{A}$ is a tolerance on $\mathfrak{A}$, every $(p, e)$-determined congruence is a $(p, e)$-determined tolerance by the definition in [5], p. 65 (for $e = p(f, f)$). Thus, every tolerance on a group $\mathfrak{G}$ is $(p, e)$-determined for $p(x, y) = x \cdot y^{-1}$, $e = x \cdot x^{-1}$, because every tolerance on $\mathfrak{G}$ is a congruence (see [4], [7], [8]) and every congruence on a group is $(p, e)$-determined (see [5]). The next example introduces an algebra with a $(p, e)$-determined tolerance which is not a congruence.

**Example 1.** Let $G = \{a, b, c\}$ and let $\mathfrak{G} = (G, \{\circ\})$ be a groupoid prescribed by the table:

| $\circ$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $b$ | $b$ |
| $c$ | $c$ | $b$ | $a$ |

Let $T = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, b \rangle\}$. Evidently, $T \in LT(\mathfrak{G})$ and $T$ is not a congruence because $\langle a, b \rangle \in T$, $\langle b, c \rangle \in T$ but $\langle a, c \rangle \notin T$. Let $p(x, y) = x \circ y$. Choose $e = a$. Evidently, $\langle x, y \rangle \in T$ implies $p(x, y) = a$ or

$p(x, y) = b$, thus $\langle p(x, y), e\rangle \in T$. If $\langle x, y\rangle \notin T$, then $\{x, y\} = \{a, c\}$ and $p(x, y) = p(a, c) = c$. Hence $\langle p(x, y), e\rangle = \langle c, a\rangle \notin T$. Accordingly, $T$ is a $(p, e)$-determined tolerance on $\mathfrak{G}$.

Let $\mathfrak{A} = (A, F)$ be an algebra and $T \in LT(\mathfrak{A})$. We call $B \subseteq A$, $B \neq \emptyset$, a *block of* $T$ if

(i) $x, y \in B$ implies $\langle x, y\rangle \in T$, i.e. $B \times B \subseteq T$,

(ii) $B$ is a maximal subset of $A$ with respect to (i).

For the properties of relational blocks the reader is referred to [1].

**Proposition.** *Let $\mathfrak{A} = (A, F)$ be an algebra, $p(x, y)$ a binary polynomial over $F$ and $e \in A$. The following conditions are equivalent:*

(1) *$T \in LT(\mathfrak{A})$ is $(p, e)$-determined,*

(2) *$\langle a, b\rangle \in T$ if and only if there exists a block $B$ of $T$ containing $e$ such that $p(x, y) \in B$.*

Proof. The implication (2) $\Rightarrow$ (1) is evident. Prove (1) $\Rightarrow$ (2). If $T$ is $(p, e)$-determined and $\langle a, b\rangle \in T$, then $\langle p(a, b), e\rangle \in T$. Since $T$ is symmetric and reflexive, we have also $\langle e, p(a, b)\rangle \in T$, $\langle e, e\rangle \in T$ and $\langle p(a, b), p(a, b)\rangle \in T$, thus the two-element set $\{e, p(a, b)\}$ satisfies (i). By Zorn's lemma, there exists a block $B$ of $T$ such that $\{e, p(a, b)\} \subseteq B$. Conversely, if $p(a, b) \in B$, where $B$ is a block of $T$ containing $e$ and $T$ is $(p, e)$-determined, then $\langle p(a, b), e\rangle \in T$ implies $\langle a, b\rangle \in T$.

**Definition 2.** Let $\mathbf{A} = (A, F)$ be an algebra, $p(x, y)$ a binary polynomial over $F$ and $\emptyset \neq M \subseteq A$. The set $M$ is said to be $(p, e)$-*admissible* on $\mathfrak{A}$ if there exists a $(p, e)$-determined $T \in LT(\mathfrak{A})$ such that

$$\langle a, b\rangle \in T \quad \text{if and only if} \quad p(a, b) \in M .$$

**Example 2.** Let $G, p, T, e$ be the same as in Example 1. Then $M = \{a, b\}$ is $(p, e)$-admissible.

The following theorem gives a characterization of $(p, e)$-admissible sets.

