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POLYNOMICALLY DETERMINED TOLERANCES

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By a tolerance \( T \) on an algebra \( \mathfrak{A} = (A, F) \) we mean a reflexive and symmetric binary relation on \( A \) satisfying the Substitution Property with respect to all operations from \( F \), i.e. for each \( n \)-ary \( f \in F \) the validity of \( \langle a_{i}, b_{i} \rangle \in T \) \( (i = 1, \ldots, n) \) implies \( \langle f(a_{1}, \ldots, a_{n}), f(b_{1}, \ldots, b_{n}) \rangle \in T \). Denote by \( LT(\mathfrak{A}) \) the set of all tolerances on \( \mathfrak{A} \). Evidently, \( LT(\mathfrak{A}) \) is an algebraic lattice with respect to the set inclusion (see [2]).

The concept of a polynomially determined congruence was introduced in [5] and [6]. The aim of this paper is to generalize this concept for tolerances and to give examples of such algebras.

**Definition 1.** Let \( \mathfrak{A} = (A, F) \) be an algebra and \( p(x, y) \) a binary polynomial over \( F \). A tolerance \( T \in LT(\mathfrak{A}) \) is called \( (p, e) \)-determined if there exists an element \( e \in A \) such that

\[ \langle a, b \rangle \in T \; \text{ if and only if } \; \langle p(a, b), e \rangle \in T. \]

**Remark.** Since every congruence \( \theta \) on \( \mathfrak{A} \) is a tolerance on \( \mathfrak{A} \), every \( (p, e) \)-determined congruence is a \( (p, e) \)-determined tolerance by the definition in [5], p. 65 (for \( e = p(f, f) \)). Thus, every tolerance on a group \( \mathfrak{G} \) is \( (p, e) \)-determined for \( p(x, y) = x \cdot y^{-1} \). Because every tolerance on \( \mathfrak{G} \) is a congruence (see [4], [7], [8]) and every congruence on a group is \( (p, e) \)-determined (see [5]). The next example introduces an algebra with a \( (p, e) \)-determined tolerance which is not a congruence.

**Example 1.** Let \( G = \{a, b, c\} \) and let \( \mathfrak{G} = (G, \circ) \) be a groupoid prescribed by the table:

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
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<tbody>
<tr>
<td>( a )</td>
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</table>

Let \( T = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, b \rangle\} \). Evidently, \( T \in LT(\mathfrak{G}) \) and \( T \) is not a congruence because \( \langle a, b \rangle \in T, \langle b, c \rangle \in T \) but \( \langle a, c \rangle \notin T \). Let \( p(x, y) = x \circ y \). Choose \( e = a \). Evidently, \( \langle x, y \rangle \in T \) implies \( p(x, y) = a \) or
\[ p(x, y) = b, \] thus \( \langle p(x, y), e \rangle \in T. \) If \( \langle x, y \rangle \notin T \), then \( \{x, y\} = \{a, c\} \) and \( p(x, y) = p(a, c) = c. \) Hence \( \langle p(x, y), e \rangle = \langle c, a \rangle \notin T. \) Accordingly, \( T \) is a \( (p, e) \)-determined tolerance on \( \mathcal{G}. \)

Let \( \mathfrak{G} = (A, F) \) be an algebra and \( T \in LT(\mathfrak{G}). \) We call \( B \subseteq A, B \neq \emptyset, \) a block of \( T \) if

(i) \( x, y \in B \) implies \( \langle x, y \rangle \in T, \) i.e. \( B \times B \subseteq T, \)

(ii) \( B \) is a maximal subset of \( A \) with respect to (i).

For the properties of relational blocks the reader is referred to [1].

