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Functional separation of inductive limits and representation of presheaves by sections. Part IV: Representation of presheaves by sections

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FUNCTIONAL SEPARATION OF INDUCTIVE LIMITS
AND
REPRESENTATION OF PRESHEAVES BY SECTIONS
PART FOUR:
REPRESENTATION OF PRESHEAVES BY SECTIONS

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INTRODUCTION

In this paper some representation theorems for certain presheaves are proven. They state that in the covering space of the presheaf in question there is a closure such that the set of all continuous sections over any open set, endowed with the topology of pointwise convergence, is precisely the set of those that canonically correspond to the sets of the presheaf, and that the natural maps of the spaces of the presheaf onto the spaces of the sections are homomorphisms. In the final section we find out when there is even a topology with the above mentioned properties. That gives us a representation theorem in terms of topological spaces.

This is the fourth and last part of the paper “Functional Separation of Inductive Limits and Representation of Presheaves by Sections”. The basic definitions and notation were introduced at the beginning of Part One which together with the other foregoing parts is very often referred to. If we refer, say, to 3.2.7 or 0.5, we mean Remark 3.2.7 of the second section of Part Three or Definition 0.5 at the beginning of Part One, respectively.

4. REPRESENTATION OF PRESHEAVES BY SECTIONS

1. PRELIMINARY LEMMAS

4.1.1. Notation. A. The set of all open nonempty subsets of a topological space \( X \) is denoted by \( \mathcal{B}(X) \). If \( x \in X \), we put \( \mathcal{B}(X) x = \{ U \in \mathcal{B}(X) \mid x \in U \} \). Throughout this chapter the inverse inclusion order in \( \mathcal{B}(X) \) is denoted by \( \leq \) (given \( U, V \) open, then
$U \subseteq V \iff V \subset U$. If $\mathcal{X} = (X, t)$ is a uniform (proximal, ...) space, then $\text{cl} \mathcal{X} = (X, \text{cl} t)$, where $\text{cl} t$ is the closure in $X$ generated by $t$.

B. A presheaf over $X$ from a category $\mathcal{R}$ in an inductive family $\mathcal{S} = \{S_U|q_{UV}\}$. From $\mathcal{S}$ over $\langle \mathcal{B}(X) \subseteq \rangle$ (see 0.2). We denote it by $\mathcal{S} = \{S_U|q_{UV}\}$. Thus, $\{S_U|q_{UV}\} \subseteq \mathcal{S}$.

C. Given an inductive category $\mathcal{R}$ (see 0.5) and a presheaf $\mathcal{S} = \{S_U|q_{UV}\}$ over $X$ from $\mathcal{R}$, $x \in X$, then we put $\mathcal{S}_x = \{S_U|q_{UV}\} \langle \mathcal{B}x \subseteq \rangle$. As $\langle \mathcal{B}x \subseteq \rangle$ is right directed, $\mathcal{S}_x$ is a presheaf from $\mathcal{R}$ in the sense of 0.2. Thus there is $\mathcal{S}_x = \lim \mathcal{S}_x$, and the $\mathcal{R}$-object $\mathcal{S}_x$ is called a stalk over $x$. By 0.4, for each $U \in \mathcal{S}$ there is a canonical $\mathcal{R}$-morphism $\xi_{Ux} : S_U \rightarrow \mathcal{S}_x$.

D. Let an i.c. category $\mathcal{L}$ with the union property (see 0.19) and a presheaf $\mathcal{S} = \mathcal{S}_x = \{\mathcal{S}_x|q_{UV}\} \subseteq \mathcal{S}_x$ be given. For every $x \in X$ we have the stalk $\mathcal{S}_x = \lim \mathcal{S}_x$, and there is an $\mathcal{L}$-object $\mathcal{P} = \bigcup \{\mathcal{S}_x| x \in X\}$ (see 0.19), which is called a covering space of $\mathcal{S}$. If $U \subset X$ is open then the section over $U$ in $\mathcal{P}$ is map $r : U \rightarrow \mathcal{P}$ such that $r(x) \in \mathcal{S}_x$ for all $x \in U$. Recall that $\mathcal{P}$ is an object from $\mathcal{L}$, thus $\mathcal{P}$ is also from CLOS or from SEM or from PROX (see 0.10). Regarding $\mathcal{P}$ as an element of that of these to which $\mathcal{P}$ belongs, we have $\mathcal{P} = (P, t)$, where $P = |\mathcal{P}|$ is a set and $t$ is a closure or semiuniformity or proximity, respectively (see 0.9). Thus $r$ is a map of $U$ into the set $P$.

E. If $U \in \mathcal{B}(X, a) \in [\mathcal{S}_x] = I_x$ for all $x \in U$. Setting $\tilde{a}(x) = \xi_{Ux}(a)$ for $x \in U$, we get a section $\tilde{a}$ over $U$. Putting $A_U = \{\tilde{a} | a \in X\}$, $p_U(a) = \tilde{a}$, we get a set $A_U$ of sections corresponding canonically to $X_U$. The map $p_U : X_U \rightarrow A_U$ is onto. It is 1–1 iff the following condition is fulfilled:

COND. If $a, b \in X_U$ so that there is an open cover $\mathcal{V}$ of $U$ with $q_{UV}(a) = q_{UV}(b)$: for all $V \in \mathcal{V}$, then $a = b$.

(The proof is straightforward.)

If $x \in U$, $\tilde{a} \in A_U$, we put $\eta_{Ux}(\tilde{a}) = \xi_{Ux}(a)$, where $\tilde{a} = p_U(a)$. We get a map $\eta_{Ux} : A_U \rightarrow I_x$. Clearly, if $p_U$ is 1–1, we have $\eta_{Ux} = \xi_{Ux} \circ p_U^{-1} : A_U \rightarrow I_x$.

F. Given $\mathcal{S} = \{\mathcal{S}_x|q_{UV}\} \subseteq \mathcal{S}_x$ from $\mathcal{L}$, where $\mathcal{L}$ is one of the categories mentioned in 0.5, and the covering space $\mathcal{P} = \bigcup \{\mathcal{S}_x| x \in X\}$ of $\mathcal{S}$, it is known that $P = |\mathcal{P}| = \bigcup \{I_x = |\mathcal{S}_x| | x \in X\}$. If in every $I_x$ we have a closure (topology, ...) $s_x$, then by $ss_x$ we denote the closure (topology, ...) inductively defined in $P$ by the canonical embeddings $j_x : (I_x, s_x) \rightarrow P$. Further, if $s_x^*$ is the closure (topology, ...) of $\mathcal{S}_x = \lim \mathcal{S}_x$, then $\mathcal{P} = (P, ss_x^*)$ (see 0.19). If $u$ is a closure (topology, ...) in $P$ — for example, if $\mathcal{L} = \text{UNIF}$, then $u$ is a uniformity in $P = \mathcal{P}$, $x \in X$, then we denote by $u_x$ the closure (topology, ...) in $I_x$ projectively defined by the canonical embeddings $j_x : I_x \rightarrow (P, u)$. The following statement holds: The identical map $i : (P, s_{u_x}) \rightarrow (P, u)$ is an $\mathcal{L}$ — morphism (for example, if $u$ is a uniformity, then $i$ is uniformly continuous) and $(s_{u_x})_x = u_x$.

Proof. Look at the commutative diagram.

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Here \( i(i_x, i_x^{-1}) \) is continuous iff the same is true for \( ij_x^2 = j_x^1(j_x^2i_x = j_x^2, j_x^2i_x^{-1} = ij_x^3). \) But the latter holds by the definition of \( u_x(su_x, (su_x)_x) \) and the continuity of \( i). \)

If \( u \) is a closure (topology, ...) in \( P, U \in \mathcal{B}(X) \), then by \( b_U(u) \) and \( \tau_U(u) \) we denote the closure (topology, ...) projectively defined in \( A_U \) and in \( X_U \) by the canonical maps \( \{ \eta_{ux} : A_U \to (I_x, u_x) \mid x \in U \} \) and \( \{ \xi_{ux} : X_U \to (I_x, u_x) \mid x \in U \} \), respectively. If \( t \) is a closure in \( P \) and \( U \in \mathcal{B}(X) \), then \( I(U, t) \) is the set of all continuous sections \( r : U \to (P, t) \) (see 4.1.1D).

**4.1.2. Notation.** Let \( \mathcal{T} = \{ \mathcal{T}_V \mid q_{UV} | X \} \) be from an i.c. category \( \mathcal{Q} \), let \( \mathcal{H} = \mathcal{T} \), \( \mathcal{H}_x = (H_x, h_x^*) = \lim \mathcal{H}_x \), \( \mathcal{R}_x = (H_x, h_x^*) = \lim \mathcal{H}_x \) be the stalks of \( \mathcal{H}_x \), \( \mathcal{H} = \bigcup \{ \mathcal{H}_x \mid x \in X \} = \bigcup \{ \mathcal{R}_x \mid x \in X \} = \bigcup \{ \mathcal{J}_x \mid x \in X \} \), respectively, where \( \mathcal{J}_x = \bigcup \{ I_x, x \in X \}, \mathcal{R}_x = \bigcup \{ H_x, x \in X \} \) (see 4.1.1A, 4.9). Let \( \mathcal{J}_x = (I_x, t_x^*) = \lim \mathcal{J}_x \). \( \mathcal{R}_x = (H_x, h_x^*) = \lim \mathcal{R}_x \). \( \mathcal{J}_x = (I_x, t_x^*) = \lim \mathcal{J}_x \). \( \mathcal{R}_x = (H_x, h_x^*) = \lim \mathcal{R}_x \). \( \mathcal{J}_x = (I_x, t_x^*) = \lim \mathcal{J}_x \). \( \mathcal{R}_x = (H_x, h_x^*) = \lim \mathcal{R}_x \) if \( \mathcal{J}_x \), \( \mathcal{R}_x \in \text{CLOS} \). Given \( U \in \mathcal{B}(X), x \in U \), let \( A_U \) and \( A'_U \) be the sets of the sections over \( U \) in \( \mathcal{J}_x \) and in \( \mathcal{R}_x \), that is, \( \{ \mathcal{J}_x \mid x \in X \} \) and \( \{ \mathcal{R}_x \mid x \in X \} \), respectively. Let \( p_U : X_U \to A_U \), \( p'_U : H_U \to A'_U \), \( \xi_{ux} : X_U \to I_x, \xi'_{ux} : H_U \to H_x \) be the canonical maps (see 4.1.1C, 4.9). As \( \mathcal{H} \) is a hull of \( \mathcal{T} \) (see 2.1.2B) so there is a \( 1 \)-\( 1 \) continuous map \( \mathcal{J}_x \to \mathcal{R}_x \) for every \( x \in X \). If \( t \) and \( h \) are closures in \( \mathcal{J}_x \) and in \( \mathcal{R}_x \), then \( b_U(t) \) and \( b'_U(h) \) denote the closures projectively defined in \( A_U \) and \( A'_U \) by the maps \( \eta_{ux} : A_U \to \mathcal{J}_x \) and \( \eta'_{ux} : A'_U \to \mathcal{R}_x \), respectively (see 1.4.1F). The next commutative diagram shows the situation.

\[
(A_U, b_U(t)), \quad (X_U, \tau_U), \quad (H_U, h_U), \quad (A'_U, b'_U(h))\]

The maps \( p_U, p'_U \) are onto (see 0.5). Further, \( p_U \) and \( p'_U \) are \( 1 \)-\( 1 \) if \( \mathcal{T} \) and \( \mathcal{H} \), respectively, fulfil the condition \( \text{COND} \) from 4.1.1E; \( e_U \) are continuous, open and \( 1 \)-\( 1 \) maps into \( (H_U, h_U) \), hence homeomorphisms of \( (X_U, m\tau_U) \) into \( (H_U, h_U) \) (\( m\tau_U \) is the topological modification of \( \tau_U \) — see 2.1.2B, 0.9), \( \eta_{ux}, \eta'_{ux} \) are continuous; \( e_x \) is \( 1 \)-\( 1 \) for all \( x \in U \).

The following lemma will be useful:
4.1.4. Lemma. A. Let $A$ be a set and for every $a \in A$ let us have a commutative diagram of closure spaces and maps such that $v$ is projectively defined by $\{g_a : Y \to (Z_a, w_a) \mid a \in A\}$. Then $f$ is continuous iff all the $h_a$ are.

\[
\begin{array}{ccc}
(X, u) & \xrightarrow{f} & (Y, v) \\
\downarrow h_a & & \downarrow g_a \\
(Z_a, w_a) & \xleftarrow{h_a} & (Y, v)
\end{array}
\]

B. Let $U \subseteq X$ be open. Suppose $\xi_{Ux}$ in the diagram 4.1.3 are continuous for all $x \in U$. If the closure $t_x$ in $I_x$ coincides with that projectively defined by $e_x : I_x \to (H_x, h_x)$, then $\xi_{Ux}$ in 4.1.3 are continuous for all $x \in U$. Thus, if for some $x \in X$, the closure $h_x$ is coarser than $h_x^*$ (i.e. $\xi_{Ux}^*$ in 4.1.3 is continuous for all $U \in \mathcal{B}(X)$) and the closure $t_x$ is projectively defined by $e_x : I_x \to (H_x, h_x)$, then $\xi_{Ux}$ from 4.1.3 is continuous for all $U \in \mathcal{B}(X)$.

**Proof.** A: $f$ is continuous iff so is $g_a f = h_a$ for all $a \in A$. But $h_a$ is continuous for such $a$. B: The continuity of $\xi_{Ux}^* e_U = e_x^* \xi_{Ux}$ yields that of $\xi_{Ux}$.

4.1.5. Lemma. With the same symbols as in 4.1.2, let us consider the diagram 4.1.3 for $U \in \mathcal{B}(X)$, $x \in U$.

A. If $a, b \in X_U$, then $p_U(a) = p_U(b)$ iff $p_U e_U(a) = p_U e_U(b)$. Thus $p_U$ is 1–1 if $p_U^*$ is (therefore if COND from 4.1.1E holds for $\mathcal{H}$, then $p_U$ are 1–1).

B. Assume that $\xi_{Ux}, \xi_{Ux}^*, e_x$ in 4.1.3 are continuous for all $x \in U$ (which, by 4.1.4B, holds if, for all $x \in U$, $h_x$ is coarser than $h_x^*$ and $t_x$ is projectively defined by $e_x : I_x \to (H_x, h_x)$). Then the maps $p_U$, $p_U^*$ in 4.1.3 are continuous.

C. If $e_U$ is a homeomorphism into $(H_U, h_U)$ (which holds if $(X_U, \tau_U)$ is topological — see 2.1.2B) together with $p_U^*$, then so is $p_U$ (recall that $e_U$ maps $X_U$ into $H_U$ while $p_U^*$ maps $H_U$ onto $A_U'$). Especially, setting $t = s t_x^*, h = s h_x^*$ (see 4.1.1E), we get from 4.1.3

\[
\begin{array}{ccc}
(A_U, b_U (s t_x^*)) & \xrightarrow{p_U} & (X_U, \tau_U) \\
\downarrow \eta_{Ux} & & \downarrow \xi_{Ux} \\
(l_x, t_x^*) & \xrightarrow{e_x} & (H_x, h_x^*) \\
\end{array}
\]

where all the maps are continuous for all $U \in \mathcal{B}(X), x \in U$.

**Proof.** If $a, b \in X_U$, $p_U(a) = p_U(b)$, then $\xi_{Ux}(a) = \xi_{Ux}(b)$ for all $x \in U$. Thus $\xi_{Ux} e_U(a) = e_x \xi_{Ux}(a) = e_x \xi_{Ux}(b) = \xi_{Ux}^* e_U(b)$ for all $x \in U$, hence $p_U e_U(a) = e_x \xi_{Ux}(a) = \xi_{Ux}^* e_U(b)$. Conversely, if $p_U e_U(a) = e_x \xi_{Ux}(a) = \xi_{Ux}^* e_U(b)$ then $e_x \xi_{Ux}(a) = \xi_{Ux}^* e_U(b)$. 514
\[
\zeta_{Ux} e_x(b) = e_x \xi_{Ux}(b) \text{ for all } x \in U, \text{ so } \xi_{Ux}(a) = \xi_{Ux}(b) \text{ for all } x \in U \text{ which shows that } p_U(a) = p_U(b).
\]

If \( h_x \) is coarser than \( h^*_x \text{ and } t_x \) is projectively defined by \( e_x : I_x \to (H_x, h_x) \), then clearly \( \zeta_{Ux} : (H_U, h_U) \to (H_x, h_x) \) and \( e_x : (I_x, t_x) \to (H_x, h_x) \) are continuous while \( \xi_{Ux} : (X_U, \tau_U) \to (I_x, t_x) \) is continuous by 4.1.4B.

The closure \( b_U(st_x) \) is projectively defined by \( \eta_{Ux} : A_U \to (I_x, t_x) \mid x \in U \) (see 4.1.2B), so \( p_U \) is continuous by 4.1.4A. The same argument works for \( p'_U \).

If \( p'_U \) and \( e_U \) are homeomorphisms, then \( p^{-1}U \) exists. Further, \( p^{-1}U \) is continuous if so is \( k_U = p'_U \circ e_U \circ p^{-1}_U \), for \( e_U \) is a homeomorphism into \( (H_U, h_U) \) (see 0.15).

As \( e_x \) is continuous and \( b_U'(sh^*_x) \) is projectively defined by \( \eta_{Ux} : A'_U \to (H_x, h^*_x) \mid x \in U \), the continuity of \( k_U \) follows from 4.1.4A. The proof is thereby complete.