**Theorem 1.** *Let $\mathbf{A} = (A, F)$ be an algebra, $p(x, y)$ a polynomial over $F$, $e \in A$ and $\emptyset \neq M \subseteq A$. A subset $M$ is $(p, e)$-admissible on $\mathfrak{A}$ if and only if:*

(1) *For each $a \in A$, $p(a, a) \in M$;*

(2) *$p(a, b) \in M$ implies $p(b, a) \in M$;*

(3) *for every $n$-ary $f \in F$, $p(a_i, b_i) \in M$ $(i = 1, \ldots, n)$ implies $p(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in M$;*

(4) *$p(p(a, b), e) \in M$ if and only if $p(a, b) \in M$.*

Proof. Let $M \subseteq A$ satisfy (1), (2), (3) and (4). Define a binary relation $T$ on $A$ such that $\langle a, b\rangle \in T$ if and only if $p(a, b) \in M$. Then $T$ is reflexive by (1) and symmetric by (2). The condition (3) implies the Substitution Property and thus $T \in LT(\mathfrak{A})$. Further, $\langle x, y \rangle \in T$ if and only if $p(x, y) \in M$ which is equivalent to $p(p(x, y), e) \in M$ by (4), i.e. $\langle p(x, y), e \rangle \in T$. Hence $T$ is $(p, e)$-determined which implies that $M$ is $(p, e)$-admissible.

Conversely, let $M$ be $(p, e)$-admissible and let $T \in LT(\mathfrak{A})$ be the corresponding $(p, e)$-determined tolerance with $p(a, b) \in M$ and only if $\langle a, b \rangle \in T$. Thus $M = \{p(a, b); \langle a, b \rangle \in T\}$. Clearly (1), (2), (3) are valid because $T$ is reflexive, symmetric and has the Substitution Property. Further, $p(p(a, b), e) \in M$ is equivalent to $\langle p(a, b), e \rangle \in T$, i.e. $\langle a, b \rangle \in T$ (since $T$ is $(p, e)$-determined), which means $p(a, b) \in M$. Thus (4) holds, too.

**Theorem 2.** *Let $\mathfrak{A} = (A, F)$, $\mathfrak{B} = (B, F')$ be algebras of the same type, $\varphi$ a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, $M$ a $(p, e)$-admissible set on $\mathfrak{A}$ and $\{M_\gamma; \gamma \in \Gamma\}$ a system of $(p, e)$-admissible subsets on $\mathfrak{A}$ for some binary polynomial $p(x, y)$ over $F$ and $e \in A$. Denote by $p^*$ the polynomial over $F'$ corresponding to $p$ in $\varphi$. Then:*

(a) $\bigcap\{M_\gamma; \gamma \in \Gamma\}$ *is a $(p, e)$-admissible set on $\mathfrak{A}$.*

(b) $\varphi(M)$ *is a $(p^*, \varphi(e))$-admissible set on $\mathfrak{B}$.*

Proof. The first statement is clear. Prove (b). Put $e^* = \varphi(e)$, then $e^* \in \varphi(M)$ If $b \in B$, there exists $b' \in A$ such that $b = \varphi(b')$. Since $p(b', b') \in M$, also $p^*(b, b) = p^*(\varphi(b'), \varphi(b')) = \varphi(p(b', b')) \in \varphi(M)$ and thus (1) of Theorem 1 is valid for $\varphi(M)$ and $e^*$. The condition (2) of Theorem 1 is evident and (3) can be proved in the same way as (1). Prove (4). Let $a, b \in B$ and $p^*(p^*(a, b), e^*) \in \varphi(M)$. Then there exist $a', b' \in A$ with $\varphi(a') = a$, $\varphi(b') = b$. Suppose $p^*(a, b) \notin \varphi(M)$. Then $p(a', b') \notin M$, i.e. $p(p(a', b'), e) \notin M$. Since $\varphi$ is a homomorphism, this implies $p^*(p^*(a, b), e^*) \notin \varphi(M)$, which is a contradiction. Thus $p^*(a, b) \in \varphi(M)$. By Theorem 1, $\varphi(M)$ is a $p$-admissible set on $B$.