**Proposition.** Let \( \mathfrak{G} = (A, F) \) be an algebra, \( p(x, y) \) a binary polynomial over \( F \) and \( e \in A. \) The following conditions are equivalent:

1. \( T \in LT(\mathfrak{G}) \) is \( (p, e) \)-determined,
2. \( \langle a, b \rangle \in T \) if and only if there exists a block \( B \) of \( T \) containing \( e \) such that \( p(x, y) \in B. \)

**Proof.** The implication (2) \( \Rightarrow \) (1) is evident. Prove (1) \( \Rightarrow \) (2). If \( T \) is \( (p, e) \)-determined and \( \langle a, b \rangle \in T, \) then \( \langle p(a, b), e \rangle \in T. \) Since \( T \) is symmetric and reflexive, we have also \( \langle e, p(a, b) \rangle \in T, \langle e, e \rangle \in T \) and \( \langle p(a, b), p(a, b) \rangle \in T, \) thus the two-element set \( \{e, p(a, b)\} \) satisfies (i). By Zorn’s lemma, there exists a block \( B \) of \( T \) such that \( \{e, p(a, b)\} \subseteq B. \) Conversely, if \( p(a, b) \in B, \) where \( B \) is a block of \( T \) containing \( e \) and \( T \) is \( (p, e) \)-determined, then \( \langle p(a, b), e \rangle \in T \) implies \( \langle a, b \rangle \in T. \)

**Definition 2.** Let \( A = (A, F) \) be an algebra, \( p(x, y) \) a binary polynomial over \( F \) and \( \emptyset \neq M \subseteq A. \) The set \( M \) is said to be \( (p, e) \)-admissible on \( \mathfrak{G} \) if there exists a \( (p, e) \)-determined \( T \in LT(\mathfrak{G}) \) such that

\( \langle a, b \rangle \in T \) if and only if \( p(a, b) \in M. \)

**Example 2.** Let \( G, p, T, e \) be the same as in Example 1. Then \( M = \{a, b\} \) is \( (p, e) \)-admissible.

The following theorem gives a characterization of \( (p, e) \)-admissible sets.

**Theorem 1.** Let \( A = (A, F) \) be an algebra, \( p(x, y) \) a polynomial over \( F, e \in A \) and \( \emptyset \neq M \subseteq A. \) A subset \( M \) is \( (p, e) \)-admissible on \( \mathfrak{G} \) if and only if:

1. For each \( a \in A, p(a, a) \in M; \)
2. \( p(a, b) \in M \) implies \( p(b, a) \in M; \)
3. for every \( n \)-ary \( f \in F, \) \( p(a_i, b_i) \in M \) \((i = 1, \ldots, n)\) implies \( p(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in M; \)
4. \( p(p(a, b), e) \in M \) if and only if \( p(a, b) \in M. \)

**Proof.** Let \( M \subseteq A \) satisfy (1), (2), (3) and (4). Define a binary relation \( T \) on \( A \) such that \( \langle a, b \rangle \in T \) if and only if \( p(a, b) \in M. \) Then \( T \) is reflexive by (1) and sym-
metric by (2). The condition (3) implies the Substitution Property and thus $T \in L T(\mathfrak{A})$. Further, $\langle x, y \rangle \in T$ if and only if $p(x, y) \in M$ which is equivalent to $T(p(x, y), e) \in M$ by (4), i.e., $\langle p(x, y), e \rangle \in T$. Hence $T$ is $(p, e)$-determined which implies that $M$ is $(p, e)$-admissible.

Conversely, let $M$ be $(p, e)$-admissible and let $T \in L T(\mathfrak{A})$ be the corresponding $(p, e)$-determined tolerance with $p(a, b) \in M$ and only if $\langle a, b \rangle \in T$. Thus $M = \{p(a, b), \langle a, b \rangle \in T\}$. Clearly (1), (2), (3) are valid because $T$ is reflexive, symmetric and has the Substitution Property. Further, $p(p(a, b), e) \in M$ is equivalent to $\langle p(a, b), e \rangle \in T$, i.e. $\langle a, b \rangle \in T$ (since $T$ is $(p, e)$-determined), which means $p(a, b) \in M$. Thus (4) holds, too.

**Theorem 2.** Let $\mathfrak{A} = (A, F)$, $\mathfrak{B} = (B, F')$ be algebras of the same type, $\varphi$ a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, $M$ a $(p, e)$-admissible set on $\mathfrak{A}$ and $\{M_\gamma; \gamma \in \Gamma\}$ a system of $(p, e)$-admissible subsets on $\mathfrak{A}$ for some binary polynomial $p(x, y)$ over $F$ and $e \in A$. Denote by $p^*$ the polynomial over $F'$ corresponding to $p$ in $\varphi$. Then:

(a) $\bigcap \{M_\gamma; \gamma \in \Gamma\}$ is a $(p, e)$-admissible set on $\mathfrak{A}$.

