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STABILIZATION OF SOLUTIONS OF ABSTRACT PARABOLIC EQUATIONS

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In this paper we investigate the stabilization and the rate of stabilization for $t \to \infty$ of the solutions of the equations

$$u'(t) + A(t)u(t) = f(t) \quad (0 < t < \infty), \quad u(0) = u_0,$$

where $A(t)$ ($t \geq 0$) are monotone, coercive, in general non-linear operators from a real, reflexive $B$-space $V$ into its dual space $V^*$. Let $H$ be a real Hilbert space. We assume that the imbedding $V \subset H$ is continuous and that $V$ is dense in $H$. Under sufficiently general conditions which guarantee the existence and uniqueness of the solution $u(t)$ of (1) (see Remarks 1 and 2) we prove in §1 that $u(t) \to 0$ in $H$ for $t \to \infty$ provided $f(t)$ decays for $t \to \infty$ in some sense. If $A(t)$ is a strictly or strongly monotone operator (see (131), (132), (133)) then $u(t) \to u_\alpha$ in $H$ for $t \to \infty$ provided $f(t)$ tends to $f_\alpha$ and $A(t)$ tends to $A_\alpha$ for $t \to \infty$ (see (92), (12)), where $u_\alpha$ is the solution of the stationary equation $A_{\alpha}u_\alpha = f_\alpha$. (If $A(t) \equiv A$, then $A_\alpha = A$). In §1 we obtain results which are modifications of those in [5], [6], [11]. In §2 we study the rate of the stabilization of $u(t)$ for $t \to \infty$. For a certain class of stationary operators $A$ we prove that the solution $u(t)$ stabilizes in finite time, i.e., there exists $t_0 = t_0(u_0)$ such that $u(t) = 0$ for $t \geq t_0$ provided $A(t) \equiv A$ and $f(t) \equiv 0$. If $f : (0, T) \to H$ is continuously differentiable in $t$ and of bounded variation on $[0, \infty)$ then we prove that $u(t) \to u_\alpha$ also in the norm of the space $V$. In §3 we present some applications of the results from §1 and §2 to parabolic initial-boundary value problems.

NOTATION AND DEFINITIONS

Denote by $\| \cdot \|$, $\| \cdot \|_*$ and $\| \cdot \|$ the norms in $V$, $V^*$ and $H$, respectively. If we identify $H$ with its dual $H^*$ then we have

$$V \subset H \subset V^*.$$
The duality between \( v \in V \) and \( f \in V^* \) will be denoted by \((f, v)\). If \( f, v \in H \) then 
\((f, v)\) coincides with the scalar product in \( H \).

Let \( X \) be an arbitrary Banach space (\( X^* \) its dual space) and \( 0 < T \leq \infty \). By 
\( L_p(0, T; X) \equiv Z (1 \leq p \leq \infty) \) we denote the Banach space (see, e.g., [15], [7]) of 
all measurable abstract functions \( v : (0, T) \to X \) satisfying 
\[
\left\| v \right\|_Z = \int_0^T \left\| v(t) \right\|_X \, dt < \infty \quad \text{for} \quad 1 \leq p < \infty
\]
and 
\[
\left\| v \right\|_X = \sup_{t \in (0, T)} \left\| v(t) \right\|_X < \infty \quad \text{for} \quad p = \infty.
\]

Henceforth, let \( p > 1, q \geq 1 \) be conjugate numbers \((p^{-1} + q^{-1} = 1)\). The dual 
space \( Z^* \) to \( Z \) is \( L_q(0, T; X^*) \) (see, e.g., [7]). By \( C(0, T; X) \) \((C^1(0, T; X))\) we denote 
the space of continuous (continuously differentiable) abstract functions \( v : (0, T) \to X \). By \( C_q(0, T; X) \) we denote the set of all abstract functions \( v : (0, T) \to X \) satisfying 
\((x^*, v(t)) \in C(0, T) \) for all \( x^* \in X^* \). The abstract function \( du/dt : (0, T) \to X \) is 
the weak derivative of \( u(t) \), iff 
\( (d/dt)(x^*, u(t)) = (x^*, du(t)/dt) \) for all \( x^* \in X^* \). 
We denote \( C^q_u(0, T; X) = \{ v : (0, T) \to X \) for which \( dv/dt \in C_q(0, T; X) \}. \) If \( dv/dt \in 
\in L_p(0, T, X) \) then there exists \( v' \) (the strong derivative) and \( v'(t) = dv(t)/dt \) for a.e. 
\( t \in (0, T) \).