**4.1.6. Remark.** If \( (H_U, h_U) \) from 4.1.3 is a compact topological space and if \( (A'_U, b'_U(h)) \) is a Hausdorff topological space, then \( p'_U \) is a homeomorphism if it is 1–1 and continuous. Clearly, \( (A'_U, b'_U(h)) \) is Hausdorff and topological if so is \( (H_x, h_x) \) for all \( x \in U \). It can happen that \( (H_x, h^*_x) \) from 4.1.5 is not topological or not Hausdorff, but that there is a Hausdorff topology \( h^*_x \) in \( H_x \) coarser than \( h^*_x \). Then we can projectively define the topology \( b'_U(sh^*_x) \) in \( A'_U \) by the maps \( \eta_{Ux} : A'_U \to (H_x, h^*_x) \mid x \in U \), getting a Hausdorff topological space \( (A'_U, b'_U(sh^*_x)) \). So we get

**4.1.7. Corollary.** The same symbols as in 4.1.2 are used. For every \( x \in X \) let \( h^*_x \) be a Hausdorff topology in \( H_x \) coarser than \( h^*_x \) (such an \( h^*_x \) exists provided \( (H_x, h^*_x) \) is f.s. — see 1.1.2). Let \( t \) be a closure in \( P \) such that for any \( U \in \mathcal{B}(X) \), \( x \in U \), the maps \( \xi_{Ux} : (X_U, \tau_U) \to (I_x, t_x) \) and \( e_x : (I_x, t_x) \to (H_x, h^*_x) \) are continuous (by 4.1.4B, this holds provided \( t = st^*_x \text{ or } t = st^*_x \), where \( t^*_x \) is projectively defined in \( I_x \) by \( e_x : I_x \to (H_x, h^*_x) \)).

Let us consider the diagram

(4.1.8)

\[
\begin{align*}
(A_U, b_U(t)) & \xrightarrow{p_U} (X_U, \tau_U) \xrightarrow{e_U} (H_U, h_U) \xrightarrow{p'_U} (A'_U, b'_U(s^*h^*_x)) \\
\eta_{Ux} & \downarrow \quad \xi_{Ux} \downarrow \\
(I_x, t_x) & \xrightarrow{e_x} (H_x, h^*_x) \xrightarrow{\eta_{Ux}'}
\end{align*}
\]

(here \( b_U(t) = b_U(st_x) \) for \( (st_x)_x = t_x \) — see 4.1.1F). Every map here is continuous and \( (A'_U, b'_U(s^*h^*_x)) \) is topological and Hausdorff. If \( p'_U \) is a homeomorphism (which holds provided \( (H_U, h_U) \) is compact and \( \mathcal{H} \) fulfills COND from 4.1.1E — then \( p'_U \) is 1–1) together with \( e_U \) (which holds provided \( (X_U, \tau_U) \) is topological — recall that \( e_U \) maps \( X_U \) into \( H_U \)), then so is \( p_U \) as well. Especially, \( p_U : (X_U, \tau_U) \to (A_U, b_U(st^*_x)) \) is a homeomorphism if \( (H_U, h_U) \) is compact, \( (X_U, \tau_U) \) topological and COND holds for \( \mathcal{H} \).

Proof follows directly from 4.1.5.

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4.1.9. Remark. Let $S = \{(X_U, \tau_U) \mid \rho_{uv} \mid X\}$ be from UNIF (see 0.5), let $\mathcal{H} = \{(H_U, h_U) \mid r_{uv} \mid X\}$ be a compact hull of $S$ so that $\mathcal{H} \in \text{TOP}$ (see 2.1.2B, C, D). As $(H_U, h_U)$ are compact, there is a unique uniformity $n_U$ in $H_U$ so that $cl\ n_U = h_U$ (see 0.9) and $\mathcal{H}' = \{(H_U, n_U) \mid r_{uv} \mid X\} \subseteq \text{UNIF}$. We put $(H_x, n_x^*) = \lim \mathcal{H}_x$, $(I_x, s_x^*) = \lim S_x$ for all $x \in X$ ($n_x^*$ is a uniformity). By 4.1.1E, we can take the semi-uniformities $ss_x^*, sn_x^*$ in $S_x = \bigcup\{I_x \mid x \in X\}$ and $P_x = \bigcup\{H_x, x \in X\}$, respectively. Then the uniformities $b_U(ss_x^*), b_U(sn_x^*)$ can be made as in 4.1.2E. We get

But we cannot get the statements of 4.1.5 or 4.1.7 in terms of UNIF unless $e_U$ and $p_U'$ are uniform embeddings (see 0.15). For example, if the complete hulls $(H_U, c_U)$ of $(X_U, \tau_U)$ are compact (i.e. $(H_U, cl\ c_U)$ are compact; $c_U$ is a uniformity), we can set $n_U = c_U$ (in this case $e_U : (X_U, \tau_U) \to (H_U, n_U)$ are uniform embeddings). If $\mathcal{R}_x = (H_x, n_x^*)$ is f.s. by $U(\mathcal{R}_x) \to R$ then there is a separated uniformity $n_x^*$ in $H_x$. Replacing $s_x^*, b_U(sn_x^*)$ in (D) by $n_x^*, b'_U(sn_x^*)$, we get that $p_U : (X_U, n_U) \to (A_U, b_U(sn_x^*))$ is a uniform embedding if so is $p_U' : (H_U, n_U) \to (A_U, b_U(sn_x^*))$ (the both maps are onto).

We have been dealing with the question whether the map $p_U$ is a homeomorphism. If this should be true, then $p_U$ must be 1–1, which holds if so is $p_U'$. The map $p_U'$ is 1–1 iff $\mathcal{H}$ fulfills COND from 4.1.1E. If this is not the case then we cannot use the same tools as above. Nevertheless, we still can deal with the question whether the identity $i_U : (X_U, \tau_U) \to (X_U, \tau_U(t))$, where $\tau_U(t)$ is defined in 4.1.1F, is a homeomorphism. We do it in the next remark.

4.1.10. Remark. Let a presheaf $S = \{(X_U, \tau_U) \mid \rho_{uv} \mid X\}$ from CLOS and its hull $\mathcal{H} = \{(H_U, h_U) \mid r_{uv} \mid X\}$ from TOP be given. As usual, let $\mathcal{I}_x = (I_x, t_x^*) = \lim S_x$, $\mathcal{R}_x = (H_x, h_x^*) = \lim \mathcal{H}_x$, $\mathcal{P} = (P_S, st_x^*)$, $\mathcal{P}_x = (P_x, sh_x^*)$, $A_U, A'_U$ be the stalks, the covering spaces and the sections in the covering spaces of $S$ and $\mathcal{H}$, respectively. If $t(h)$ is a closure in $P_x(P_{\mathcal{P}_x})$, then we can projectively define the closure $\tau_U(t)(h_U(h))$ in $A_U(A'_U)$ by the maps $\xi_{ux} : X_U \to (I_x, t_x^*)$ in $U$ — see 4.1.1E — as the following commutative diagram shows ($i_U, i'_U$ are identities):

$$\begin{align*}
(X_U, \tau_U(t)) &\xrightarrow{i_U} (X_U, \tau_U) \xrightarrow{e_U} (H_U, h_U) \xrightarrow{i_U} (H_U, h_U(h)) \\
I_x, t_x^* &\xrightarrow{e_x} (H_x, h_x) \xrightarrow{\xi_{ux}} X_U \\
(I_x, t_x^*) &\xrightarrow{\xi_{ux}} (H_x, h_x) \xrightarrow{\xi_{ux}'} X_U
\end{align*}$$

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Assume that \( e_x, \xi_{Ux}, \xi'_{Ux} \) are continuous for all \( x \in U \) (by 4.1.1,B this holds if \( \xi'_{Ux} \) is continuous — in particular, if every \( h_x \) is coarser than \( h^*_x \) — and if \( t_x \) is projectively defined by \( e_x : I_x \to (H_x, h_x) \)). Then clearly \( i_U, i'_U \) are continuous by the definition of \( \tau_U(a) \); \( h_t(h) \). If \( e_U \) is a homeomorphism (this holds if \( (X_U, \tau_U) \) is topological — recall that \( e_U \) maps \( X_U \) into \( H_U \) together with \( i_U \) (this holds in particular if \( (H, h_U) \) is compact and \( (H_U, h_U(h)) \) Hausdorff), then so is \( i_U \). Especially, if for every \( x \in U \) there is a Hausdorff topology \( h^*_x \) in \( H_x \) coarser than \( h_x \) (this holds if \( (H, h^*_x) \) is f.s.) and if \( t^*_x \) is projectively defined in \( I_x \) by \( e_x : I_x \to (H_x, h^*_x) \), then \( i_U : (X_U, \tau_U) \to (X_U, \tau_U(st^*_x)) \) is a homeomorphism if \( (H_U, h_U) \) is compact and \( (X_U, \tau_U) \) topological.

Proof. The continuity of \( i_U^{-1} \) can be proved as the continuity of \( p_U^{-1} \) in 4.1.5. If \( (X_U, \tau_U) \) is topological then \( e_U \) is a homeomorphism (see 2.1.2.B). If \( (H_U, h_U) \) is compact, then \( i'_U : (H_U, h_U) \to (H_U, h_U(st^*_x)) \) is a homeomorphism, from which our statement easily follows.

### 4.1.12. Lemma. If \( i_U \) in 4.1.11 is continuous or open (i.e., \( i_U(M) \) is \( h_U(h) \)-open if \( M \) is \( h_U \)-open), then \( p_U : (H_U, h_U) \to (A_U, b_U(h)) \) is continuous or open, respectively (we do not assume that \( p_U \) is 1-1). Thus if \( i_U \) is a homeomorphism, then \( (A_U, b_U(h)) \) is topological and if \( p_U \) is 1-1 then it is a homeomorphism.

Proof. Openness: Given \( a \in H_U \), a finite set \( F = \{x_1, \ldots, x_n\} \subset U \) and some \( h_{x_i} \), nbds \( N_i \) of \( \xi_{Ux_i}(a) \) in \( H_{x_i} \), \( i = 1, \ldots, n \), then put \( N_F = \bigcap \{N_i \mid i = 1, \ldots, n \} \). Then \( N_F \) and \( M_F = \bigcap \{N_i \mid i = 1, \ldots, n \} \) is \( h_U(h) \) — nbds of \( a \), respectively, and \( p_U N_F \subset M_F \). If \( q \in M_F \) then there is \( a \in H_U \) with \( p_U(a) = q \). As \( \eta_{Ux_i} q \in N_i \), \( i = 1, \ldots, n \), we get \( \xi_{Ux_i}(a) = \eta_{Ux_i} p_U(a) = \eta_{Ux_i} q \in N_{x_i} \) for \( i = 1, \ldots, n \), so \( p_U N_F = M_F \). Let \( N \) be an \( \tau_U \)-open set. Since \( i_U(N) \) is open, there is a family \( \mathcal{F} \) of finite subsets \( F \subset U \) such that \( i_U(N) = \bigcup \{N_F \mid F \in \mathcal{F} \} \), where \( N_F \) are the sets described above. Then \( p_U(N) = \bigcup \{p_U N_F = M_F \mid F \in \mathcal{F} \} \), which is open.

To prove the continuity of \( p_U \), look at the commutative diagram

\[
\begin{array}{ccc}
(H_U, h_U) & \xrightarrow{i'_{U}} & (H_U, h_U) \\
\downarrow p_U & & \downarrow \xi_{Ux} \\
(A'_U, b'_U(h)) & \xrightarrow{\eta_{Ux}} & (H_x, h_x)
\end{array}
\]

Here the continuity of \( i_U \) implies the continuity of \( \xi_{Ux} i'_U = \eta_{Ux} p_U \) for all \( x \in U \), so \( p_U \) is continuous.

### 4.1.13. Notation. If the hull \( \mathcal{H} \) of \( \mathcal{F} \) does not satisfy COND from 4.1.1E then the map \( p_U : H_U \to A_U \) is not 1-1. If not even \( \mathcal{F} \) satisfies COND, then \( p_U \) is not 1-1, either. But we can make the factorspace \( (X_U[p_U], \tau'_U) \) of \( (X_U, \tau_U) \) or \( (H_U[p_U], h_U) \) of \( (H_U, h_U) \) by the equivalence \( \{a, b \in X_U, a \sim b \iff p_U(a) = p_U(b)\} \) or \( \{p, q \in H_U, \)
then $p \sim q$ iff $p'_U(p) = p'_U(q)$, endowed with the closure $\tau'_U$ or $h'_U$ inductively defined in $X_U/p_U$ or $H_U/h'_U$ by the canonical map $k_U : (X_U, \tau_U) \to (X_U/p_U)$ or $k'_U : (H_U, h_U) \to (H_U/p_U)$, respectively. By 4.1.5, if $a, b \in X_U$, then $a \sim b$ iff $e_U(a) \sim e_U(b)$. Therefore, there is a $1 \sim 1$ map $e'_U : X_U/p_U \to H_U/p_U$ such that $e'_uk_U = k'_Ue_U$. Further, there are canonical maps $q'_U : X_U/p_U \to A_U$, $q'_U : H_U/p_U \to A_U$ which are $1 \sim 1$. If moreover $t$ and $h$ are closures in $P_{\kappa}$ and in $P_{\kappa}$ (so that we can make $b_U(t), b'_U(h)$) respectively, we have the following commutative diagram:

$$
\begin{array}{c}
\begin{array}{ccc}
(X_U/p_U, \tau'_U) & \xrightarrow{e'_U} & (H_U/p_U, h'_U) \\
| & & |
\downarrow{k'_U} & \downarrow{k'_U} & \downarrow{e_U} & \downarrow{e_U} & \downarrow{e_U}
\end{array}
\begin{array}{ccc}
(A_U, b_U(t)) & \xrightarrow{p_U} & (X_U, \tau_U) & \xrightarrow{e_U} & (H_U, h_U) & \xrightarrow{p'_U} & (A'_U, b'_U(h))
\end{array}
\begin{array}{ccc}
(l_x, t_x) & \xrightarrow{\eta_{Ux}} & (l_x, h_x) & \xrightarrow{\eta_{Ux}} & (l_x, h_x)
\end{array}
\end{array}
$$

Here $e'_U$ is continuous. Indeed, the continuity of $e'_U \circ k_U = k'_U \circ e_U$ yields that of $e'_U$.

A subset $M$ of $H_U$ is called saturated if $\{p \in H_U |$ there is $q \in M$ such that $p \sim q\} = M$.

4.1.15. Proposition. Suppose that there is a closure $h$ in $P_{\kappa}$ such that the identity $i_U : (H_U, h_U) \to (H_U, h_U(h))$ is open and continuous (this holds if $(H_U, h_U)$ is compact and if there is a Hausdorff topology $h_{x}^{*}$ in every stalk $H_{x}$, coarser than $h_{x}^{*}$, particularly, if the stalks $(H_{x}, h_{x}^{*})$ are f.s. — then we can put $h = s h_{x}^{*}$ — see 4.1.10).

If $e_U(X_U)$ is saturated, then $e'_U$ is open. Further, the map $q'_U$ is $h'_U - b'_U(h)$ continuous.

Proof. Look at 4.1.14, where $h$ is the closure from the assumption. If $B \subset X_U/p_U$ is $\tau'_U$-open then $C = k'_U^{-1}(B)$ is $\tau'_U$-open and $e_U(C)$ is open in $(e_U(X_U), \text{ind } h_U)$ — see 0.14. There is an $h'_U$-open set $D$ such that $e_U(C) = D \cap e_U(X_U)$. Clearly, $E = \{q \in H_U |$ there is $p \in D$ such that $p \sim q\}$ is saturated. Moreover, $E \cap e_U(X_U) = e_U(C)$. Indeed, if $q \in e_U(X_U) \cap E$ then there is $p \in D$ with $p \sim q$. We have $p \in e_U(X_U)$, for $q$ is from the saturated set $e_U(X_U)$. There are $a, b \in X_U$ with $e_U(a) = p$, $e_U(b) = q$. We have $a \in C$ for $p \in D$. From $p \sim q$ we get by 4.1.6 that $a \sim b$. Thus $k_U(b) = k_U(a) \in B$, so $b \in C$. Therefore $q = e_U(b) \in e_U(C)$.