**Definition 3.** Let $\mathfrak{A} = (A, F)$ be an algebra, $p(x, y)$ a binary polynomial over $F$ and $e \in A$. We say that $\mathfrak{A}$ has *$(p, e)$-determined tolerances* if each $T \in LT(\mathfrak{A})$ is $(p, e)$-determined.

Let $\mathfrak{A} = (A, F)$ be an algebra, $x, y \in A$. Denote $T(x, y) = \bigcap\{T \in LT(\mathfrak{A}); \langle x, y \rangle \in T\}$. Clearly, $T(x, y) \in LT(\mathfrak{A})$ and it is called the *principal tolerance on $\mathfrak{A}$ generated by $\langle x, y \rangle$* (see [3]). It is a generalization of the principal congruence on $\mathfrak{A}$ (see [5]).

We give a characterization of $\mathfrak{A}$ having $(p, e)$-determined tolerances:

**Theorem 3.** *An algebra $\mathfrak{A} = (A, F)$ has $(p, e)$-determined tolerances (for a binary polynomial $p(x, y)$ over $F$ and $e \in A$) if and only if:*

(1) $p(a, a) = e$ *for each* $a \in A$,

(2) $\langle a, b \rangle \in T(p(a, b), e)$ *for each* $a, b \in A$.

Proof. Denote by $\Delta$ the identity relation on $A$. Clearly $\Delta$ is the least element in the lattice $LT(\mathfrak{A})$. If $\mathfrak{A}$ has $(p, e)$-determined tolerances, then also $\Delta$ is $(p, e)$-determined, i.e. $\langle a, a\rangle \in \Delta$ if and only if $\langle p(a, a), e\rangle \in \Delta$. Since $\langle a, a\rangle \in \Delta$ for each $a \in A$, we have $\langle p(a, a), e\rangle \in \Delta$ which means $p(a, a) = e$. Thus (1) is proved. Since $\langle p(a, b), e\rangle \in T(p(a, b), e)$ for each $a, b \in A$, and for each $T \in LT(\mathfrak{A})$ we have $\langle a, b\rangle \in T$ if and only if $\langle p(a, b), e\rangle \in T$, we conclude $\langle a, b\rangle \in T(p(a, b), e)$ and also (2) is proved.

Conversely, let (1), (2) be true and $T \in LT(\mathfrak{A})$. Suppose $\langle a, b\rangle \in T$. By the Substitution Property, also $\langle p(a, b), p(a, a)\rangle \in T$ and, by (1), $\langle p(a, b), e\rangle \in T$. If, conversely, $\langle p(a, b), e\rangle \in T$, then $T(p(a, b), e) \subseteq T$ and, by (2), also $\langle a, b\rangle \in T$. Thus $T$ is $(p, e)$-determined.

**Remark.** Clearly every congruence on a group $\mathfrak{G}$ is a $(p, e)$-determined tolerance for $p(x, y) = x \cdot y^{-1}$, $e = x \cdot x^{-1}$. Since $\mathfrak{G}$ has no tolerance different from a congruence [8], $\mathfrak{G}$ is an example of an algebra with $(p, e)$-determined tolerances. The next example introduces an algebra with $(p, e)$-determined tolerances some of which are not congruences.