We shall assume that \( f(t) \) is an abstract function \( f : (0, \infty) \to V^* \) such that 
\( f \in L_q(0, T; V^*) \) (for all \( T < \infty \)) and \( u_0 \) from (1) is an element of \( H \). In some special 
cases \( f \) and \( u_0 \) will be supposed to be more regular.

Under the solution of (1) we understand an abstract function \( u : (0, \infty) \to V \) 
with the following properties: \( u \in L_p(0, T; V) \), \( u' \in L_q(0, T; V^*) \), \( u(0) = u_0 \) and \( u(t) \) 
satisfies (1) for a.e. \( t \in (0, \infty) \).

In the following remarks we introduce some results concerning existence and 
uniqueness of the solution of (1).

**Remark 1.** From [1], [2], [3], the following results follows: If the following 
assumptions hold:
\[\begin{align*}
  &a_1) \; A(t) : V \to V^* \text{ (for } t \geq 0) \text{ is demicontinuous}, \\
  &b_1) \; (A(t)v, w) \text{ is measurable in } t \text{ for all fixed } v, w \in V, \\
  &c_1) \; (A(t)v - A(t)w, v - w) \geq 0 \text{ for all } t > 0 \text{ and } v, w \in V, \\
  &d_1) \; \|A(t)v\| \geq C_1 \|v\|^p - C_2, \; C_1 > 0, \; 1 < p < \infty, \\
  &e_1) \; \|A(t)v\| \leq C(1 + \|v\|^{p^*}) \text{ for all } t > 0, \\
  &f_1) \; f \in L_q(0, T; V^*) \text{ for all } T < \infty, \\
  &g_1) \; u_0 \in H, \\
\end{align*}\]
then there exists a unique solution of (1).

**Remark 2.** Existence of a more regular solution of (1) can be guaranteed by 
stronger assumptions on \( f(t), A \) and \( u_0 \) as in Remark 1. Let \( V \) and \( H \) be separable 
spaces and let \( A(t) \equiv A \).

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If the following assumptions are satisfied:

a. \( A : V \to V^\ast \) is demicontinuous and bounded,
b. \( (Av - Aw, v - w) \geq 0 \) for all \( v, w \in V \),
c. \( \|v\|^{-1}(Av, v) \to \infty \) for \( \|v\| \to \infty \),
d. \( u_0 \in V \) and \( Au_0 \in H \),
e. \( f : (0, T) \to H \) is Lipschitz continuous on each compact subset of \( (0, \infty) \),

then there exists a unique solution \( u(t) \) of (1) (see, e.g., [5], [6]) with the following properties:

\( u : (0, \infty) \to H \) is Lipschitz continuous on each compact subset of \( (0, \infty) \), \( u \in L_\infty(0, T; V) \), \( u' \in L_\infty(0, T; H) \) and \( Au \in L_\infty(0, T; H) \).

Moreover, if \( f \in C^1(0, T; H) \) then \( u \in C_\infty(0, T; H) \), \( Au \in C_\infty(0, T; H) \) and if we replace \( u'(t) \) by \( du(t)/dt \) then (1) is valid for all \( t > 0 \) (see [8], [9]). The estimate

\[
\left| \frac{du(t)}{dt} \right| \leq |f(0)| + |Au|_0 + \int_0^T |f'(t)| \, dt
\]

holds (see [8], Remark 2 and Lemma 5). A similar result (but under some additional assumptions) is proved also for the nonstationary case \( A(t) \neq A \) in [10].

Positive constants will be denoted by \( C \) and the dependence of \( C \) on the parameter \( \epsilon \) by \( C(\epsilon) \). Constants \( C \) and \( C(\epsilon) \) may denote also various constants in the same discussion.

In this paper we assume that there exists a unique solution (in the previously defined sense) \( u(t) \) of (1). Since \( u \in L_p(0, T; V) \) and \( u' \in L_\infty(0, T; V^\ast) \), we have \( u \in C(0, T; H) \) for all \( T < \infty \) and

\[
|u(r)|^2 - |u(s)|^2 = 2 \int_s^r (u'(t), u(t)) \, dt
\]

for all \( 0 \leq r, s < \infty \) (see [1], [7]).

Let \( \gamma(t) \) be a continuous function satisfying: \( \gamma(0) = 0 \), \( \gamma(t) > 0 \) for \( t > 0 \) and there exists \( \delta > 0 \) and \( t_0 > 0 \) such that \( \gamma(t) > \delta \) for \( t \geq t_0 \).