Now we prove that $E$ is open. If $q \in E$, then there is $p \in D$ such that $p'_U(p) = p'_U(q)$. There is an open $h - \text{nbd } N$ of $p$ such that $N \subset D$ for $D$ is open (recall that $(H_U, h_U)$ is topological). Let $h$ be the closure mentioned in the assumptions, for which $i'_U : (H_U, h_U) \to (H_U, h_U(h))$ is open. By 4.1.12, $p'_U : (H_U, h_U) \to (A_U, b'_U(h))$ is open and continuous. Thus $M = p'_U(N)$ is $b'_U(h)$-open, $L = p'_U^{-1}(M)$ is $h'_U$-open, and $L$ is an $h'_U$-nbd of $q$ and $L \subset E$ as desired. Further, $e'_U(B) = k'_U e_U(C) = q'_U^{-1} p'_U e_U(C)$. The first equality and $k'_U e_U(C) \subset q'_U^{-1} p'_U e_U(C)$ is clear. To prove the other inclusion, take $a \in e_U(C)$. Therefore $a \in q'_U^{-1} p'_U e_U(C)$. There is $b \in e_U(C)$ with $p'_U(b) = q'_U(a)$. If there were $k'_U(b) \neq a$ then
we should have $p'_v(b) = q'_v k'_v(b) = q'_v(a)$, for $q'_v$ is 1-1. This contradicts $p'_v(b) = = q'_v(a)$, so $a \in k'_v e_v(C)$ as desired. Furthermore, $p'_v(E \cap e_v(X_U)) = p'_v(E) \cap p'_v e_v(X_U)$. Indeed if $a$ is from the right hand side, then there is $u \in E$ and $v \in e_v(X_U)$ with $p'_v(u) = = p'_v(v) = a$. As $u \sim v$ and $E$ with $e_v(X_U)$ are saturated, we get $u \in E, u \in e_v(X_U)$, hence $a \in p'_v(E \cap e_v(X_U))$, which proves the inclusion $\supset$ while $\subset$ is clear. Finally, $q'_v : (H_U/p'_v, h'_v) \rightarrow (A'_v, b'_v(h))$ is continuous for so is $q'_v \circ k'_v = p'_v$ in 4.1.14. (By 4.1.12 $p'_v$ is continuous.) Thus $p'_v(E) = b'_v(h)$ open for $p'_v$ is open. Further, $q'_v^{-1} p'_v(E)$ is $h'_v$-open for $q'_v$ is continuous. Thus $e'_v(B) = q'_v^{-1} p'_v e_v(C) = q'_v^{-1} p'_v(E \cap e_v(X_U)) = = q'_v^{-1} (p'_v E \cap p'_v e_v(X_U)) = q'_v^{-1} p'_v E \cap q'_v^{-1} p'_v e_v(X_U)$. We have $q'_v^{-1} p'_v e_v(X_U) = = e'_v k'_v(X_U) = e'_v(X_U/p'_v)$. Indeed, if $a$ is from the left hand side, then there is $b \in X_U$ with $q'_v(a) = p'_v e_v(b)$. Since $q'_v = 1-1$, we get $e'_v k'_v(b) = a$ for $q'_v e'_v k'_v(b) = = p'_v e_v(b) = q'_v(a)$. This proves the inclusion $\supset$ from the left equality, while the others are clear. Thus $e'_v(B) = q'_v^{-1} p'_v E \cap e'_v(X_U/p'_v)$. The porposition is proved.

4.1.16. Corollary. Let $(H_U, h_U)$ be compact and $e_v(X_U)$ saturated. Suppose that there is a closure $h$ in $P_\mathcal{S}$ such that $(A'_v, b'_v(h))$ and $(H_U, h_U(h))$ are Hausdorff and topological and that $p'_v : (H_U, h_U) \rightarrow (A'_v, h'_v(h))$ is continuous. (This holds if there is a Hausdorff topology $h_x$ in every $H_x$ coarser than $h_x^*$, particularly if $(H_x, h_x^*)$ are f.s. — then we can put $h = sh_x^*$. If $t$ is a closure in $P_\mathcal{S}$ such that $\xi_U : (X_U, \tau_U) \rightarrow (I_x, t_x)$ and $\xi_x : (I_x, t_x) \rightarrow (H_x, h_x)$ are continuous for all $x \in U$ (in particular, if $t_x = t_x^*$ or if $t_x$ is projectively defined by $e_x : I_x \rightarrow (H_x, h_x)$ for all $x \in U$ then $q_U : (X_U/p'_v, m\tau'_v) \rightarrow (A_U, m b'_v(t))$ is a homeomorphism (see 0.9).

Proof. Look at 4.1.14, where $h$ is the closure mentioned in the assumptions. By [1, Chap. 1, sec. 10(b), Cor. 1 of Prop. 8, p. 97], $(H_U/p'_v, m h'_v)$ is Hausdorff and hence compact. By 4.1.15, $q'_v$ is 1-1 and continuous, hence an $m h'_v - b'_v(h)$ homeomorphism. Further, $i'_v : (H_U, h_U) \rightarrow (H_U, h_U(h))$ is a homeomorphism, hence by 4.1.15 so is $i'_v : (X_U/p'_v, m\tau'_v) \rightarrow (H_U/p'_v, m h'_v)$ as well. The continuity of all $\xi_Ux$ for all $x \in U$ gives that of $p'_v$. From this the continuity of $q_U$ follows. As $e_x$ and $\eta_Ux$ are continuous for all $x \in U$, we get the continuity of $q_U^{-1}$ as in 4.1.5. It remains to prove that $\xi_Ux$ and $e_x$ are continuous if $t_x = t_x^*$ or if $t_x$ is projectively defined by $e_x : I_x \rightarrow \rightarrow (H_x, h_x)$. But this follows from 4.1.4, which completes the proof.

Suppose that the hull $\mathcal{H} = \{(H_U, h_U)|r_{UV}|X\}$ of $\mathcal{S} = \{(X_U, \tau_U)|q_{UV}|X\}$ is an $\mathcal{E}$ — compact hull $(\mathcal{E} - \text{compact hull})$ of $\mathcal{S}$ by a strongly separating family $\mathcal{E} = = \{F_U \subset C((X_U, \tau_U) \rightarrow Q) \mid U \in \mathcal{B}(X)\}$ $(\mathcal{E} = \{\mathcal{A}_U \subset C^*((X_U, \tau_U) \rightarrow C) \mid U \in \mathcal{B}(X)\})$ (see 2.1.6, 2.2.6; $Q$ is the compact unit interval and $C$ is the field of complex numbers). If the maps $p_U$ and $p'_U$ are not 1-1, then we can use 4.1.16 if $e_v(X_U)$ are saturated. This means that if $\phi \in H_U, a \in X_U, \phi \sim e_v(a)$, then there is $b \in X_U$ with $e_v(b) = \phi$. Here $H_U$ are the sets $Q^{F_U}$ (the sets $ML(\mathcal{A}_U \rightarrow C)$ of all continuous multiplicative linear functionals on $\mathcal{A}_U$) and $r_{UV} = q_{UV}^\ast$.

4.1.17. Remark. The saturatedness of $e_v(X_U)$ is equivalent to the following condition $K$: "Given $\phi \in H_U, a \in X_U$ and an open cover $\mathcal{V}$ of $U$ such that $\phi$ coincides
with $e_V(a)$ on $M_V = \{\varphi^*_V F_v \mid V \in \mathcal{V}\}$ (on $N_V = \{\varphi^*_V \mathcal{A}_V \mid V \in \mathcal{V}\}$) — or equivalently $\varphi^*_V \varphi(f_V) = f_V \varphi^*_V(a)$ for all $V \in \mathcal{V}$ and all $f_V \in F_V$ ($f_V \in \mathcal{A}_V$) — then there is $b \in X_U$ with $\varphi = e_U(b)$.” Let $\mathcal{S}$ fulfill COND from 4.1.1E so that all the $p_U$ are 1—1.

A. Since $e_U(a) \sim \varphi = e_U(b)$ implies $a = b$, we conclude that if $e_U(X_U)$ is saturated, then $\varphi \sim e_U(a)$ iff $\varphi = e_U(a)$. Thus $e_U(X_U)$ is saturated iff $\varphi \sim e_U(a)$ implies $\varphi = e_U(a)$.

B. If $\mathcal{H}$ is the $\mathcal{S}$ — hull of $\mathcal{S}$ then the following are equivalent:

1) $p_U$ is 1—1,

2) $e_U(X_U)$ is saturated,

3) $M_V = F_V$ for any open cover $\mathcal{V}$ of $U$.

If $\mathcal{H}$ is the $\mathcal{S}_\beta$ — hull of $\mathcal{S}$, the conditions

4) $\mathcal{N}_V$ is norm dense in $\mathcal{A}_U$ ($\mathcal{N}_V$ is the smallest subalgebra of $\mathcal{A}_U$ such that $N_V \subseteq \mathcal{N}_V$),

5) for any open cover $\mathcal{V}$ of $U$ and any $a \in X_U$ there is a unique extension $\varphi$ of the restriction $e_U(a)/\mathcal{N}_V$ of $e_U(a) \in ML(\mathcal{A}_U \to C)$ to the whole $\mathcal{A}_U$ (if this is the case then $\varphi = e_U(a)$) satisfy $4 \Rightarrow 1 \Rightarrow 5 \Rightarrow 2$. Further, each of the conditions 3 and 4 implies COND; thus $3 \Rightarrow 4$ and $4 \Rightarrow 1$ even without assuming COND beforehand.

Proof. If $\varphi, \psi \in H_U$ then $\mathcal{S} \sim \psi$ means that $\xi'_{UX}(\varphi) = \xi'_{UX}(\psi)$ for all $x \in U$. Thus for every $x \in U$ there is an open nbd $V_x \subseteq U$ of $x$ such that $\varphi^*_U \varphi(f_x) = \varphi^*_U \xi'_{UX}(f_x) = \psi^*_U \psi(f_x)$ for all $x \in X$ and all $f_x \in F_{V_X}$. So $\varphi = \psi$ on $M_V$ (on $N_V$), where $V = \{V_x \mid x \in X\}$.

Let COND hold for $\mathcal{S}$. If $a, b \in X_U$, $e_U(a) \sim e_U(b)$ then there is an open cover $\mathcal{V}$ of $U$ such that $\varphi^*_V e_U(a)(f) = f \varphi^*_V(a) = \varphi^*_V e_U(b)(f) = f \varphi^*_V(b)$ for all $V \in \mathcal{V}$ and all $f \in F_V$ ($f \in \mathcal{A}_V$). Let $V \in \mathcal{V}$. As $\mathcal{S}(\mathcal{S}_\beta)$ is separating, we get $\varphi^*_V(b) = \varphi^*_V(a)$. So $\varphi^*_U(a) = \varphi^*_U(b)$ for all $V \in \mathcal{V}$ which, by COND, gives $a = b$.

Let $e_U(X_U)$ be saturated, $a \in X_U$, $\varphi \in H_U$, $\varphi \sim e_U(a)$. Then there is $b \in X_U$ with $\varphi = e_U(b)$, so $a = b$ and $\varphi = e_U(a)$. Conversely, if $\varphi \sim e_U(a)$ implies $\varphi = e_U(a)$ then $e_U(X_U)$ is saturated, which proves A.

B. Let $\mathcal{H}$ be the $\mathcal{S}$ — hull of $\mathcal{S}$. As $\varphi \sim \psi$ iff $p_U(\varphi) = p_U(\psi)$ we get $1 \Rightarrow 2$. If $\varphi \sim \psi$ then there is $\mathcal{V}$ such that $\varphi = \psi$ on $M_V$. If $M_V = F_V$ we have $\varphi = \psi$ which proves $1 \Rightarrow 1$ (here we have not used COND). Let $f \in F_V \setminus M_V$. Take $a \in X_U$ and set $\varphi(g) = g(a)$ for all $g \in F_V \setminus \{f\}$, $\varphi(f) = c$ where $c = 0$ if $f(a) = 0$ and $c = 1$ if $f(a) = 0$. Then $\varphi$ is a map of $F_V$ into $C$, hence $\varphi \in H_U$. As $\varphi = e_U(a)$ on $M_V$, we have $\varphi \sim e_U(a)$. Since $\varphi \neq e_U(a)$, we have $\varphi \neq e_U(b)$ for all $b \in X_U$, so $e_U(X_U)$ is not saturated which gives $2 \Rightarrow 3$. If $\mathcal{H}$ is the $\mathcal{S}_\beta$ — hull of $\mathcal{S}$, $\varphi \sim \psi$, then there is an open cover $\mathcal{V}$ of $U$ with $\varphi = \psi$ on $N_V$. As $\varphi, \psi$ are continuous maps of $\mathcal{H}$ into $C$, we have $\varphi = \psi$ if $\varphi \sim \psi$ and if $N_V$ is dense in $\mathcal{H}$. This proves $4 \Rightarrow 1$ (here we have not used COND). Given $a \in X_U$, an open cover $\mathcal{V}$ of $U$ and an extension $\varphi \in ML(\mathcal{A}_U \to C)$ of $e_U(a) \mid N_V$, then $e_U(a) \sim \varphi$. If $p_U$ is 1—1, we have $\varphi = e_U(a)$, so $1 \Rightarrow 5$. If $\varphi \in H_U$, $a \in X_U$, $\varphi \sim e_U(a)$, then there is $\mathcal{V}$ with $\varphi = e_U(a)$ on $N_V$. 

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If the extension of \( e_v(a) \) from \( N_x \) is unique, we have \( \varphi = e_v(a) \), hence \( e_v(X_u) \) is saturated, so \( 5 \Rightarrow 2 \). Given \( a \in X_u \) and \( \mathcal{V} \) such that \( e_v(a)|N_x \) has an extension \( \varphi \in \text{ML}(\mathcal{A}_u \to C) \), \( \varphi \equiv e_v(a) \), then \( \varphi \in H_u \), \( \varphi \sim e_v(a) \), so \( e_v(X_u) \) is not saturated and \( 2 \Rightarrow 5 \) follows. The remark is proved.

2. REPRESENTATION THEOREMS

4.2.1. Definition. Let a presheaf \( \mathcal{P} = \{ S_u | e_u \mid X \} \), \( U \in \mathfrak{B}(X) \) and an open cover \( \mathcal{V} \) of \( U \) be given. A family \( \mathcal{H} = \{ a_v \in S_v | V \in \mathcal{V} \} \) is called \( \mathcal{V} \)-smooth if \( e_{uvw} \equiv e_v(a_v) \equiv e_{uw}(a_w) \) for all \( V, W \in \mathcal{V} \) with \( V \cap W = \emptyset \). \( \mathcal{P} \) is called projective if for every \( U \in \mathfrak{B}(X) \), any open cover \( \mathcal{V} \) of \( U \) and any \( \mathcal{V} \)-smooth family \( \mathcal{H} \) there is \( a \in S_U \) with \( e_{uv}(a) = a_v \) for all \( V \in \mathcal{V} \).

4.2.2. Theorem. Let \( \mathcal{P}' = \{ S'_u | e'_u \mid X \} \) be a presheaf from an i.c. category \( \mathfrak{Q} \) such that \( \mathcal{P} = \text{cl} \mathcal{P}' = \{ S_U = (X_u, \tau_u) \mid e'_u \mid X \} \) is \( T_i \) (see 2.1.2A), which is endowed with a strongly separating family \( \mathfrak{E} = \{ F_U \subset C^*(X_u \to R | \mathfrak{Q}) | U \in \mathfrak{B}(X) \} \) (see 1.1.5) so that all the \( e'_{uv} \) send \( F_v \) into \( F_U \) (see 4.1.5A). Further, let every \( x \in X \) have a filter base \( Ax \) of open nbds of \( x \) such that

1. \( \langle Ax \leq \rangle \) is well ordered (see 4.1.1A),
2. a) the family \( \mathfrak{E}_x = \{ F_U | U \in Ax \} \) is leftward smooth;
   b) either \( \mathfrak{E}_x \) is connected (see 1.1.5A) or \( x \) is of a countable local character and \( e'_{uv} \) maps \( F_v \) onto \( F_U \) for any \( U, V \in Ax, U \leq V \);
3. if \( U \subset X \) is open and if \( \mathcal{V} \) is an open cover of \( U \) then \( F_U = \bigcup \{ e'_{uv}F_v \mid V \in \mathcal{V} \} \).

If \( \mathcal{P} = (P, \mathfrak{I}) \) is the covering space of \( \mathcal{P} \) and \( A_U \) the set of the sections in \( \mathfrak{P} \) which corresponds canonically to \( X_u \) (see 4.1.1D, E, F), then:

a) For any \( x \in X \) the stalk \( \mathfrak{S}_x = (I_x, t^*_x) = \lim \mathfrak{S}_x \) of \( \mathcal{P} \) (see 4.1.1C) is f.s. by \( C^*(\mathfrak{S}_x \to R) \). Thus there is a separated topology \( t^*_x \) in \( I_x \) coarser than \( t^*_x \) (see 1.1.2). The topology, projectively defined by any separating family \( D_x \subset C^*(\mathfrak{S}_x \to R) \), may be taken as \( t^*_x \).

b) For any open \( U \subset X \) the map \( p_U : (X_u, m\tau_u) \to (A_U, b_U(st^*_x)) \) is a homeomorphism (see 4.1.1E; \( m\tau_u \) is the topological modification of \( \tau_u \) — see 0.9; \( st^*_x \) and \( b_U(st^*_x) \) are closures in \( P = \bigcup \{ I_x | x \in X \} \) and in \( A_U \), respectively — see 4.1.1F).

c) Let \( \mathcal{P} \) be projective (see 4.2.1). There is a separated closure \( \mathfrak{I} \) in \( P \) such that \( b_U(\mathfrak{I}) = b_U(st^*_x) \) (thus \( p_U : (X_u, m\tau) \to (A_U, b_U(\mathfrak{I})) \) is a homeomorphism), and \( \Gamma(U, \mathfrak{I}) = A_U \) for any open \( U \) (see 4.1.1F).

d) There is a separated topology \( \mathfrak{I} \) in \( P \) with \( A_U \subset \Gamma(U, \mathfrak{I}) \) and \( b_U(\mathfrak{I}) = b_U(st^*_x) \) for all \( U \in \mathfrak{B}(X) \), so that each canonical map \( p_U : (X_u, \tau_u) \times U \to (P, \mathfrak{I}) \) is continuous (the joint continuity of \( p_U \) is meant — see [9, Ch. 7, p. 233] — we have \( p_U(a, x) = \xi_{uv}(a) = (p_U(a))(x) \) if \( U \in \mathfrak{B}(X), a \in X_u, x \in U \) — see 4.1.1C, D).