(b) $\varphi(M)$ is a $(p^*, \varphi(e))$-admissible set on $\mathfrak{B}$.

**Proof.** The first statement is clear. Prove (b). Put $e^* = \varphi(e)$, then $e^* \in \varphi(M)$ if $b \in B$, there exists $b' \in A$ such that $b = \varphi(b')$. Since $p(b', b') \in M$, also $p^*(p(b', b')) = \varphi(p(b', b')) \in \varphi(M)$ and thus (1) of Theorem 1 is valid for $\varphi(M)$ and $e^*$. The condition (2) of Theorem 1 is evident and (3) can be proved in the same way as (1). Prove (4). Let $a, b \in B$ and $p^*(p(a, b), e^*) \in \varphi(M)$. Then there exist $a', b' \in A$ with $\varphi(a') = a, \varphi(b') = b$. Suppose $p^*(a, b) \notin \varphi(M)$. Then $p(a', b') \notin M$, i.e. $p(p(a', b'), e) \notin M$. Since $\varphi$ is a homomorphism, this implies $p^*(p(a', b'), e^*) \notin \varphi(M)$, which is a contradiction. Thus $p^*(a, b) \in \varphi(M)$. By Theorem 1, $\varphi(M)$ is a $(p, e)$-admissible set on $B$.

**Definition 3.** Let $\mathfrak{A} = (A, F)$ be an algebra, $p(x, y)$ a binary polynomial over $F$ and $e \in A$. We say that $\mathfrak{A}$ has $(p, e)$-determined tolerances if each $T \in L T(\mathfrak{A})$ is $(p, e)$-determined.

Let $\mathfrak{A} = (A, F)$ be an algebra, $x, y \in A$. Denote $T(x, y) = \bigcap \{T \in L T(\mathfrak{A}); \langle x, y \rangle \in T\}$. Clearly, $T(x, y) \in L T(\mathfrak{A})$ and it is called the principal tolerance on $\mathfrak{A}$ generated by $\langle x, y \rangle$ (see [3]). It is a generalization of the principal congruence on $\mathfrak{A}$ (see [5]).

We give a characterization of $\mathfrak{A}$ having $(p, e)$-determined tolerances:

**Theorem 3.** An algebra $\mathfrak{A} = (A, F)$ has $(p, e)$-determined tolerances (for a binary polynomial $p(x, y)$ over $F$ and $e \in A$) if and only if:

1. $p(a, a) = e$ for each $a \in A$,
2. $\langle a, b \rangle \in T(p(a, b), e)$ for each $a, b \in A$.

472
Proof. Denote by $\Delta$ the identity relation on $A$. Clearly $\Delta$ is the least element in the lattice $LT(\mathcal{U})$. If $\mathcal{U}$ has $(p,e)$-determined tolerances, then also $\Delta$ is $(p,e)$-determined, i.e. $\langle a, a \rangle \in \Delta$ if and only if $\langle p(a, a), e \rangle \in \Delta$. Since $\langle a, a \rangle \in \Delta$ for each $a \in A$, we have $\langle p(a, a), e \rangle \in \Delta$ which means $p(a, a) = e$. Thus (1) is proved. Since $\langle p(a, b), e \rangle \in T(p(a, b), e)$ for each $a, b \in A$, and for each $T \in LT(\mathcal{U})$ we have $\langle a, b \rangle \in T$ if and only if $\langle p(a, b), e \rangle \in T$, we conclude $\langle a, b \rangle \in T(p(a, b), e)$ and also (2) is proved.

Conversely, let (1), (2) be true and $T \in LT(\mathcal{U})$. Suppose $\langle a, b \rangle \in T$. By the Substitution Property, also $\langle p(a, b), p(a, a) \rangle \in T$ and, by (1), $\langle p(a, b), e \rangle \in T$. If, conversely, $\langle p(a, b), e \rangle \in T$, then $T(p(a, b), e) \subseteq T$ and, by (2), also $\langle a, b \rangle \in T$. Thus $T$ is $(p, e)$-determined.