Coerciveness of \( A(t) \) will be assumed in some of the following forms:

\( (3_1) \) \( (A(t) v, v) \geq 0 \),
\( (3_2) \) \( (A(t) v, v) \geq \gamma(\|v\|) \),
\( (3_3) \) \( (A(t) v, v) \geq C \|v\|^p \) \( (1 < p < \infty) \).

Clearly, \( (3_2) \) implies \( (3_1) \). We shall assume \( f(t) \) to have the following properties:

\( (4_1) \) \( f \in L_1(0, T; H) \),
\( (4_2) \) \( f \in L_\infty(0, \infty; V^\ast) \),
\( (4_3) \) \( f \in L_\infty(0, T; V^\ast) \) for all \( T < \infty \).
Lemma 1. Let one of the assumptions i) or ii) be satisfied, where

i) \((3_1), (4_1)\).

ii) \((3_3), (4_2)\).

Then \(u \in L_\infty(0, \infty; H)\).

Proof. i) From (1) we deduce

\[
|u(t)|^2 - |u(0)|^2 \leq 2 \int_0^t |f(s)| |u(s)| ds,
\]

which implies \((u \in C(0, t), H))\)

\[
\max_{0 \leq \xi \leq t} |u(\xi)|^2 \leq |u(0)|^2 + 2 \max_{0 \leq \xi \leq t} |u(\xi)| \int_0^t |f(s)| ds.
\]

From this inequality we easily obtain

\[
|u(t)| \leq |u(0)| + 2 \int_0^\infty |f(s)| ds
\]

for all \(t \geq 0\) which proves the assertion.

ii) In this case (5) and (3_3) imply

\[
(u'(t), u(t)) + C \|u(t)\|^p \leq \|f(t)\| \|u(t)\| \leq \frac{\varepsilon - q}{q} \|f(t)\|^q + \frac{\varepsilon^p}{p} \|u(t)\|^p,
\]

where Young’s inequality has been used (\(\varepsilon > 0\)). Integrating (6) over \(s, r\) for a suitable \(\varepsilon\) we obtain

\[
|u(r)|^2 - |u(s)|^2 + C \int_s^r \|u(t)\|^p dt \leq C_1 \int_s^r \|f(t)\|^p dt
\]

where \(C_1 = C_1(\varepsilon)\). From (7) (for \(s = 0\)) and (4_2) we deduce the required result.

Theorem 1. Let one of the assumptions i) or ii) be satisfied, where

i) \((3_2), (4_1)\).

ii) \((3_3), (4_2)\).

Then \(u(t) \to 0\) in \(H\) for \(t \to \infty\).

Proof. i) Integrating (5) over the interval \(s, r\) and using (3_2) we obtain

\[
|u(r)|^2 - |u(s)|^2 + 2 \int_s^r \gamma(\|u(t)\|) dt \leq 2 \int_s^r |f(t)| |u(t)| dt.
\]
Using Lemma 1 and (4.1), we deduce from (8) that
\[ \int_0^\infty \gamma(\|u(t)\|) \, dt < \infty \]
which implies: There exists a subsequence \( \{t_n\} \), \( t_n \to \infty \) for \( n \to \infty \), such that \( \|u(t_n)\| \to 0 \) for \( n \to \infty \). Thus, \( |u(t_n)| \to 0 \) for \( n \to \infty \) since \( V \subset H \). From this fact and from Lemma 1, (4.1) and (8) we obtain the required result.

ii) From (7) (for \( s = 0 \)), (4.2) and Lemma 1 we deduce
\[ \int_0^\infty \|u(t)\|^p \, dt < \infty . \]

Hence, using (7) and (4.2), by the same argument as in Assertion i) we deduce the required result.

Let \( f_\infty \) be an element of the space \( H \) or \( V^* \). We shall assume that \( f(t) \) tends to \( f_\infty \) for \( t \to \infty \) in the following sense:

(9.1) \[ \int_0^\infty |f(t) - f_\infty| \, dt < \infty , \]
(9.2) \[ \int_0^\infty \|f(t) - f_\infty\|^q \, dt < \infty . \]

Let \( A_\infty \) be an operator from \( V \) into \( V^* \) and let \( u_\infty \in V \) be a solution of the equation

(10) \[ A_\infty u_\infty = f_\infty . \]

We shall assume that \( A(t) \) tends to \( A_\infty \) for \( t \to \infty \) in the following sense:

(11) \[ \int_0^\infty \|A(t) u_\infty - A_\infty u_\infty\|^q \, dt < \infty . \]

Assumption (11) is clearly satisfied, if

(12) \[ \int_0^\infty \|A(t) v - A_\infty v\|^q \, dt < \infty \]
holds for all \( v \in V \). In particular, if \( A(t) \equiv A \) for \( t > 0 \), then \( A \equiv A_\infty \).