Further, if the topology \( t^*_x \) in every stalk is metrisable and \( X \) is metrisable then so is \( \mathfrak{I} \).
Proof. Let $\mathcal{F} = \{\mathcal{C}_U = (C_U, t_U) \mid \mathcal{C}_{U+1} \times X\}$ be the $\mathcal{C}$-hull of $\mathcal{F}$ — see 2.1.6. By 2.1.4, $\mathcal{F}_x = \mathcal{F}_{Ax}$ is the $\mathcal{C}$-compact hull of $\mathcal{F}_{Ax}$ by $\mathcal{C}_x = \{F_U \mid U \in Ax\}$. If $x$ has countable Local character then there is a countable filter base $Bx \subset Ax$ of nbds of $x$.

By Th. 2.1.7, $(H_x, h_x^*) = \lim \mathcal{T}_x$ is f.s. (we put $\mathcal{T}_x = \mathcal{T}_{Ax}$) — see 4.1.1A, C, and 1.1.1.

By 1.1.2, there is a Hausdorff topology $h_x^*$ in $H_x$ coarser than $t_x^*$. For each open $U$ let $A'_U$ be the set of the sections in the covering space of $\mathcal{F}$, which corresponds canonically to $C_U$, and for $x \in U$ let $e_{Ux}$ be the canonical maps of $C_U$ and $A'_U$ into $H_x$, respectively. As in 4.1.2B, let $b_U^*(h_x^*)$ be the topology projectively defined in $A'_U$ by $\{\eta_{Ux} : A'_U \to (H_x, h_x^*) \mid x \in U\}$ (see 4.1.1F). By 4.1.6, $b_U^*(h_x^*)$ is Hausdorff, and from the condition (3) and 4.1.1B it follows that $p_U : \mathcal{C}_U \to (A'_U, b_U^*(h_x^*))$ is 1–1, hence it is a homeomorphism. By 4.1.5, all the $p_U$ are 1–1. As $\mathcal{F}$ is a hull of $\mathcal{F}$, the canonical embeddings $e_U : (X_U, m_{\tau_U}) \to \mathcal{C}_U$ are homeomorphisms (see 2.1.1B). Now, let $t^*_x$ be the topology projectively defined in $I_x$ by $e_x : I_x \to (H_x, h_x^*)$ — see 4.1.7. Then the maps $\xi_{Ux} : X_U \to I_x \to (H_x, h_x^*)$ are continuous. Continuous, so they are also $m_{\tau_U} \to t^*_x$ continuous. By 4.1.7, the map $p_U : (X_U, m_{\tau_U}) \to (A'_U, b_U^*(h_x^*))$ is a homeomorphism. The statements (a), (b) are proved.

If $U \in \mathfrak{B}(X)$, $x \in U$, $a \in X_U$, $\alpha = \xi_{Ux}(a)$, we set graph $(a, U) = \{\xi_{Ux}(a) \mid y \in U\}$. If $x \in P$, $s \in I_x$ we set $H(x) = \{\text{graph} \ (a, U) \mid \cup U \in \mathfrak{B}(X), a \in X_U \mid x \in U, \xi_{Ux}(a) = x; N \text{ is a } t^*_x \text{-nbhd of } x\}, K(x) = \{\cup I_y \mid y \in U, y + x \in U \in \mathfrak{B}(X), a \in X_U \mid x \in U, \xi_{Ux}(a) = x; N \text{ is a } t^*_x \text{-nbhd of } x\}$. Then $H(x), K(x)$ are separate round $x \in P$. They make a separated closure $t, \bar{t}$ in $P$. If $t \in I_x \times t^*_x$ then clearly $t_x = t^*_x$ (where $t_x$ is the closure induced in $I_x$ by $t$) for all $x \in X$ (so $b_U(t) = b_U(t^*_x)$, and $A_U \subset \subset I(U, t)$ for all $U \in \mathfrak{B}(X)$. By [11, Chap. 2, Sec. 4, Prop. 2.4.3, p. 608], $I(U, t) = A_U$ which proves (c). Clearly $\bar{t}$ is a topology and (d) holds (compare also with [11, Chap. 2, Sec. 3, Prop. 2.3.4, p. 607]).

Let every $t^*_x$ be metrisable by a metric $d_x$ and let $X$ be metrisable by $D'$. Given $\alpha, \beta \in P$, $x \in I_x$, $\beta \in I_y$, we set $D(\alpha, \beta) = D'(x, y)$ if $x = y$, $D(\alpha, \beta) = d_x(\alpha, \beta)$ if $x \neq y$. Clearly $D$ is a metric in $P$ which makes $\bar{t}$. The theorem is proved.

4.2.3. Theorem. Let $\mathcal{F}' = \{\mathcal{F}_U \mid U \in X\}$ be a presheaf from an i.c. category $\mathfrak{C}$ such that $\mathcal{F} = \text{cl } \mathcal{F}' = \{\mathcal{F}_U \mid U \in X\}$ is $T_1$ (see 2.1.2A). Suppose that for each open $U \subset X$ we have a Banach algebra $\mathcal{A}_U \subset C^\ast(\mathcal{F}_U \to C) \mid \mathfrak{C}$ (see 0.11, 2.1.2) with the sup-norm which separates points from closed sets of $\mathcal{F}_U$, so that $\mathcal{A}_{U'}$ maps $\mathcal{A}_U$ into $\mathcal{A}_U$ if $V \subset U$. Let every $x \in X$ have a filter base $A_x$ of open nbds such that $\langle A_x \leq\rangle$ is well ordered (see 4.1.1A) and

1. a) $\mathcal{A}_{U+1} \subset \mathcal{A}_{U+1}$ is norm-dense in $\mathcal{A}_U$ if $U \in Ax$ (U + 1 is the follower of $U$ in $\langle A_x \leq\rangle$);

b) either $x$ is of countable Local character and $\mathcal{A}_{Ax}$ is norm-dense in $\mathcal{A}_U$ for all $U, V \in Ax$, $U \leq V$, or the family $\mathcal{C}_x = \{\mathcal{A}_U \mid U \in Ax\}$ is connected (see 1.1.5) and $\mathcal{A}_U$ is symmetric (see 2.2.2B) for all $U \in Ax$;

2. if $U \subset X$ is open and if $\mathcal{V}$ is an open cover of $U$ then the smallest algebra in $\mathcal{A}_U$ containing $\cup \{\mathcal{A}_{U'} \mid V \in \mathcal{V}\}$ is norm dense in $\mathcal{A}_U$. 522
Then the statements (a)–(d) of Th. 4.2.2 hold.

Proof. Put $\mathcal{E} = \{\mathcal{A}_U \mid U \in \mathcal{B}(X)\}$ and let $\mathcal{F} = \{\mathcal{G}_U = (\mathcal{F}_U, t_U) \mid \mathcal{G}_U(X) \}$ be the $\mathcal{E}$-hull of $\mathcal{F}$ by $\mathcal{E}$—see 2.2.5B, 2.2.6. If $x \in X$ then by Th. 2.2.7 (or by 2.2.8 if $x$ is of countable local character), $(H_x, h_x) = \lim \mathcal{F}_x$ is f.s. (here $\mathcal{F}_x = \mathcal{F}_{Ax}$), so by 1.1.2 there is a Hausdorff topology $h^*_x$ in $H_x$ coarser than $h_x^x$. By 4.1.6, $b'_U(sh^*_x)$ is Hausdorff (see 4.1.2B). From the condition (2) and 4.1.17B it follows that $p'_U : \mathcal{G}_U \to (\mathcal{A}_U, b'_U(sh^*_x))$ is $1-1$ so it is a homeomorphism (see 4.1.1E, 4.1.2). The rest of the proof is the same as in Th. 4.2.2.

4.2.4. Corollary. Given a presheaf $\mathcal{F} = \{\mathcal{X}'_U \mid \mathcal{G}_U|X \}$ from UNIF such that all the $\mathcal{X}'_U$ are separated (see 0.5, 0.17), suppose that every $x \in X$ has a filter base $Ax$ of open nbds such that $\langle Ax \leq \rangle$ is well ordered (see 4.1.1A) and

1) a) $q_{VV+1}$ is a uniform embedding of $\mathcal{X}'_U$ into $\mathcal{X}'_{V+1}$ for all $U \in Ax$ (see 0.15);  
   b) if $\mathcal{G} = \{F_U = U^*(\mathcal{X}'_U \to R) \mid U \in \mathcal{B}(X)\}$—see 0.14, then either the family $\mathcal{E}_x$ is connected or $q_{UV}$ is a uniform embedding of $\mathcal{X}'_U$ into $\mathcal{X}'_V$ for all $V \in Ax, U \leq V$ and $x$ is of countable local character;

2) if $U \subset X$ is open and $\mathcal{V}$ is an open cover of $U$ then $\bigcup\{q_{UV}F_V \mid V \in \mathcal{V} \} = F_U$
   (this holds if $q_{UV} : \mathcal{X}'_U \to \mathcal{X}'_V$ are uniform embeddings for all $V \subset U$—see 1.3.4b).

Then the statements (a)–(d) of Th. 4.2.2 hold.

The conditions (1b) and (2) may be replaced by the following ones:

1b') If $\mathcal{E} = \{\mathcal{A}_U = U^{*}(\mathcal{X}'_U \to C) \mid U \in \mathcal{B}(X)\}$ (see 0.14; $C$ is the field of complex numbers), then either $E_x = \{\mathcal{A}_U \mid U \in Ax\}$ is connected, or $q_{UV}$ is a uniform embedding of $\mathcal{X}'_U$ into $\mathcal{X}'_V$ for all $U, V \in Ax, U \leq V$, and $x$ is of countable local character.

2') The smallest algebra in $\mathcal{A}_U$ containing $\bigcup\{q_{UV}^*A_V \mid V \in \mathcal{V} \}$ is norm-dense in $\mathcal{A}_U$ for every open $U \subset X$ and any open cover $\mathcal{V}$ of $U$.

(1b) is equivalent to (1b') for $\mathcal{A}_U = \{f + ig \mid f, g \in F_U\}$ for all $U$; notice that (2) yields (2') but (2) does not follow from (2').

Proof. By 1.3.4B, 1.3.7B, the conditions of Th. 4.2.2 are fulfilled for $\mathcal{F}$ and $\mathcal{E}$. If (1b') and (2') hold then by the same argument Th. 4.2.3 works.

4.2.5. Corollary. Let $\mathcal{F} = \{\mathcal{X}_U \mid \mathcal{G}_U|X \}$ be a normal and $T_1$ presheaf from TOP (see 2.1.2A, 0.5). Suppose that every $x \in X$ has a filter base $Ax$ of open nbds such that $\langle Ax \leq \rangle$ is well ordered and

1) a) $q_{VV+1}$ is a homeomorphism of $\mathcal{X}_U$ into $\mathcal{X}_{V+1}$ and $q_{VV+1}(\mathcal{X}_U)$ is closed in $\mathcal{X}_{V+1}$ for all $U \in Ax$;  
   b) if we put $\mathcal{E} = \{F_U = C^*(\mathcal{X}_U \to R) \mid U \in \mathcal{B}(X)\}$ then either $\mathcal{E}_x = \{F_U \mid U \in Ax\}$ is fully connected, or (1a) is fulfilled for any pair $U, V \in Ax, U \subset V$, instead of for $U, U + 1$ only and $x$ is of countable local character;

2) $F_U = \bigcup\{q_{UV}^*F_V \mid V \in \mathcal{V} \}$ for every $U \in \mathcal{B}(X)$ and any open cover $\mathcal{V}$ of $U$.  

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This holds if \( q_{UV} : \mathcal{X}_U \rightarrow \mathcal{X}_V \) is a homeomorphism into \( \mathcal{X}_V \) for all \( V \subset U \) — see 1.3.4a.) Then the statement of Th. 4.2.2 holds.

The conditions (1b) and (2) may be replaced by the following ones:

(1b') If \( \mathcal{E} = \{ \mathcal{A}_U = C(X_U \rightarrow C) \mid U \in \mathcal{B}(X) \} \) then either \( \mathcal{E}_x = \{ \mathcal{A}_U \mid U \in \mathcal{A}_x \} \) is connected or (1a) is fulfilled for any pair \( U, V \in \mathcal{A}_x, U \leq V \), instead of for \( U, U + 1 \) only. ((1b) is equivalent to (1b'), \( C \) is the field of complex numbers.)

(2') The smallest algebra in \( \mathcal{A}_U \) containing \( \bigcup \{ q^*_V \mathcal{A}_V \mid V \in \mathcal{V} \} \) is norm-dense in \( \mathcal{A}_U \) for every \( U \in \mathcal{B}(X) \) and any open cover \( \mathcal{V} \) of \( U \).

Proof. By 1.3.1A, 1.3.7A, the conditions of Th. 4.2.2 are fulfilled for \( \mathcal{S} \) and \( \mathcal{E} \). If we have (1b'), (2') then by the same argument 4.2.3 works.

4.2.6. Corollary. Let \( \mathcal{S} = \{ \mathcal{X}_U = (X_U, \tau_U) \mid q_{UV} \mid X \} \) be a locally compact presheaf from TOP. Suppose that every \( x \in X \) has a filter base \( \mathcal{A}_x \) of open nbds such that \( \langle \mathcal{A}_x \leq \rangle \) is well ordered and

1) a) if \( U, V \in \mathcal{A}_x, U \leq V \), then \( q_{UV} \) is a homeomorphism into \( \mathcal{X}_V \) such that the filter base \( B_{UV} = \{ q_{UV}(X_U - K) \mid K \subset X_U \ \text{compact} \} \) either has no cluster point or has a limit point in \( \mathcal{X}_V \);

b) if \( \mathcal{E} = \{ \mathcal{A}_U = \mathcal{L}^+_U \mid U \in \mathcal{B}(X) \} \) (see 2.3.1), then either \( \mathcal{E}_x = \{ \mathcal{A}_U \mid U \in \mathcal{A}_x \} \) is connected, or \( x \) is of countable Local character;

2) the algebra generated in \( \mathcal{A}_U \) by \( \bigcup \{ q^*_V \mathcal{A}_V \mid V \in \mathcal{V} \} \) is norm-dense in \( \mathcal{A}_U \) in the usual sup-norm for all open \( U \) and any open cover \( \mathcal{V} \) of \( U \).

Then the statement of Th. 4.2.2 holds.

Proof. By 2.3.2, 2.3.5A, B, the conditions of Th. 4.2.3 hold for \( \mathcal{S}, \mathcal{E} \).

4.2.7. Corollary. Let \( \mathcal{S} = \{ \mathcal{X}_U = (X_U, \tau_U) \mid q_{UV} \mid X \} \) be a compact presheaf from CLOS which fulfills COND from 4.1.1D. Suppose that each \( x \in X \) has a filter base \( \mathcal{A}_x \) of open nbds so that \( \langle \mathcal{A}_x \leq \rangle \) is well ordered and

1) a) \( q_{U+1 \cup U+2} = 1 - 1 \) on \( q_{U+1}(X_U) \) for all \( U \in \mathcal{A}_x \) (\( U + 1 \) is the follower of \( U \) in \( \langle \mathcal{A}_x \leq \rangle \));

b) either \( x \) is of countable Local character, or the family \( \mathcal{E}_x = \{ F_U = C(X_U \rightarrow R) \mid U \in \mathcal{A}_x \} \) fulfills the following condition: "Given \( U \in \mathcal{A}_x \) such that the predecessor \( U - 1 \) of \( U \) in \( \langle \mathcal{A}_x \leq \rangle \) does not exist, \( V \in \mathcal{A}_x \), \( V < U \), and a thread \( \{ f_W \in F_W \mid W \in \mathcal{A}_x, V \leq W < U \} \) through \( \mathcal{E}_x \), then there is \( f \in F_{U+1} \) with \( q^*_{U+1} f = f_W \) for all \( W \in \mathcal{A}_x, V \leq W < U \). (This always holds for countable \( \mathcal{A}_x \)).

Then the statement of Th. 4.2.2 holds.

Proof. By 3.1.3, \( (I_x, t^*_x) = \lim \mathcal{S}_{\mathcal{A}_x} \) is f.s. Thus if \( t^*_x \) is a Hausdorff topology in \( I_x \) coarser than \( t^*_x \), \( U \in \mathcal{B}(X) \), then it easily follows from 4.1.5 that \( p_U : \mathcal{X}_U \rightarrow \)}
\( (A_U, b_U(st^*_x)) \) is continuous, hence it is a homeomorphism, which proves that the statements (a), (b) of 4.2.2 hold for \( \mathcal{S} \). The rest can be proved as in 4.2.2.

4.2.8. Theorem. Let \( \mathcal{S}' = \{ X_U \mid q_{UV} \} \) be a presheaf from an i.c. category \( \mathcal{O} \) such that \( \mathcal{S}' = \text{cl } \mathcal{S}' = \{ X_U = (X_U, \tau_U) \mid q_{UV} \} \) is \( T_1 \), which is endowed with a strongly separating family \( \mathcal{E}' = \{ F_U \subset C_U = C^*(X_U \to C \mid \mathcal{O}) \mid U \in \mathcal{B}(X) \} \) such that all \( q_{UV}^* \) send \( F_V \) into \( F_U \). Let every \( x \in X \) have a filter base \( Ax \) of open nbds so that \( \langle Ax \rangle \leq \) is well ordered (see 4.1.1A) and

1. a) \( q_{UV}^* F_{U+1} \) is norm-dense in \( F_U \) if \( U \in Ax \) (\( U + 1 \) is the follower of \( U \) in \( \langle Ax \rangle \));

b) either (i): \( x \) is of countable local character and \( q_{UV}^*(F_V) \) is norm-dense in \( F_U \) for all \( U, V \in Ax, U \leq V \),

or (ii): the family \( \mathcal{E}_x = \{ F_U \mid U \in Ax \} \) is connected (see 1.1.5) and for every \( U \in Ax, F_U \) is either a symmetric Banach algebra or an algebra of real functions over the field of real numbers, complete in the sup-norm;

2. if \( U \subset X \) is open and if \( V \) is an open cover of \( U \) then \( M_V = \{ q_{UV}^* F_V \mid V \in V \} \) is norm-dense in \( F_U \). Then the statements (a)–(d) of Th. 4.2.2 hold.