**Example 2.** Let $\mathfrak{G} = \{a, b, c, d, e\}$, $F = \{\circ\}$ and let $\mathfrak{G} = (G, F)$ be a groupoid with the table

| $\circ$ | $e$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $e$ | $e$ | $c$ | $b$ | $a$ | $d$ |
| $a$ | $c$ | $e$ | $d$ | $d$ | $c$ |
| $b$ | $b$ | $d$ | $e$ | $d$ | $b$ |
| $c$ | $a$ | $d$ | $d$ | $e$ | $a$ |
| $d$ | $d$ | $c$ | $b$ | $a$ | $e$ |

1°. Prove $T(a, e) = G \times G$. Clearly $\langle a, e\rangle, \langle e, a\rangle \in T(a, e)$. Further

$$\begin{aligned}
\langle c, e\rangle &= \langle a \circ e, e \circ e\rangle \in T(a, e), \quad \text{i.e. also} \quad \langle e, c\rangle \in T(a, e),\\
\langle a, c\rangle &= \langle c \circ e, e \circ a\rangle \in T(a, e),\\
\langle c, d\rangle &= \langle e \circ a, c \circ a\rangle \in T(a, e),\\
\langle b, d\rangle &= \langle e \circ b, c \circ b\rangle \in T(a, e),\\
\langle a, d\rangle &= \langle e \circ c, a \circ c\rangle \in T(a, e),\\
\langle b, a\rangle &= \langle e \circ b, c \circ d\rangle \in T(a, e), \quad \text{hence} \quad \langle a, b\rangle \in T(a, e),\\
\langle c, b\rangle &= \langle a \circ e, b \circ e\rangle \in T(a, e),\\
\langle b, e\rangle &= \langle b \circ e, a \circ a\rangle \in T(a, e),\\
\langle e, d\rangle &= \langle a \circ a, b \circ a\rangle \in T(a, e).
\end{aligned}$$

Since $T(a, e)$ is symmetric and reflexive, we conclude $T(a, e) = G \times G$.

2°. Since

$$\langle a, e\rangle = \langle c \circ e, e \circ e\rangle \quad \text{and} \quad \langle a, e\rangle = \langle d \circ e, e \circ e\rangle,$$

it is also $\langle a, e\rangle \in T(c, e)$, $\langle a, e\rangle \in T(d, e)$ and, by 1°, $T(c, e) = T(d, e) = G \times G$.

3°. Clearly $\langle b, e\rangle$ and $\langle e, b\rangle \in T(b, e)$. Hence

$$\langle c, d\rangle = \langle e \circ a,\ b \circ a\rangle \in T(b, e)\,, \quad \langle d, b\rangle = \langle b \circ c,\ b \circ d\rangle \in T(b, e)\,.$$

4°. Put $p(x, y) = x \circ y$ and let $e$ be an element of $G$. To prove that $\mathfrak{G}$ has $(p, e)$-determined tolerances, it suffices, by Theorem 4, only to prove

$$(*) \qquad \langle x, y\rangle \in T(x \circ y, e) \quad \text{for each} \quad x, y \in G\,,$$

because $x \circ x = e$ is evident.

If $p(x, y) = x \circ y = b$, then either $\{x, y\} = \{b, e\}$ or $\{x, y\} = \{d, b\}$. By 3°,

$$\langle b, e\rangle \in T(b, e)\,, \quad \langle e, b\rangle \in T(b, e)\,,$$
$$\langle d, b\rangle \in T(b, e)\,, \quad \langle b, d\rangle \in T(b, e)\,;$$

thus $(*)$ is true for these $x, y$.

If $\{x, y\} \neq \{b, e\}$ and $\{x, y\} \neq \{b, d\}$, then $p(x, y) \neq b$. In this case 1° or 2° implies $(*)$ trivially. Thus $\mathfrak{G}$ has $(p, e)$-determined tolerances.

5°. Let $T = \Delta \cup \{\langle e, d\rangle, \langle d, e\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$. Then, clearly, $T = T(a, b) = T(b, c) \in LT(\mathfrak{G})$. However, $T$ is not a congruence, because $\langle a, b\rangle$, $\langle b, c\rangle \in T$ but $\langle a, c\rangle \notin T$.

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