Monotonicity of \( A(t) \) will be considered in the form

(13.1) \[ (A(t) v - A(t) w, v - w) > 0 \] for all \( v, w \in V, \ v \neq w \),
(13.2) \[ (A(t) v - A(t) w, v - w) \geq \gamma(\|v - w\|) \] for all \( v, w \in V \),
(13.3) \[ (A(t) v - A(t) w, v - w) \geq C\|v - w\|^p \] (1 < \( p < \infty \))
for all \( v, w \in V \). Clearly, (13.2) implies (13.1).
Theorem 2. Suppose (10). Let one of the assumptions i) or ii) be satisfied, where
i) \((9_1), (13_2), A(t) \equiv A\),
ii) \((9_2), (13_3), (11)\).
Then \(u(t) \to u_\infty \) in \(H\) for \(t \to \infty\).

Proof. i) From (10) and (1) we obtain

\[
(u'(t), u(t) - u_\infty) + \langle A u(t) - A u_\infty, u(t) - u_\infty \rangle =
(f(t) - f_\infty, u(t) - u_\infty).
\]

Integrating (14) over \(\langle s, r \rangle\) and using (13_2) we deduce

\[
|u(r) - u_\infty|^2 - |u(s) - u_\infty|^2 + 2 \int_s^r \gamma(\|u(t) - u_\infty\|) \, dt \leq
\]

\[
\leq 2 \int_s^r |f(t) - f_\infty| \, |u(t) - u_\infty| \, dt.
\]

From (15) and (9_1) similarly as in Lemma 1, we deduce \(u \in L_\infty(0, \infty; H)\). Hence, from (15) we conclude

\[
\int_0^\infty \gamma(\|u(t) - u_\infty\|) \, dt < \infty.
\]

From this fact, analogously as in Theorem 1, the required result follows.

ii) From (1) and (10) we have

\[
(u'(t), u(t) - u_\infty) + \langle A(t) u(t) - A(t) u_\infty, u(t) - u_\infty \rangle =
(f(t) - f_\infty, u(t) - u_\infty) - \langle A(t) u_\infty - A_\infty u_\infty, u(t) - u_\infty \rangle.
\]

Using (13_3), (9_2), Hölder's and Young's inequalities in (16) we obtain

\[
(u'(t), u(t) - u_\infty) + C\|u(t) - u_\infty\|^p \leq \frac{2\varepsilon^p}{p} \|u(t) - u_\infty\|^p +
\]

\[
+ \frac{\varepsilon^{-q}}{q} (\|f(t) - f_\infty\|^q + \|A(t) u_\infty - A_\infty u_\infty\|^q).
\]

Integrating (17) over the interval \(\langle s, r \rangle\) for a suitable \(\varepsilon > 0\) we deduce

\[
|u(r) - u_\infty|^2 - |u(s) - u_\infty|^2 + C_1 \int_s^r \|u(t) - u_0\|^p \, dt \leq
\]

\[
\leq C_2 \int_s^r (\|f(t) - f_\infty\|^q + \|A(t) u_\infty - A_\infty u_\infty\|^q) \, dt
\]

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\((C_1 = C_1(e))\). Hence, analogously as in the previous part we successively deduce

\[
u \in L_\infty(0, \infty; H) , \int_0^\infty \|u(t) - u_\infty\|^p \, dt < \infty
\]

and then the required result.

**Consequence.** Theorem 2 implies that the solution \(u_\infty\) of (10) is unique in \(V\).

**Remark 3.** If in (3.2), (13.2) \(\|v - w\|\) is replaced by \(|v - w|\), then Theorems 1 and 2 remain true. Moreover, in this case the assumption \(V \subset H\) can be weakened to the assumption that \(V \cap H\) is dense in \(V\) and \(H\).

**Theorem 3.** Suppose \(A(t) = A, (9.1), (10)\) and \(c_2, d_2\) (from Remark 2). Assume that \(f : (0, \infty) \to H\) is continuously differentiable and satisfies

\[
\int_0^\infty |f'(t)| \, dt < \infty .
\]

i) If the imbedding \(V \subset H\) is compact and (13.1) holds then \(u(t) \to u_\infty\) in \(H\) for \(t \to \infty\).

ii) If (13.2) holds then \(u(t) \to u_\infty\) in \(V\) for \(t \to \infty\).