Proof. Let us denote by \( \mathcal{A}_U \) the symmetric algebraic hull of \( F_U \) and set \( \mathcal{E}' = \{ X_U \mid U \subset X \text{ open} \} \). Let \( \mathcal{S}' = \{ (F_U, \mu_U) \mid q_{UV}^* X \} \) be the \( \mathcal{E}' \)-hull of \( \mathcal{S}' \) by \( \mathcal{E}' \) — see 2.2.5B, 2.2.6. By 2.2.5A, \( q_{UV}^*(\mathcal{A}_U) \) is norm-dense in \( \mathcal{A}_U \) whenever so is \( q_{UV}^*(F_U) \) in \( F_U \). Thus if \( x \in X \) then \( H_x, h_x^* = \lim \mathcal{S}_{Ax} \) is f.s. by Th. 2.2.8 if the case (i) occurs, and by Th. 2.2.7 if (ii) occurs. Putting \( S = T = X_U, F = F_U, G = \bigcup \{ q_{UV}^*(F_V) \mid V \in V \} \), \( h = \text{identity} \) in 2.2.5A we get that the symmetric algebraic hull \( \mathcal{S}(G) \) is norm-dense in \( \mathcal{A}_U \). Thus also the smallest algebra \( Z \) generated by \( G \) is dense in \( \mathcal{A}_U \) as \( \mathcal{S}(G) \) is the norm-closure of \( Z \). But \( G \subset N \) is \( \bigcup \{ q_{UV}^*(\mathcal{A}_U) \mid V \in V \} \), so the smallest algebra generated by \( N \) is dense in \( \mathcal{A}_U \) which yields that \( p_U : \mathcal{S}_U \to \mathcal{A}_U \) is 1–1 (see 4.1.1E, 4.1.2). Henceforward the proof proceeds as that of Th. 4.2.3 and of Th. 4.2.2.

4.2.9. Corollary. Let \( \mathcal{S}' = \{ (X_U, \tau_U) \mid q_{UV} \} \) be a presheaf from CLOS such that there is a metric \( d_U \) in every \( X_U \) which generates \( \tau_U \), and that all \( q_{UV} : (X_U, \tau_U) \to (X_V, \tau_V) \) are homeomorphisms into \( (X_V, \tau_V) \). Assume that every \( x \in X \) has countable local character. Let \( (P, st^*_x) \) be the CLOS-covering space of \( \mathcal{S} \) — see 4.1.1D and let \( \mathcal{S} \) satisfy COND from 4.1.1E. Then there is a metric \( v \) in \( P \) such that for the topology \( t \) generated in \( P \) by \( v \) we have

a) the map \( p_U : (X_U, \tau_U) \to (A_U, b_U(t)) \) is a homeomorphism for every \( U \) (see 4.1.1E);

b) the statements (c), (d) of Th. 4.2.2 hold, if we write \( t \) instead of \( st^*_x \).

Proof. We have \( \mathcal{S}_x = \lim \mathcal{S}_x \) for all \( x \in X \) (here \( \mathcal{S}_x = \mathcal{S}_{Ax} \) and \( \mathcal{S}_x \) is the stalk of \( \mathcal{S} = (P, st^*_x) \) — see 4.1.1D). As \( Ax \) is countable, the metric \( D_x \) in every \( I_x = |\mathcal{S}_x| \) can be defined by 3.4.1.
If \(a, b \in P\) then there are \(x, y \in X\) with \(a \in I_x, b \in I_y\). We set \(v(a, b) = 1\) if \(x = y\) and \(v(a, b) = D_x(a, b)\) if \(x \neq y\). Clearly \(v\) is a metric in \(P\).

Let \(U \subset X\) be open. By 3.4.1B, we may assume that \(U\) is the smallest element of \(\mathcal{A} \subseteq X\) for all \(x \in U\). As \(\mathcal{A}\) fulfills COND, (see 4.1.1E) \(p_U^{-1}\) exists. The continuity of \(\xi_{U, x} : (X_U, d_U) \rightarrow (I_x, d_x)\) or of \(\xi_{U, x}^{-1} : (\xi_{U, x}(X_U), D_x) \rightarrow (X_U, d_U)\) for all \(x \in U\) yields that of \(p_U : (X_U, d_U) \rightarrow (A_U, b_U(t))\) (see 4.1.4A) or of \(p_U^{-1} : (A_U, b_U(t)) \rightarrow (X_U, d_U)\), respectively (we have \(p_U^{-1} = \xi_{U, x}^{-1} \circ \eta_{U, x}\) for all \(x \in U\), where \(\eta_{U, x} : (A_U, b_U(t)) \rightarrow (I_x, d_x)\), with continuous \(\xi_{U, x}^{-1} \circ \eta_{U, x}\) and \(\eta_{U, x}(A_U) = \xi_{U, x}(X_U)\)), which proves (a), while (b) can be proved as in 4.2.2.

3. TOPOLOGISATION

In Theorems 4.2.1 – 4.2.6 we proved the existence of such a closure \(t\) in \(P\) that the maps \(p_U : \text{cl} \mathcal{R}_U \rightarrow (A_U, b_U(t))\) are homeomorphisms and the sets \(I(U, t)\) of all continuous sections over \(U\) in \((P, t)\) are precisely the sets \(A_t\) for all \(U \in \mathcal{A}(X)\). In this section we seek conditions for the existence of a topology in \(P\) with the mentioned properties.

### 4.3.1. Definition

A subset \(M\) of a closure space \(X\) is called a zero set if there is an \(f \in C(X \rightarrow R)\) such that \(M = f^{-1}(0)\). We say that \(X\) has the zero property (ZP) if each \(x \in X\) is a zero set. If \(x \in X\) is a zero set, \(f \in C(X \rightarrow R)\) with \(f(x) = 0, f > 0\) on \(X - \{x\}\), then \(f\) is called the \(x\) – function on \(X\).

Clearly, we get the same if we replace \(C(X \rightarrow R)\) by \(C(X \rightarrow Q)\) in the definition (\(Q\) being the compact unit interval).

An inductive family \(\mathcal{A} = \{X_a|q_{ab}| \langle A \subseteq X \rangle\}\) from CLOS is said to have ZP if every \(X_a\) has ZP.

A presheaf \(\mathcal{A} = \{S_U|q_{UV}| X\}\) from a category \(\mathcal{A}\) is said to have the unique Continuation property (UCP) if \(X\) is locally connected and the maps \(q_{UV}\) are 1 – 1 for every connected \(U\) and any \(V \subset U\).

### 4.3.2. Lemma

A necessary condition for a closure space \((X, t)\) to have ZP is that every \(x \in X\) has a countable set \(S_x\) of \(t\)-nbds of \(x\) with \(\bigcap\{N|N \in S_x\} = \{x\}\). This condition is also sufficient if \((X, t)\) is topological and completely regular.

B. ZP is hereditary.

Proof. If \((X, t)\) has ZP, \(a \in X\), then there is an \(a\) – function \(f\) on \(X\). If \(N_k = \{x \in X|f(x) < 2^{-k}\}, S_a = \{N_k|k = 1, 2, \ldots\}\), then \(S_a\) has the desired properties.

If \((X, t)\) is topological and completely regular, \(a \in X, S_a = \{N_1, N_2, \ldots\}\), then for every \(j = 1, 2, \ldots\) there is \(f_j \in C((X, t) \rightarrow Q)\) with \(f_j \geq 0, f_j(a) = 0, f_j = 1\) on \(X - N_j\), as we may assume that \(N_j\) are open. Then \(f = \sum_{j=1}^{\infty} 2^{-j}f_j\) is the desired \(a\) – function, which proves A; B is clear.
4.3.3. Lemma. Let \((X, t)\) be normal and have ZP.

A. Given a nonnegative continuous function \(g\) on \((X, t)\), a closed set \(Y \subseteq X\) and \(a \in Y\) such that \(Y \cap \{x \in X | g(x) = 0\} = \{a\}\), then there is an \(a -\) function \(f\) on \(X\) with \(f \geq g, f = g\) on \(Y\).

B. Let a nonnegative continuous function \(g\) on \((X, t)\), \(a \in X, \varepsilon > 0, \delta \geq 0\) be given. Let \((X, t)\) be \(T_1\). Then there is an \(a -\) function \(f\) on \(X\) such that \(f \geq \max (g, \delta)\) on \(M(g, \varepsilon) = \{x \in X | g(x) - g(a) \geq \varepsilon\}\).

C. Given a closed set \(Y \subseteq X\), \(a \in Y\) and an \(a -\) function \(g\) on \(Y\), then there is an \(a -\) function \(f\) on \(X\) with \(f = g\) on \(Y\). Moreover, if \(h\) is a nonnegative continuous function on \(X\) and \(g \geq \max (h, \delta)\) on \(\{x \in Y | h(x) - h(a) \geq \varepsilon\}\) then there is an \(a -\) function \(f'\) on \(X\) such that \(f' = g\) on \(Y\) and \(f' \geq \max (h, \delta)\) on \(M = \{x \in X | h(x) - h(a) \geq \varepsilon\}\).

Proof. A. By 4.3.2B, there is an \(a -\) function \(f_1\) on \(N = \{x \in X | g(x) = 0\}\), so there is a nonnegative continuous function \(f_2\) on \(X\) with \(f_2 = f_1\) on \(N, f_2 = 0\) on \(Y\). Then \(f = g + f_2\) is the desired function.

B. Let \((X, t)\) be \(T_1\). As \(M = M(g, \varepsilon)\) is closed, \(\max (g, \delta)\) continuous on it and \(a \notin M\), there is a nonnegative continuous function \(f_1\) on \(X\) with \(f_1(a) = 0, f_1 = \max (g, \delta)\) on \(M\) since \(Y = M \cup \{a\}\) is closed. If \(N = \{x \in X | f_1(x) = 0\}\), then \(Y \cap N = \{a\}\). Applying A to \(Y\) and \(f_1\) we get the function we desired.

C. There is a continuous nonnegative extension \(f_1\) of \(g\) to the whole \(X\). Applying A to \(f_1\) and \(Y\) we get the function \(f\) we wanted. If \(h\) is the function mentioned in C, we can put \(f_1' = \max (f, h, \delta)\) on \(M\). As \(f_1' = g\) on \(M \cap Y\), there is a continuous nonnegative function \(f_2\) on \(X\) with \(f_2 = f_1'\) on \(M\) and \(f_2 = g\) on \(Y\). Applying A to \(M \cup Y\) and \(f_1, f_2\) we get the desired function.

4.3.4. Definition. Let a presheaf \(\mathcal{S} = (X_U, (X_U, \tau_U, \mathcal{P}))\) from CLOS and its covering space \(\mathcal{S} = (P, (\mathcal{P}, S^x_P))\) over \(x \in X\) be given.

A. If \(x, y \in X, \alpha \in I_x, \beta \in I_y\) then \(\alpha\) and \(\beta\) are said to be relative (\(\alpha \simeq \beta\)) if there is a connected \(U \in \mathcal{B}(X)\) with \(x, y \in U\) and \(a \in X_U\) so that \(\xi_{U, \alpha}(a) = \alpha, \xi_{U, \beta}(a) = \beta\). The relation \(\simeq\) is not necessarily an equivalence. Clearly, if \(x = y\) and \(\alpha \simeq \beta\) then \(\alpha = \beta\).

B. If \(M \subseteq X\), we put \(P_M = \bigcup \{I_x | x \in M\}\). If \(f\) is a function on \(P_M, N \subseteq M, x \in M,\) we set \(f_N = f|_{P_N, f_x = f(x)}\). An \(M -\) function with respect to the relation \(\simeq\) is defined to be a nonnegative function \(f\) on \(P_M\) such that

1) \(f_x\) is continuous on every stalk \(\mathcal{S}_x\) with \(x \in M,\)

2) if \(\gamma, \delta \in P_M\), \(\gamma \simeq \delta\), then \(f(\gamma) = f(\delta)\) (it means the function \(\varphi(x) = f \circ \xi_{U, \alpha}(a)\) is constant on \(M \cap U\) for every connected \(U \in \mathcal{B}(X)\) and any \(a \in X_U\)).

If \(\alpha \in P_M\) then an \(M -\) function \(f\) with respect to \(\simeq\) is called the \(M -\) function for \(\alpha\) with respect to \(\simeq\), if \(f_x\) is a \(\gamma -\) function in \(\mathcal{S}_x\) (see 4.3.1) whenever \(x \in M, \gamma \in I_x, \gamma \simeq \alpha\).
If there is no danger of misunderstanding we write only $M - \text{function}$, $M - \text{function}$ for $\alpha$, leaving out the words "with respect to the relation $\simeq$".

C. $\mathcal{S}$ is said to be connectively projective if the following proposition holds:
"Given a connected $U \in \mathcal{B}(X)$, an open cover $\mathcal{V}$ of $U$ and a $\mathcal{V} - \text{smooth family}$ \{\(a_{V} \in X_{V} \mid V \in \mathcal{V}\) (see 4.2.1), then there is $a \in X_{U}$ with $q_{UV}(a) = a_{V}$ for all $V \in \mathcal{V}$.

4.3.5. Remark. A. If $f, g$ are $M$-functions for $\alpha$, then so are $\max (f, g), \min (f, g)$.

B. The relation $\simeq$ is an equivalence in $P$ if $\mathcal{S}$ is connectively projective and either all $\xi_{UV}$ are $1-1$ or the topology in $X$ is made by an order and $q_{UV}$ are $1-1$ for connected $U$.

C. If all the $q_{UV}$ are $1-1$ then $\mathcal{S}$ is not projective (see 4.2.1).

D. If $\mathcal{S}$ fulfills COND from 4.1.1E and $\simeq$ is an equivalence then $q_{UV}$ are $1-1$ for all connected $U \in \mathcal{B}(X)$.

Proof. B. If $\alpha \simeq \beta \simeq \gamma, \alpha \in I_{x}, \beta \in I_{y}, \gamma \in I_{z}$, then there are connected $U, V \in \mathcal{B}(X)$ provided $x \in U, y \in U \cap V, z \in V$ and $a \in X_{U}, b \in X_{V}$ with $\xi_{u}(a) = \alpha, \xi_{u}(a) = \xi_{v}(b) = \beta, \xi_{z}(b) = \gamma$. Setting $U \cap V = W, a' = q_{WV}(a), b' = q_{WV}(b)$, we have $\xi_{w}(a') = \beta = \xi_{w}(b')$. If all the $q_{UV}$ are $1-1$ we get $a' = b'$. If the topology in $X$ is made by an order then $W$ is connected, so again $a' = b'$. As $Z = U \cup V$ is connected and $\mathcal{S}$ is connectively projective, there is $c \in X_{Z}$ with $q_{ZV}(c) = a, q_{ZV}(c) = b$. Since $\xi_{Z}(c) = \alpha, \xi_{Z}(c) = \gamma$ we have $\alpha \simeq \gamma$.

C. If $\mathcal{S}$ is projective, $U, V \in \mathcal{B}(X)$, $U \cap V = \emptyset, a \in X_{U}, b \in X_{V}, b \not\equiv c, Z = U \cup V$, then there are $d, e \in X_{Z}$ with $q_{ZV}(d) = b, q_{ZV}(e) = c, q_{ZV}(d) = q_{ZV}(e) = a$. As $b \not\equiv c$, we have $d \not\equiv e$, so $q_{ZV}$ is not $1-1$.

D. If $U \in \mathcal{B}(X)$ is connected, $V \subset U$ open, $a, b \in X_{U}$, $q_{UV}(a) = q_{UV}(b), x \in U$, $y \in V$, $\alpha = \xi_{u}(a), \beta = \xi_{u}(b), \gamma = \xi_{v}(a) = \xi_{v}(b)$ then $\alpha \simeq \gamma \simeq \beta$, hence $\alpha = \beta$, as $\alpha \simeq \beta$ (see 4.3.4A). Thus $\xi_{u}(a) = \xi_{u}(b)$ for all $x \in U$. By COND, $a = b$ which proves D. A is clear.

4.3.6. Lemma. Let $\mathcal{S} = \{\mathcal{X}_{U} = (X_{U}, \tau_{U}) \mid q_{UV}(X) \}$ be a presheaf from CLOS with UCP (see 4.3.1), which is endowed with a family $\mathcal{E} = \{F_{U} \subset C(\mathcal{X}_{U}) \rightarrow R) \mid U \in \mathcal{B}(X)\}$ such that $\mathcal{S}$ fulfills the conditions P1-P3 and $\mathcal{E}$ the conditions F1-F3 below:

P1: Every $x \in X$ has a well ordered filter base $\langle Ax \leq \rangle$ of its open nbds.

P2: If $\mathcal{S} = (P, st^{*})$ is the covering space of $\mathcal{S}$ then the relation $\simeq$ is an equivalence in $P$.