Remark. Clearly every congruence on a groupoid $\mathcal{G}$ is a $(p, e)$-determined tolerance for $p(x, y) = x \cdot y^{-1}$, $e = x \cdot x^{-1}$. Since $\mathcal{G}$ has no tolerance different from a congruence [8], $\mathcal{G}$ is an example of an algebra with $(p, e)$-determined tolerances. The next example introduces an algebra with $(p, e)$-determined tolerances some of which are not congruences.

Example 2. Let $\mathcal{G} = \{a, b, c, d, e\}$, $F = \{\circ\}$ and let $\mathcal{G} = (G, F)$ be a groupoid with the table

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
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<td>e</td>
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<td>b</td>
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</tbody>
</table>

1°. Prove $T(a, e) = G \times G$. Clearly $\langle a, e \rangle$, $\langle e, a \rangle \in T(a, e)$. Further $\langle c, e \rangle = \langle e \circ e, e \circ e \rangle \in T(a, e)$, i.e. also $\langle e, c \rangle \in T(a, e)$, $\langle a, c \rangle = \langle e \circ e, e \circ a \rangle \in T(a, e)$, $\langle c, d \rangle = \langle e \circ a, c \circ a \rangle \in T(a, e)$, $\langle b, d \rangle = \langle e \circ b, c \circ b \rangle \in T(a, e)$, $\langle a, d \rangle = \langle e \circ c, a \circ c \rangle \in T(a, e)$, $\langle b, a \rangle = \langle e \circ b, c \circ d \rangle \in T(a, e)$, hence $\langle a, b \rangle \in T(a, e)$, $\langle c, b \rangle = \langle a \circ e, b \circ e \rangle \in T(a, e)$, $\langle b, e \rangle = \langle b \circ e, a \circ a \rangle \in T(a, e)$, $\langle e, d \rangle = \langle a \circ a, b \circ a \rangle \in T(a, e)$.

Since $T(a, e)$ is symmetric and reflexive, we conclude $T(a, e) = G \times G$.

2°. Since $\langle a, e \rangle = \langle c \circ e, e \circ e \rangle$ and $\langle a, e \rangle = \langle d \circ e, e \circ e \rangle$, it is also $\langle a, e \rangle \in T(c, e)$, $\langle a, e \rangle \in T(d, e)$ and, by 1°, $T(c, e) = T(d, e) = G \times G$. 473
3°. Clearly $\langle b, e \rangle$ and $\langle e, b \rangle \in T(b, e)$. Hence

$$\langle c, d \rangle = \langle e \circ a, b \circ a \rangle \in T(b, e), \quad \langle d, b \rangle = \langle b \circ c, b \circ d \rangle \in T(b, e).$$

4°. Put $p(x, y) = x \circ y$ and let $e$ be an element of $G$. To prove that $\mathfrak{G}$ has $(p, e)$-determined tolerances, it suffices, by Theorem 4, only to prove

$$(*) \quad \langle x, y \rangle \in T(x \circ y, e) \quad \text{for each} \quad x, y \in G,$$

because $x \circ x = e$ is evident.

If $p(x, y) = x \circ y = b$, then either $\{x, y\} = \{b, e\}$ or $\{x, y\} = \{d, b\}$. By 3°,

$$\langle b, e \rangle \in T(b, e), \quad \langle e, b \rangle \in T(b, e),$$

$$\langle d, b \rangle \in T(b, e), \quad \langle b, d \rangle \in T(b, e);$$

thus $(*)$ is true for these $x, y$.

If $\{x, y\} = \{b, e\}$ and $\{x, y\} = \{b, d\}$, then $p(x, y) = b$. In this case 1° or 2° implies $(*)$ trivially. Thus $\mathfrak{G}$ has $(p, e)$-determined tolerances.

5°. Let $T = \Delta \cup \{\langle e, d \rangle, \langle d, e \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, b \rangle\}$. Then, clearly, $T = T(a, b) = T(b, c) \in LT(\mathfrak{G})$. However, $T$ is not a congruence, because $\langle a, b \rangle, \langle b, c \rangle \in T$ but $\langle a, c \rangle \notin T$.

References


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