P3: Given a component $M$ of $X, x, y \in M$ and $W \in Ax, W \subset M$, then for $B(W, y) = = \{U \in B(X) \mid U \text{ connected, } W \subset U, y \in U\}$ we have $\mathcal{X}_{W} = \text{lim } \{\mathcal{X}_{U} \mid q_{UV}(X), \leq{\rangle} \}$ (notice that $B(W, y)$ is not always right directed: the inductive limit is meant in the sense of 0.12).

F1: If $x \in X, V \subset Ax, a \in X_{V}$ then there is $W \in Ax$ with $W \subset V$ such that there is a $q_{WV}(a) - \text{function}$ $f \in F_{W}$ (see 4.3.1; this holds namely if $\mathcal{S}_{Ax}$ has ZP and $F_{V} \supset C^{*}(\mathcal{X}_{U} \rightarrow R)$ for any $x \in X$ and any $V \in Ax$).
F2: Given \( x \in X, \ V \in AX, \ a \in X_V \) and an \( a \) - function \( f \in F_V \), then there is a \( \varphi_{V+1}(a) \) - function \( g \in F_{V+1} \) with \( \varphi_{V+1}^* g = f \) (\( V + 1 \) is the follower of \( V \) in \( \langle AX \leq \rangle \); this holds namely if \( \mathcal{F} \) is normal, has ZP, \( F_V \) \( \Rightarrow \) \( C^* (X_V \rightarrow R) \) and \( \varphi_{V+1} X_V = \tau_{V+1} \) - closed for every \( x \in X \) and any \( V \in AX \) - see 4.3.3C).

F3: Either \( x \) is of countable local character or the following holds: Given \( U \in AX \) such that the predecessor \( U - 1 \) of \( U \) in \( \langle AX \leq \rangle \) does not exist, \( W \in AX \) with \( U \subset W \), \( a \in X_W \) and a thread \( \mathcal{F} = \{ f_V \mid V \in AX, \ U \subset V \subset W \} \) through \( \mathcal{F} \) such that every \( f_V \) is a \( \varphi_W(a) \) - function on \( X_V \), then there is a \( \varphi_W(a) \) - function \( f \) on \( X_U \) with \( \varphi_U f = f_V \) for all \( V \in AX \) with \( U \subset V \subset W \).

F4: If \( x \in X, \ V \in AX, \ U \in \mathcal{B}(X) \) connected with \( V \subset U \), then \( \varphi_U^* \) sends \( F_V \) into \( F_U \).

F5: Let \( M \) be a component of \( X, \ x, \ y \in M, \ W \in AX \). If \( \mathcal{F} = \{ f_V \mid V \in B(W, y) \} \) is a thread through \( \mathcal{F} \) (see P2) then \( \lim \mathcal{F} \in F_W \). (F4, F5 hold if \( F_V = C^* (X_Y \rightarrow R) \) or \( F_V = C(X_Y \rightarrow Q) \) for all \( V \in \mathcal{B}(X) \).)

Let \( M \) be a component of \( X, \ x \in M, \ \alpha \in I_x \). Then

A: There is an \( M \) - function \( f \) for \( \alpha \) such that \( \{ f_x \circ \xi_{U,x} \mid U \in Az \} \) is a thread through \( \mathcal{F}_A \) for all \( z \in M \). Further, given an \( \alpha \) - function \( g \) on \( I_x \) such that \( \{ g \circ \xi_{V,x} \mid V \in AX \} \) is a thread through \( \mathcal{F}_A \), then there is a unique \( M \) - function \( f \) for \( \alpha \) with \( f_x = g \). Furthermore, given \( \varepsilon > 0, \ \delta \geq 0 \), \( y \in M \), \( \beta \in I_y \), and an \( M \) - function \( g' \) with \( f \geq \max (g', \delta) \) on \( \{ y \in I_y \mid |g'(y) - g'(\beta)| \geq \varepsilon \} \), then \( f \geq \max (g', \delta) \) on \( \{ y \in I_y \mid |g'(y) - g'(\beta)| \geq \varepsilon \} \).

B: Let \( S(\alpha) \) be the set of all the \( M \) - functions for \( \alpha \) such that \( \{ f \circ \xi_{U,x} \mid U \in AX \} \) is a thread through \( \mathcal{F}_A \). If for every \( V \in AX \) the set \( F_V \) is closed under maximum and minimum of two functions from \( F_V \), then so is \( S(\alpha) \).

C: If \( y \in M \) then there is a unique \( \beta = h_{xy}(\alpha) \in I_y \) with \( \alpha \simeq \beta \). The map \( h_{xy} : \mathcal{F}_x \rightarrow \mathcal{F}_y \) is a homeomorphism. This statement follows only form P2, P3.

Proof. Let \( U \in AX \) be such that there is an \( a \) - function \( f_U \in F_U \). As P1, we can assume that there is an \( a \) - function \( f_U \in F_U \). Then \( \mathcal{F} = \{ f_V \mid V \in AX, \ U \subset V \} \) through \( \mathcal{F}_A \) can be made by induction similarly as in 1.1.7, so that every \( f_V \in \mathcal{F} \) is a \( \varphi_V(a) \) - function on \( X_V \). Then \( g = \lim \mathcal{F} \) is an \( \alpha \) - function on \( I_x \) and \( \{ g \circ \xi_{V,x} \mid V \in AX \} \) is a thread through \( \mathcal{F}_A \).

Now, let \( g \) be any \( \alpha \) - function on \( I_x \) such that \( \{ g_V = g \circ \xi_{V,x} \mid V \in AX \} \) is a thread through \( \mathcal{F}_A \). By the maximality principle, there is a maximal set \( N \subset M \) and an \( N \) - function \( h \) for \( \alpha \) with \( h_x = g \) (see 4.3.4B) such that \( \{ h^*_V = h_x \circ \xi_{V,x} \mid V \in Az \} \) is a thread through \( \mathcal{F}_A \) for every \( z \in N \). If \( y \in M - N, \ Vy \in Ay \) with \( Vy \subset M, \ a \in X_{Vy} \), \( z \in N \), then by P3, there is a connected \( U \in \mathcal{B}(X) \) with \( Vy \subset U \), \( z \in U \) and \( b \in X_U \) so that \( \varphi_{UV}(b) = a \) (such \( aVy \) exists, for the components of \( X \) are open since \( X \) is locally connected as \( \mathcal{F} \) has UCP - see 4.3.1). Further, there is \( Vz \in Az \) with \( Vz \subset U \). We can put

\[
 f^{U,z}_{Vy}(a) = h^*_U(b) = h_x \circ \xi_{U,z}(b)
\]
which defines the function $f_{V,y,z,b}(h_U)$ on $q_{UV,Y}(X_U)(X_U)$. As $h_U = q_{UV,Y} h_{V,Y}$ and $h_{V,Y} \in F_{V,Y}$ we have $h_U \in F_U$ by F4. We shall prove that $f_{V,y,z,b}$ does not depend on $U$, $z$, $y$.

If $U' \in \mathcal{B}(X)$ is connected, $v \in N$ and $c \in X_{U'}$ then $q_{UV,Y}(c) = a$, then for $\gamma = \xi_{U,V}(b)$, $\delta = \xi_{U',V}(c)$, $\eta = \xi_{U,V}(b) = \xi_{U',Y}(c)$ we have $\gamma \simeq \eta \simeq \delta$. By P2, $\gamma \simeq \delta$, so $h_{U}(\gamma) = h_{U}(\delta)$. Thus by (E), $f_{V,y,z,b}(a) = h_{U}(\delta) = h_{U}(\gamma) = f_{V,y,z,b}(a)$ as desired.

Thus we can define a function $f_{V,y}$ on $X_{V,Y}$ in this way: If $a \in X_{V,Y}$ then we can take an arbitrary $z \in N$. By P3, there is a connected $U \in \mathcal{B}(X)$ with $z \in U$, $V \in U$, $b \in X_{U}$ with $q_{UV,Y}(b) = a$. Then we can put $f_{V,y}(a) = f_{V,y,z,b}(a) = h_{U}(b)$. We have just shown that this choice does not depend on $U$, $b$ and $z$.

Now we shall show that $\mathcal{F}(V_{Y}, z) = \{h_{U} \mid U \in B(V_{Y}, z)\}$ is a thread through $\mathcal{E}_{B(V_{Y}, z)}$. By (E), if $U, U' \in B(V_{Y}, z)$, $U \subseteq U'$ then $q_{UV,Y} h_{U} = h_{U'} h_{U} = q_{UV,Y} h_{U} = q_{UV,Y} h_{U} = q_{UV,Y} h_{U}$ as desired.

Now we show that $f_{V,Y} \in F_{V,Y}$. By (E), $f_{V,Y} \circ q_{UV,Y} = h_{U} \in F_{U}$ for all $U \in B(V_{Y}, z)$ (notice that we have $a = q_{UV,Y}(b)$ in (E)), which together with P3 gives $f_{V,Y} = \lim_{V \in A_{Y}} \mathcal{F}(V_{Y}, z)$. As $\mathcal{F}(V_{Y}, z)$ is a thread through $\mathcal{E}_{B(V_{Y}, z)}$, we get $f_{V,Y} \in F_{V,Y}$ by F5.

Now we show that $\mathcal{F}_{y} = \{f_{y} \cup \mathcal{V} = A_{y}\}$ is a thread through $\mathcal{E}_{A_{y}}$. If $V, W \in A_{y}$, $\mathcal{V} \in V$, $a \in X_{V}$, $b \in X_{W}$, $q_{UV}(b) = a$, then by (E), $f_{y}(a) = h_{U}(b) = f_{W}(q_{VW}(a)) = q_{VW}(w(a)) = q_{VW}(w(a))$, so $q_{VW}(w(a)) = f_{V} = f_{W}$ as desired. If $f' = \lim_{V \in A_{y}} \mathcal{F}_{y}$, $f = g$ on $N$, $f' = f'$ on $I_{y}$, then $f$ is an $N \cup \{y\}$ function for $x$, so $M = N$.

In order to prove that $f$ is unique we need to prove the statement C. By P3, there is a connected $U \subset M$ with $x, y \in U$ and $a \in X_{U}$ with $\xi_{U,Y}(a) = a$. Then $\beta = \xi_{U,Y}(a) \simeq \alpha$. If $\gamma \in I_{y}$, $\gamma \simeq \alpha$ then $\gamma \simeq \alpha \simeq \beta$, so $\gamma \simeq \beta$, hence $\gamma = \beta$. We set $h_{xy}(\alpha) = \beta$. Since $h_{xy}(\alpha)$ is unique and the relation $\simeq$ is symmetric, we get $h_{xy} \circ h_{yx} = \text{identity}$. Inter changing $x$ and $y$ we get $h_{yx} \circ h_{xy} = \text{identity}$, so $h_{xy}$ is a 1-1 map onto $I_{y}$. Further, $h_{xy} : \mathcal{F}_{x} \rightarrow \mathcal{F}_{y}$ is continuous iff so is $h_{xy} = h_{xy} \circ h_{yx} : \mathcal{F}_{x} \rightarrow \mathcal{F}_{y}$ for all $V \in A_{y}$. If $V \in A_{x}$, then by P3, $h_{V}$ is continuous iff so is $g_{U} = h_{V} \circ q_{UV} : \mathcal{F}_{U} \rightarrow \mathcal{F}_{y}$ for all $U \in B(V, y)$. But $g_{U} = \xi_{U,Y}$ and it is continuous, so C follows.

Now the uniqueness of $f$ follows from the following statement: "If $g, h$ are two $M-$ functions such that there is $z \in M$ with $g = h$ on $I_{z}$, then $g = h$ on $P_{M}$. "Indeed, if $y \in M$, $x \in I_{y}$, then by C, there is $\beta \in I_{z}$ with $\alpha \simeq \beta$, so $g(\alpha) = g(\beta) = g(\beta) = h(\alpha)$ as desired.

The last statement we need is S: "Given $M-$ functions $f, g, f_{1}, g_{1}$ and $z \in M$ such that $f_{1} \geq g_{1}$ on $\{y \in I_{z} \mid f(y) \geq g(y)\}$, then $f_{1} \geq g_{1}$ on $\{y \in P_{M} \mid f(y) \geq g(y)\}\}."

Indeed, if $\gamma \in P_{M}$ with $f(\gamma) \geq g(\gamma)$ then by C, there is $\beta \in I_{z}$ with $\gamma \simeq \beta$. As $f(\gamma) = f(\gamma) \geq g(\gamma) = g(\beta)$, we have $f_{1}(\beta) \geq g_{1}(\beta)$, so $f_{1}(\gamma) \geq g_{1}(\gamma)$ as desired. Now the last statement of (A) follows from the fact that $|g' - g'(\beta)|$, $e, f$, max $(g', \delta)$ are $M-$ functions, and from S.

We prove B. If $f, g \in S(\alpha), h = \max(f, g)$ then by 4.3.5A, $h$ is an $M-$ function for $\alpha$. If $V \in A_{x}$ then $h_{V} = h \circ \xi_{V,Y} \in \max(f \circ \xi_{V,Y}, g \circ \xi_{V,Y}) \in F_{V}$, so $h_{V} \in V \in A_{x}$ is a thread through $\mathcal{E}_{A_{x}}$. By A, there is a unique $h' \in S(\alpha)$ with $h' = h$ on $I_{x}$, so $h' = h$ which completes the proof.

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4.3.7. Remark. A. In 4.3.6 we assume that $\mathcal{S}$ has UCP. It means that $X$ is locally connected and that $q_{uv}$ are $1-1$ for connected $U$. But in the proof we never used the fact that these $q_{uv}$ are $1-1$ or that $M$ are components. Thus it can be easily seen that the conditions of 4.3.6 can be weakened. Instead of UCP we can only assume that every $x \in X$ has a connected open nbd $K$ so that the conditions P1—P3, F1—F5 of 4.3.7 hold for $K$ instead of $M$, and in the statement as well as in the proof replace $M$ by $K$.

Another weak point of the foregoing lemma is the condition P2 which is very often not fulfilled (see 4.3.5). Nevertheless, in this point we can take the lemma as a method for formulating many other similar ones for we can take another relation $\simeq$ (for instance $\alpha \simeq \beta$ iff there is a relatively compact $U \in \mathcal{D}(X)$ and $a \in X_U$ with $\alpha = \xi_{U,x}(a), \beta = \xi_{U,y}(a)$", which is an equivalence if $\mathcal{S}$ is projective and if $q_{uv}$ are $1-1$ for relatively compact $U$) and adopt the conditions, the statement and the proof to it. In this way we can get many similar lemmas. This method can be expressed in a one piece which we do in part C of the remark. Instead of UCP the following notion will be useful there: “Given a nonempty set $\mathcal{D}(X) \subset \mathcal{D}(X)$, $\alpha \in I_x, \beta \in I_y$, then $\alpha$ and $\beta$ are $\mathcal{D}(X)$ — relative if there is $U \in \mathcal{D}(X)$ and $a \in X_U$ with $\xi_{U,x}(a) = \alpha, \xi_{U,y}(a) = \beta$.”

Instead of P2 we may assume that the $\mathcal{D}(X)$ — relation is an equivalence in $P$. Likewise as in 4.3.4 we can define the $M$ — functions for $\alpha$ and the $M$ — functions with respect to the $\mathcal{D}(X)$ — relation.

B. A question arises when the $\mathcal{D}(X)$ — relation is an equivalence. To answer it, the following property can be useful, which can be called the “$\mathcal{D}(X)$ — projectivity” and which is formulated as follows: “Given $U \in \mathcal{D}(X)$, an open cover $\mathcal{V}$ of $U$ and a $\mathcal{V}$ — smooth family $\{a_v \in X_v \mid V \in \mathcal{V}\}$ then there is $a \in X_U$ with $q_{uv}(a) = a_v$ for all $V \in \mathcal{V}$”. Likewise as in 4.3.5 we can prove this statement: Let $\mathcal{S}$ be $\mathcal{D}(X)$ — projective. Then the $\mathcal{D}(X)$ — relation is an equivalence if either $q_{uv}$ are $1-1$ for all the $U \in \mathcal{D}(X)$ and $V, W \in \mathcal{D}(X)$, $V \cap W \not= \emptyset$ implies $V \cup W, V \cap W \in \mathcal{D}(X)$, or if all the $q_{uv}$ are $1-1$ and either $\mathcal{S}$ is projective or $V \cup W \in \mathcal{D}(X)$ if $V, W \in \mathcal{D}(X)$, $V \cap W \not= \emptyset$.

C. The generalization of 4.3.6 proceeds like this: Let the family $\mathcal{S}$ and the bases $Ax$ from 4.3.6 fulfill the conditions P1, F1—F3 and P2': There is an open cover $\mathcal{K}$ of $X$ such that for every $K \in \mathcal{K}$ there is a nonempty set $\mathcal{D}(K) \subset \mathcal{D}(K)$ so that the $\mathcal{D}(K)$ — relation (see 4.3.7A) in $P_k$ is an equivalence, P3': Given $y, z \in K$, $W \in Ay$, $W \subset K$, then $x_W = \lim \{x_U \mid q_{uv} \mid B_K(W, z) = \{U \in \mathcal{D}(K) \mid W \subset U \subset K, z \in U\} \leq\}.$

F4': If $K \in \mathcal{K}$, $x \in K, V \in Ax, U \in \mathcal{D}(K)$ with $V \subset U$, then $q_{uv}^x$ sends $F_V$ into $F_U$.

F5': Given $y, z \in K$, $W \in Ay$ then $\lim \mathcal{S} \in F_W$ for any thread $\mathcal{F}$ through $\mathcal{S}_{B_K(W, z)}$.

Given $K \in \mathcal{K}$, $x \in K, \alpha \in I_x$, then there is a unique $K$ — function $f^K$ for $\alpha$ with respect to the $\mathcal{D}(K)$ — relation (see 4.3.4) such that $f^K \circ \xi_{U,z} \mid U \in Az$ is a thread through $\mathcal{S}_{Az}$ for all $z \in K$, and that the statements 4.3.6A, B, C hold if we replace the
word “$M$-function” by “$K$-function” and the equivalence $\simeq$ by the $\mathcal{D}(K)$-equivalence in $P_K$.

The proof of 4.3.7C follows directly from that of 4.3.6.

The purpose of the foregoing remark is to extend every $\alpha$-function $g$ on $\mathcal{I}_x$ to a $K$-function for $x$, where $K$ is an open nbhd of $x$ (which may depend on $x, g$).

If we have $F_U = C^*(\mathcal{I}_U \to R)$ or $F_U \subset C(\mathcal{I}_U \to Q)$ for all $U \in \mathcal{B}(X)$, $x, y \in K$ then the homeomorphism $h_{xy}$ carries every $\alpha$-function on $\mathcal{I}_x$ onto an $h_{xy}(\alpha)-$function on $\mathcal{I}_y$; thus $g$ can be extended to a $K$-function with the help of those $h_{xy}$, so that we need not use the conditions F4 F5 (F4', F5') from 4.3.6 (4.3.7C). If $F_U$ are defined in this way, we can assume directly that $\mathcal{I}_x$ and $\mathcal{I}_y$ are homeomorphic under a homeomorphism $h_{xy}$ so that $h_{xy} = h_{yx} \circ h_{xy}$, $h_{xy} \circ h_{yx}$ = identity for all $x, y, z \in K$. Clearly, a sufficient condition for this is “$\mathcal{I}_W = \lim \{\mathcal{I}_U|\mathcal{U}_W| \subseteq \langle U \in \mathcal{B}(X) \mid y \in U \rangle \leq \rangle \}$ for all sufficiently small $W \in Ax$ and any $y \in K$”. By 1.4.5, this is fulfilled if P3 or P3' from 4.3.6 or from 4.3.7, respectively, holds.

4.3.8. Proposition. Let $\mathcal{I} = (\mathcal{I}_U = (X_U, \tau_U)|\mathcal{U}_V|X)$ be a presheaf from CLOS which fulfils COND from 4.1.1E. Suppose that the following conditions hold:

1) For every $x \in X$ and any $x \in I_x$ there is a nonempty set $S(x)$ whose elements are pairs $(V, f)$, where $V$ is an open nbhd of $x$ and $f$ is a function on $P_V$ (see 4.3.4B), which satisfies $a-d$:

   a) If $(V, f) \in S(x)$ then for any open nbhd $W \subseteq V$ of $x$ we have $(W, f|W) \in S(x)$.
   b) If $(V, f), (W, g) \in S(x)$ then there is an open nbhd $Z \subseteq V \cap W$ of $x$ with $(Z, \max (f, g)), (Z, \min (f, g)) \in S(x)$.
   c) Given $(V, g) \in S(x), y \in V, \beta \in I_y, \varepsilon > 0, \delta \geq 0$, then there is $(W, f) \in S(\alpha)$ with $W \subseteq V$ such that $f \geq \max (g, \delta)$ on $\{\gamma \in P_W \mid g(\gamma) - g(\beta) \geq \varepsilon\}$.
   d) If $(K, f) \in S(x)$ then $f_x$ is an $\alpha$-function on $\mathcal{I}_x$ and for every $y, z \in K$ there is a homeomorphism $h_{yz}$ of $\mathcal{I}_x$ onto $\mathcal{I}_z$ with $h_{yz} \circ h_{xy} = \text{id}$ such that for $\beta \in I_y, \gamma \in I_x$, we have $f(\beta) = f(\gamma)$ if $\gamma = h_{yx}(\beta)$, and that for $a \in X_K$ there is an open nbhd $V \subseteq K$ of $x$ such that $y, z \in V, \beta = \xi_{K,a}(a), \gamma = \xi_{K,a}(a)$ implies $\gamma = h_{yx}(\beta)$ (clearly, this holds if there is an open cover $\mathcal{D}(K)$ of $K$ such that the $\mathcal{D}(K)$-relation (see 4.3.4, 4.3.7A) is an equivalence in $P_K$ and that $P3'$ from 4.3.7C holds).

2) There is an open cover $\mathcal{V}$ of $X$ such that every $V \in \mathcal{V}$ has ZP (see 4.3.1).

3) In every stalk $I_x \subseteq P$ there is a closure $t^*_x$ with the following properties:

   a) All the canonical maps $p_U : \mathcal{I}_x \to (A_U, b_U(st^*_{t_U}))$ are homeomorphisms.
   b) Given $x \in I_x$ and a $t^*_x$-nbhd $L$ of $x$, then there is $(V, g) \in S(x)$ and $\varepsilon > 0$ such that $M_x = \{\gamma \in I_x \mid g(\gamma) < \varepsilon\} \subseteq L$.

Then there is a Hausdorff topology $t$ in $P$ such that all the canonical maps $p_U : \mathcal{I}_U \to (A_U, b_U(t))$ are homeomorphisms and $A_U \subseteq \mathcal{I}(U, t)$ (see 4.1.1F) for all $U \in \mathcal{B}(X)$.

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If every $V \in \mathcal{V}$ is a normal topological space, $X$ has countable local character and $\mathcal{S}$ is $\mathcal{D}(X)$ - projective (see 4.3.7B) or projective, then $\Gamma(U, t) = A_U$ for all $U \in \mathcal{D}(X)$ or for all $U \in \mathcal{B}(X)$, respectively. Further, $(I_x, t_x)$ is functionally separated by $L_x = \{ f \mid$ there is $\alpha \in I_x$ and $V$ with $(V, f) \in S(\alpha) \}$ ($t_x = t/I_x$ - see 0.14).

Proof. Let $x \in I_x$, $\varepsilon > 0$. By (1), there is $(K, f) \in S(x)$. By 1a and 2, there is an $x -$ function on $K$. We put $S(x \mid f < g \mid K) = \{ \gamma \in P_K \mid f(\gamma) < g(\gamma) \}$, $S(x \mid g < f < \varepsilon \mid K) = \{ \gamma \in P_K \mid g(\gamma) < f(\gamma) < \varepsilon \}$ ($p$ is the canonical projection of $P$ onto $X$, i.e., if $x \in P$, $\alpha \in I_x$, then $p(\alpha) = x$). If $(K, f_1) \in S(x)$ and if $g_1$ is another $x -$ function on $K$, we set $N(x \mid f, g, K, \varepsilon, f_1, g_1) = S(x \mid f < g \mid K) \cup S(x \mid g_1 < f_1 < \varepsilon \mid K) \cup \{ x \}$. It is sketched in the following picture ($S$ and $T$ is the first and the second set in the union respectively):

![Diagram](image)

We can easily see from 1a, 1b that the set $N(x)$ of all $N(x \mid f, g, K, \varepsilon, f_1, g_1)$ is a filter base round $x$. This holds for every $\alpha \in P$ so that we get a closure $t$ in $P$. We shall show that the sets from $N(x)$ are open, henceforth $t$ is a topology.

A: Let $\beta \in S(x \mid f < g \mid K)$, $\beta \in I_y$. Then by 1d, $0 \leq f(\beta) < g(\gamma)$. We have to find $(K', f_1) \in S(\beta)$ and a $y -$ function $g_1$ on $K'$ so that $\gamma \in P_{K'}$, $f_1(\gamma) < g_1 \beta \gamma$ implies $f(\gamma) < g \beta \gamma$. There is $\varepsilon > 0$ such that $f(\beta) + \varepsilon < g(\gamma)$. By 1c, there is $(K', f_1) \in S(\beta)$ with $K' \subset K$ and with $f_1 \leq f$ on $\{ \gamma \in P_K \mid f(\gamma) \leq f(\beta) + \varepsilon \}$. By 2 and 1a, we may assume that there is a $y -$ function $g_1$ on $K'$ with $g_1 < f(\beta) + \varepsilon < g$ on $K'$. Thus if $\gamma \in P_{K'}$ and $f(\gamma) \geq g \beta \gamma$, then $g_1 \beta \gamma < f(\beta) + \varepsilon < g \beta \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gambar
As $0 \leq g(y) < f(\beta) < \varepsilon$, there is $\eta > 0$ with $f(\beta) + \eta < \varepsilon$, $g(y) + \eta < f(\beta)$. By 1c, there is $(K', f_1) \in S(\beta)$ with $K' \subset K$, $f_1 \geq \max(f, \varepsilon)$ on $\{ y \in P_{K'} \mid |f(y) - f(\beta)| \geq \eta \}$. By 1a, we may assume $g < f(\beta) - \eta$ on $K'$. Now, if $y \in P_{K'}$, $f(\gamma) \geq \varepsilon$ then $f(\gamma) + \eta < \varepsilon \leq f(y) \leq f(\gamma)$, so $f(y) \geq \varepsilon$. Thus $f(y) < \varepsilon$ if $y \in P_{K'}$, $f_1(y) < \varepsilon$. If $f(\gamma) \leq g(p(\gamma))$, $y \in P_{K'}$, then $f(\gamma) \leq f(\beta) - \eta$ so $f_1(\gamma) \geq \varepsilon$, hence $f_1(\gamma) < \varepsilon$ implies $g(p(\gamma)) < f(\gamma)$. Thus $S(\beta, g_1, f_1, \varepsilon) \subset S(\alpha, g < f < \varepsilon | K)$ for any $y$ - function $g_1$ on $K'$.

D: Now we will find a $y$ - function $g_1'$ on $K'$ such that $y \in P_{K'}$, $f_1(y) < g_1$, $p(y)$ implies $g(p(y)) < f(y) < \varepsilon$ ($K'$, $f_1, \varepsilon$ are from C). Take a $y$ - function $g_1$ on $K'$ with $g_1 < \varepsilon$. Then for $y \in P_{K'}$ with $f_1(y) < g_1(p(y))$ we have $f_1(y) < \varepsilon$, so $g(p(y)) < f(y) < \varepsilon$ by C. Thus $S(\beta, f_1, g_1) \subset S(\alpha, g < f < \varepsilon | K)$ which completes the proof of the openness of the $t$ - nbds of the points of $P$. Thus $t$ is a topology.

It easily follows from 1c that every $f \in L_X$ is $t_x$ - continuous. Thus $(I_x, t_x)$ is functionally separated by $L_x$, hence $t_x$ is Hausdorff. Thus so is $t$ as well.

Let $\alpha \in P$, $\alpha \in I_x$. By 3b, the topology $t_x$ induced in $I_x$ by $t$ is finer than the closure $t^+_x$, so all the $p^{-1}_0: (A_U, b_U(t)) \rightarrow \mathcal{X}_U$ are continuous. If $U \in \mathcal{B}(X)$ then $\xi_{U,x}: \mathcal{X}_U \rightarrow \mathcal{I}_x$ is continuous for all $x \in U$ as $t_x$ is coarser than $t^+_x$ (recall that if $x \in X$, $\alpha \in I_x$, $(V, f) \in S(\alpha)$, then $f_x$ is an $\alpha$ - function on $f_x$, hence it is $t^+_x$ - continuous — see 1d). By 4.1.4A or 4.1.5, all the $p_0: \mathcal{X}_U \rightarrow (A_U, b_U(t))$ are continuous, so they are homeomorphisms. The definition of $t$ directly implies $A_U \subset \Gamma(U, t)$ for all $U \in \mathcal{B}(X)$.

Let every $V \in \mathcal{Y}$ be normal, and let $X$ have countable local character. Given $U \in \mathcal{B}(X)$, $x \in U$, $a \in X_U$ and a continuous section $r: U \rightarrow (P, t)$ such that $a = \equiv r(x) = \xi_{U,x}(a)$, then $N = \{ y \in U \mid r(y) = \xi_{U,y}(a) \} \neq \emptyset$ as $x \in N$. The set $\{ \xi_{U,y}(a) \mid y \in U \}$ is closed in $(P_U, t)$ as $t$ is Hausdorff. Thus $N$ is closed in $U$. We want to show that $N = U$ if $U$ is connected. We will prove the statement S: "Every $y \in N$ has an open nbh $W \subset U$ such that $r(z) = \xi_{U,z}(a)$ in $W$" (here $U$ need not be connected).

Let $y \in N$. There is $(K, f) \in S(\beta)$, where $\beta = r(y)$. We may assume $K \subset U$. Suppose on the contrary that in every open nbhd $W \subset K$ of $y$ there is $x_W$ with $r(x_W) \equiv \neq \xi_{U,x_W}(a)$. The function $g = f \circ r$ is continuous on $K$ and by 1d, there is an open nbhd $V \subset K$ of $y$ such that $F^h(x) = f \circ \xi_{V,x}(b)$ is constant on $V$ for every $b \in X_Y$. By 1a, we may assume $V = K$. Thus by 1d, $g(z) = 0$ iff $r(z) = \xi_{U,z}(a)$, so $\{ z \in K \mid \xi_{U,z}(a) = r(z) \} = \{ z \in K \mid g(z) = 0 \}$. There is $V \subset \mathcal{Y}$ with $y \in V$. We may assume $K \subset V$. As $X$ has countable local character, we may assume by 1d that there is a sequence $\{ x_n \}$ in $K$ which tends to $y$ so that $g(x_n) > 0$ for all $n$. As $g$ is continuous on the closed set $M = \{ x_n \} \cup \{ y \}$ so by the normality of $V$ and 4.3.2C, there is a $y$ - function $h$ on $V$ with $h = g$ on $M$. Then $r(x_n) \notin N(\beta | f, h, K, \varepsilon, f, h)$ for every $n$ and any $\varepsilon > 0$. Thus $r: U \rightarrow (P, t)$ is not continuous, which is the desired contradiction. Thus every $y \in N$ has an open nbhd $V \subset K$ of $y$ with $r(z) = \xi_{U,z}(a)$ in $V$, so $N$ is open and closed hence $N = U$ if $U$ is connected.

Now, if $\mathcal{F}$ is $\mathcal{D}(X)$ - projective and $U \in \mathcal{D}(X)$ then for every $r \in \Gamma(U, t)$ there is $a \in X_U$ and $x \in U$ with $r(x) = \xi_{U,x}(a)$. Indeed, for every $x \in U$ there is an open nbhd $Ux \subset U$ of $x$ and $a_x \in X_{Ux}$ with $\xi_{Ux,x}(a_x) = r(x)$. By the statement $S$, for every
\( x \in X \) there is an open nbhd \( V_x \subset U_x \) of \( x \) with \( \xi_{U,x}(a_x) = r(z) \) for all \( z \in V_x \). Then \( \mathcal{V} = \{ V_x \mid x \in U \} \) is an open cover of \( U \). If \( V \in \mathcal{V} \), we set \( a_V = \xi_{U,V}(a_x) \). If \( V, W \in \mathcal{V} \), \( Z = V \cap W \neq 0, z \in Z \), \( b = \xi_{WZ}(a_V) \), \( c = \xi_{WZ}(a_W) \) then \( \xi_{Z,x}(b) = r(z) = \xi_{Z,x}(c) \).

As \( \mathcal{S} \) fulfills COND from 4.1.1E, we have \( b = c \). Since \( \mathcal{S} \) is \( \mathcal{D}(X) \) — projective and \( U \in \mathcal{D}(X) \), thus there is \( a \in X_U \) with \( \xi_{U,V}(a) = a_V \) for all \( V \in \mathcal{V} \) (see 4.3.7). Then \( \xi_{U,x}(z) = r(x) \) for all \( x \in U \). The proof is thereby complete.

**4.3.9. Theorem.** Given a normal and \( T_1 \) presheaf \( \mathcal{S} = \{ \mathcal{F}_U = (X_U, \tau_U) \mid \mathcal{Q}(\mathcal{V}) \} \) (see 2.1.1A) from \( \text{TOP} \), assume that the following conditions are fulfilled:

S1: \( \mathcal{S} \) fulfills the condition (2) or (2') from 4.2.5, ZP and UCP,

S2: there is an open cover \( \mathcal{V} \) of \( X \) such that every \( V \in \mathcal{V} \) has ZP,

S3: the relation \( \simeq \) from 4.3.4 is an equivalence in \( P \),

S4: every \( x \in X \) has a well ordered countable filter base \( \langle Ax \rangle \) of its open nbds whose ordinal type is \( \omega_0 \), such that \( \xi_{UW} \) maps \( \mathcal{F}_U \) onto a \( \tau_{U+1} \) — closed set \( \xi_{UW+1}(X_U) \) homeomorphically for all \( U \in Ax \).

S5: if \( M \) is a component of \( X, x, y \in M, W \in Ax, W \subset M \), then \( \mathcal{F}_W = \lim_{(W,y)} \mathcal{S}_{B(W,y)} \)

\( B(W, y) = \{ U \in \mathcal{B}(X) \mid U \text{ connected, } W \subset U, y \in U \} \) — see P3 from 4.3.6.

Then there is a topology \( t \) in \( P \) such that the canonical maps \( p_U : \mathcal{F}_U \to (A_U, b_U(t)) \) are homeomorphisms and \( A_U \subset \Gamma(U, t) \) for all \( U \in \mathcal{B}(X) \). Moreover, if every \( V \in \mathcal{V} \) is a normal topological space, \( X \) has countable local character and \( \mathcal{S} \) is \( \mathcal{D}(X) \) — projective (see 4.3.7B), then \( \Gamma(U, t) = A_U \) for all \( U \in \mathcal{D}(X) \). Especially, \( \Gamma(U, t) = A_U \) for all \( U \in \mathcal{B}(X) \) if \( \mathcal{S} \) is projective, if every \( V \in \mathcal{V} \) is normal and if \( X \) has countable local character.

**Proof.** We set \( \mathcal{E} = \{ \mathcal{F}_U = C(\mathcal{F}_U \to \mathcal{Q}) \mid U \in \mathcal{B}(X) \} \) (\( \mathcal{Q} \) is the compact unit interval). We show that the conditions of 4.3.6 are fulfilled. Clearly, P1 — P3 of 4.3.6 hold. As \( \mathcal{S} \) has ZP, so F1 from 4.3.6 holds. It follows from the normality of \( \mathcal{S} \), S4 and 4.3.3C that F2 from 4.3.6 holds while F3 holds as \( Ax \) is countable. Clearly, \( \mathcal{E} \) fulfills F4, F5. Thus the statement of 4.3.6 holds, hence every \( x \in X \) has an open nbhd \( M \) of \( x \) such that for every \( \alpha \in P_M \) there is an \( M \) — function for \( \alpha \) with respect to \( \simeq \) (\( M \) is the component of \( X \) with \( x \in M \); \( M \) is open as \( X \) is locally connected for \( \mathcal{S} \) has UCP). Given \( x \in X, \alpha \in I_x, e > 0 \), \( \delta \geq 0 \) and a function \( g \) on \( I_x \) such that \( \{ g_V = g \circ \xi_{V,x} \mid V \in Ax \} \) is a thread through \( \mathcal{E}_{Ax} \), then for the component \( M \) of \( X \) with \( x \in M \) there is an \( M \) — function \( f \) for \( \alpha \) with respect to \( \simeq \) so that \( f \geq \max \{ g, \delta \} \) on \( M_x = \{ y \in I_x \mid g(y) - g(\alpha) \geq e \} \) and that \( \{ f \circ \xi_{V,x} \mid V \in Ax \} \) is a thread through \( \mathcal{E}_{Ax} \).

Indeed, there is \( W \in Ax \) and \( a \in X_W \) with \( \xi_{W,x}(a) = \alpha \). We may assume \( W \) to be the smallest element of \( \langle Ax \rangle \) and put \( a_V = \xi_{WV}(a) \) for \( V \in Ax \). By 4.3.3C, we can
by induction construct a thread $\mathcal{F} = \{f_\nu \mid V \in A_x\}$ through $\mathcal{E}_{Ax}$ so that $f_\nu$ is an $a_\nu$ - function in $\mathcal{X}_\nu$ and that $f_\nu \geq \max (g_\nu, \delta)$ on $\{b \in X_\nu \mid |g_\nu(b) - g_\nu(a_\nu)| \geq \varepsilon\}$.

Then $f_1 = \lim \mathcal{F}$ is an $\alpha$ - function on $\mathcal{X}_x$ and $f_1 \geq \max (g, \delta)$ on $M$. By 4.3.6A, there is an $M$ - function $f$ for $\alpha$ with $f_\alpha = f_1$ as desired.

We show that the condition 1c of 4.3.8 holds. Given a component $M$ of $X$, $x \in M$, $\alpha \in I_x$, $(K, g) \in S(\alpha)$, $\beta \in I_y$, $\varepsilon > 0$, $\delta \geq 0$, then $g_1 = g/\varepsilon$ is a function on $I_\gamma$ such that $\{g_\nu = g_1 \circ \xi_{\nu, \alpha} \mid V \in A_\gamma\}$ is a thread through $\mathcal{E}_{Ax}$. By the statement T, there is $f \in S(\beta)$ with $f \geq \max (g, \delta)$ on $\{\gamma \in I_\gamma \mid |g(\gamma) - g(\beta)| \geq \varepsilon\}$. By 4.3.6A, $f \geq \max (g, \delta)$ on $\{\gamma \in P_M \mid |g(\gamma) - g(\beta)| \geq \varepsilon\}$ as desired. By S3, S5 and 4.3.6C, we get that 1d of 4.3.8 is fulfilled with $K = M$ ($M$ is connected).

By 2.2.2A and 4.2.5, there is a topology $t_x$ in every stalk $I_x$ which is projectively defined by a set $Dx \subset C^*(\mathcal{X}_x \rightarrow R)$, so that all the $p_U : \mathcal{X}_U \rightarrow (A_U, b_0(st_x^U))$ are homeomorphisms and that $\{f \circ \xi_{U, \alpha} \mid U \in A_x\}$ is a thread through $\mathcal{E}_{Ax}$ for every $f \in Dx$. We show that the condition 3b of 4.3.8 holds. Let $x \in I_x$ and let a $t_x$ - nbd $L$ of $x$ be given. As $t_x$ is projectively made by $Dx$, we may assume that there is $g \in Dx$ with $L = \{\gamma \in I_\gamma \mid |g(\gamma) - g(\alpha)| < \varepsilon\}$. Then $\mathcal{F} = \{g_\nu = g \circ \xi_{\nu, \alpha} \mid V \in A_\alpha\}$ is a thread through $\mathcal{E}$. We set $\delta = \varepsilon$. By the statement T and 4.3.6A, there is $(M, f) \in S(\alpha)$ with $f \geq \max (g, \delta) \geq \delta = \varepsilon$ on $I_\alpha - L$, so $\{\gamma \in I_\gamma \mid f(\gamma) \leq \varepsilon\} \subset M$ as desired. From the assumption S1 and from 4.1.17B it follows that the canonical maps $p_U : H_U \rightarrow A_U$ which belong to the $\mathcal{E} - \hull(\mathcal{E}_x^U - \hull)$ of $\mathcal{E}_x$ by $\mathcal{E}$ (by $\mathcal{E}_x = \{\mathcal{X}_v(F_U) \mid U \in \mathcal{B}(X)\}$) are 1-1. By 4.1.5 and 4.1.1E, $\mathcal{E}$ fulfills COND from 4.1.1E. Thus the conditions of 3.3.8 are fulfilled which proves the theorem.

The foregoing theorem can be easily strengthened in the sense of 4.3.7.

4.3.10. Definition. Let a presheaf $\mathcal{S} = \{\mathcal{X}_U = (X_U, \tau_U) \mid \mathcal{O}_{UV}(X)\}$ from TOP and a nonempty set $\mathcal{D}(X) \subset \mathcal{B}(X)$ be given. $\mathcal{S}$ is called topologically $\mathcal{D}(X)$ - projective if for every $U \in \mathcal{D}(X)$ and any open cover $\mathcal{V}$ of $U$, provided $V \cap W \in \mathcal{V}$, if $V, W \in \mathcal{V}$ we have $\mathcal{X}_U = \lim_\mathcal{V} \{\mathcal{X}_V \mid \mathcal{O}_{UV}(X) \langle \mathcal{V} \subseteq \rangle \}$ see 4.1.1A. If $\mathcal{D}(X) = \mathcal{B}(X)$ then $\mathcal{S}$ is called topologically projective. (The same definition can be given in terms of CLOS, UNIF, ...)

4.3.11. Remark. The following assertions are equivalent:

1) $\mathcal{S}$ is topologically $\mathcal{D}(X)$ - projective,

2) $\mathcal{S}$ is $\mathcal{D}(X)$ - projective (see 4.3.7B), fulfills COND from 4.1.1E, and the topology $\tau_U$ is projectively defined by the set of maps $\mathcal{X} = \{\mathcal{O}_{UV} : X_U \rightarrow \mathcal{X}_V \mid V \in \mathcal{V}\}$ for every $U \in \mathcal{D}(X)$ and any open cover $\mathcal{V}$ of $U$.

Proof. 1 $\Rightarrow$ 2: Let an open cover $\mathcal{V}$ of $U \in \mathcal{D}(X)$ and a $\mathcal{V}$ - smooth family $\mathcal{H} = \{a_\nu \in X_V \mid V \in \mathcal{V}\}$ be given (see 4.2.1). As we can add all the finite intersections of the sets from $\mathcal{V}$ to $\mathcal{V}$ and adapt $\mathcal{H}$ accordingly, we may assume $V \cap W \in \mathcal{V}$ if $V, W \in \mathcal{V}$. Setting $h_\nu(\mathcal{H}) = a_\nu$, we get a family between $\{\mathcal{H}\}$ and $\mathcal{S}_\nu$ (see 0.6), thus there is a unique map $f : \{\mathcal{H}\} \rightarrow \mathcal{X}_\nu$ so that $\mathcal{O}_{UV} \circ f = h_\nu$ for all $V \in \mathcal{V}$. Thus
for \( a = f(\mathcal{H}) \in X_U \) we have \( \varrho_{UV}(a) = a_V \) for all \( V \in \mathcal{V} \), hence \( \mathcal{S} \) is \( \mathcal{D}(X) \) – projective.

If \( a, b \in X_U, a_V = \varrho_{UV}(a) = \varrho_{UV}(b) = b_V \) for all \( V \in \mathcal{V} \), then for \( \mathcal{H} = \{ a_V \mid V \in \mathcal{V} \} \) we have \( \mathcal{H} = \{ b_V \mid V \in \mathcal{V} \} \). As \( f \) is the unique map with \( \varrho_{UV} \circ f(\mathcal{H}) = a_V \) we have \( f(\mathcal{H}) = a \), so \( a = b \). Thus \( \mathcal{S} \) fulfills COND. Clearly, \( \tau_U \) is projectively defined by \( \mathcal{S} \).

2 \( \Rightarrow \) 1: Given an open cover \( \mathcal{V} \) of \( U \in \mathcal{D}(X) \), a topological space \( \mathcal{R} = (R, t) \) and a family \( \{ f_V : R \to \mathcal{H} \mid V \in \mathcal{V} \} \) between \( \mathcal{R} \) and \( \mathcal{S} \), we may assume \( V \cap W \in \mathcal{V} \) if \( V, W \in \mathcal{V} \) as above. If \( x \in R \) then \( \{ f_V(x) = a_V \mid V \in \mathcal{V} \} \) is \( \mathcal{V} \) – smooth, so there is \( a \in X_U \) with \( \varrho_{UV}(a) = a_V \) for all \( V \in \mathcal{V} \). We set \( f(x) = a \). Then \( f : R \to X_U \) is continuous (for \( \varrho_{UV} \circ f = f_V \) are), and from COND it follows that \( f \) is unique. By 0.6, \( \mathcal{X}_U = \lim \mathcal{S}_\mathcal{V} \).

4.3.12. **Remark.** Given a topologically \( \mathcal{D}(X) \) – projective sheaf presheaf \( \mathcal{S} = \{ \mathcal{X}_U = (X_U, \tau_U) \mid \varrho_{UV} \mid V \in \mathcal{V} \} \) from CLOS, suppose that for every \( x \in X \) and any open nbhd \( V \) of \( x \) there is \( D \in \mathcal{D}(X) \) with \( x \in D \subset V \). Let us have an open cover \( \mathcal{V} \) of \( X \) such that for every \( W \in \mathcal{V} \) there is a closure (topology) \( t_W \) in \( P_W \) (see 4.3.4B; clearly, \( P_W \) is the covering space of the presheaf \( \mathcal{S}_W = \{ \mathcal{X}_U \mid \varrho_{UV} \mid W \} \). Then there is a closure (topology) \( t \) in \( P \) with the following properties:

A. If for any \( W \in \mathcal{V} \) the canonical maps \( p_U : \mathcal{X}_U \to (A_U, b_U(t_W)) \) are homeomorphisms for all \( U \in \mathcal{D}(X) \) with \( U \subset W \), then \( p_U : \mathcal{X}_U \to (A_U, b_U(t)) \) are homeomorphisms for all \( U \in \mathcal{D}(X) \). Especially, if \( \mathcal{S} \) is topologically projective, then \( p_U : \mathcal{X}_U \to (A_U, b_U(t)) \) are homeomorphisms for all \( U \in \mathcal{D}(X) \).

B. If for any \( W \in \mathcal{V} \) we have \( A_U = \Gamma(U, t_W) \) for all \( U \in \mathcal{D}(X) \) with \( U \subset W \), then \( A_u = \Gamma(U, t) \) for all \( U \in \mathcal{D}(X) \). Especially, if \( \mathcal{S} \) is topologically projective, then \( A_u = \Gamma(U, t) \) for all \( U \in \mathcal{D}(X) \).

**Proof.** A: If \( x \in X \), we put \( st x = \{ W \in \mathcal{V} \mid x \in W \} \). If \( x \in I_x \) and \( F \subset st x \) is finite, and if \( N_W \) is a \( t_W \) – nbnd of \( x \) for \( W \in F \), we set \( E_F = \bigcap N_W \mid W \in F \}, \mathcal{B}_x = \{ E_F \mid F \subset st x, F \} \). Then \( \mathcal{B}_x \) is a filter base round \( x \). These define a closure \( t \) in \( P \); if all the \( t_W \) are topologies then so is \( t \) as well.

Let \( t_x(t_W) \) be the closure or the topology induced in \( I_x \) by \( t \) (by \( t_W \) for \( W \in st x \). By 4.2.1, the closure or the topology \( b_U(t) \) is projectively defined by the canonical maps \( \{ \eta_{U,x} : A_U \to (I_x, t_x) \mid x \in U \} \) (see 4.1.1E). Thus \( p_U : \mathcal{X}_U \to (A_U, b_U(t)) \) is continuous iff so is \( \eta_{U,x} \circ p_U = \xi_{U,x} : \mathcal{X}_U \to (I_X, t_x) \) for every \( x \in U \). It follows from the definition of \( t \) that \( t_x(t_W) \) is projectively defined by the identities \( \{ i : I_x \to (I_x, t_x) \mid W \in st x \}, \mathcal{B}_x \), so \( \xi_{U,x} : \mathcal{X}_U \to (I_X, t_x) \) is continuous iff it is \( \mathcal{U}_x : \mathcal{X}_U \to (I_X, t_W) \) for all \( W \subset st x \). If \( W \subset st x, D \in \mathcal{D}(X) \), \( x \in D \subset V \) then by the definition of \( b_U(t_W) \) and 4.1.4A, the canonical maps \( \xi_{D,x} : \mathcal{X}_D \to (I_x, t_W) \) are continuous for all \( x \in D \). If \( U \in \mathcal{D}(X) \), \( x \in U \), \( W \subset st x \) then there is \( D \in \mathcal{D}(X) \) with \( x \in D \subset U \cap W \), and \( \xi_{U,x} = \xi_{D,x} \circ \varrho_{UD} : \mathcal{X}_D \to (I_x, t_W) \). As \( \varrho_{UV} : \mathcal{X}_U \to \mathcal{X}_V \) and \( \xi_{D,x} : \mathcal{X}_D \to (I_x, t_W) \) are continuous, \( \mathcal{X}_U \) is. Hence \( p_U : \mathcal{X}_U \to (A_U, b_U(t)) \) is continuous for all \( U \in \mathcal{D}(X) \). If \( U \in \mathcal{D}(X) \) then there is an open cover \( \mathcal{C} \) of \( U \subset \mathcal{C} \subset \mathcal{D}(X) \) such that for every \( C \in \mathcal{C} \) there is \( W \in \mathcal{V} \) with \( C \subset W \). Then \( p_C : \mathcal{X}_C \to (A_C, b_C(t_W)) \) is a homeomorphism for every
\[ C \in \mathcal{C} \] and any \( W \in \mathcal{V} \) with \( C \subseteq W \). Let \( W \in \mathcal{V} \), \( C \in \mathcal{C} \), \( C \subseteq W \). As \( b_c(t) \) is finer than \( b_c(t_w) \), so \( p_c^{-1} : (A_c, b_c(t)) \to \mathcal{X}_C \) is continuous. The map \( p_u^{-1} : (A_U, b_u(t)) \to \mathcal{X}_U \) is continuous iff so is \( \varrho_{UC} \circ p_u^{-1} : (A_U, b_u(t)) \to \mathcal{X}_C \) for all \( C \in \mathcal{C} \) as \( \mathcal{S} \) is topologically \( \mathcal{D}(X) \) — projective. But \( \varrho_{UC} \circ p_u^{-1} = p_c^{-1} \circ \sigma_{UC} \), where \( \sigma_{UC} = p_c \circ \varrho_{UC} \circ p_u^{-1} : (A_U, b_u(t)) \to (A_C, b_c(t)) \). By 4.1.4A, \( \sigma_{UC} \) is continuous, hence so is \( p_u^{-1} \) as desired.

B. Let \( U \in \mathcal{D}(X), r \in \Gamma(U, t) \). Take the cover \( \mathcal{C} \) of \( U \) from the proof of A. If \( C \in \mathcal{C} \) and \( W \in \mathcal{V} \) with \( C \subseteq W \) then clearly \( r|C \in \Gamma(C, t_w) \), so there is \( a_c \in X_C \) with \( \xi_{C,x}(a_c) = r(x) \) for all \( x \in C \). As \( \mathcal{S} \) fulfills COND (see 4.3.11), the family \( \{ a_c \mid C \in \mathcal{C} \} \) is \( \mathcal{C} \) — smooth. As \( \mathcal{S} \) is \( \mathcal{D}(X) \) — projective, there is \( a \in X_U \) with \( \varrho_{UC}(a) = a_c \) for all \( C \in \mathcal{C} \). Thus \( r(x) = \xi_{U,x}(a) \) for all \( x \in U \), so \( \Gamma(U, t) \subseteq A_U \). The other inclusion follows easily from the definition of \( t \). If \( \mathcal{S} \) is topologically projective, we can set \( \mathcal{D}(X) = \mathcal{B}(X) \) and the special statements follow.

References


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