**Proof.** i) From the estimates (2), (18) and the equation

\[
\frac{du(t)}{dt} + Au(t) - Au_\infty = f(t) - f_\infty \quad \text{for all } t > 0
\]

(see Remark 2) we deduce that there exist \(C_1, C_2\) such that

\[
\left| \frac{du(t)}{dt} \right| \leq C_1 \quad \text{for all } t > 0
\]

and

\[
|Au(t)| \leq C_2 \quad \text{for all } t > 0 .
\]

From (21) and \(c_2\) we conclude

\[
\|u(t)\| \leq C_3 , \quad |u(t)| \leq C_4 \quad \text{for all } t > 0
\]

\((C_3, C_4\) are suitable constants) since \(|Au(t)| \geq \|Au(t)\| \geq \|u(t)\|^{-1} (Au(t), u(t))\) and \(V \subset H\). Hence, integrating (14) over \((0, \infty)\) we obtain the estimate

\[
\int_0^\infty (Au(t) - Au_\infty, u(t) - u_\infty) \, dt \leq C_5 \left( \int_0^\infty |f(t) - f_\infty| \, dt + 1 \right) .
\]
Thus, there exists a sequence \( \{t_n\} \), \( t_n \to \infty \) with \( n \to \infty \) such that

\[
(A \ u(t_n) - Au_\infty, u(t_n) - u_\infty) \to 0 \quad \text{with} \quad n \to \infty.
\]

From (21), (22), from the reflexivity of the space \( V \) and from the compactness of the imbedding \( V \subset H \) we conclude that there exists \( y \in H \) and \( v \in V \cap H \) such that

\( A u(t_n) \to y \) in \( H \) (weak convergence in \( H \)) and \( u(t_n) \to v \) in \( H \) for \( k \to \infty \) (\( \{t_n\} \) is a suitable subsequence of \( \{t_n\} \)). From these facts and the monotonicity of \( A \) we deduce easily \( y = Av \). Then (23) implies \( (Av - Au_\infty, v - u_\infty) = 0 \) and hence (13) yields \( v = u_\infty \). From \( u(t_n) \to u_\infty \) in \( H \), (9) and the formula

\[
|u(r) - u_\infty|^2 - |u(s) - u_\infty|^2 \leq C \int_s^r |f(t) - f_\infty| \, dt
\]

we obtain the required result.

ii) From (23) we deduce \( u(t_n) \to u_\infty \) in \( V \) for \( n \to \infty \) and hence \( u(t_n) \to u_\infty \) in \( H \) for \( t \to \infty \). Thus, from (24) we conclude \( u(t) \to u_\infty \) in \( H \) for \( t \to \infty \). On the other hand, from (19), (13) and from the estimates (22) we obtain the estimate

\[
\gamma(t) = C \gamma \leq C \gamma |u(t) - u_\infty| \quad \text{for} \quad t > 0
\]

which yields the required result.

2

Estimating the rate of stabilization of the solution \( u(t) \) of (1) (for \( t \to \infty \)) we use the following assertion on the asymptotical behaviour for the solution \( y(t) \) of the equation

\[
y'(t) = -C_0 y(t)^\alpha + \varphi(t) \quad (0 < t < \infty, \ C_0 > 0)
\]

where \( y(0) \geq 0, 0 < \alpha \), and \( \varphi(t) \) is a measurable nonnegative function.

**Assertion 1.** a) If \( \varphi(t) \to 0 \) for \( t \to \infty \), then \( y(t) \to 0 \) for \( t \to \infty \).

b) Let \( 0 < \alpha < 1 \).

i) If \( \varphi(t) \equiv 0 \), then \( y(t) = 0 \) for \( t \geq y(0)^{1-\alpha}/C_0(1-\alpha) \) \( (C_0 \text{ is from (25)}) \).

ii) If \( \varphi(t) = O(t^{-\beta}) \ (\beta > 1) \), then \( y(t) = O(t^{-\beta+1}) \).

c) Let \( \alpha = 1 \).

i) If \( \varphi(t) = O(t^{-\beta}) \ (\beta > 1) \), then \( y(t) = O(t^{-\beta}) \).

ii) If \( \varphi(t) = O(e^{-\beta t}) \ (\lambda > 0) \), then \( y(t) = O(e^{-\delta t}) \) where \( \delta = \min(C_0, \lambda) \).

d) Let \( 1 < \alpha < \infty \). If \( \varphi(t) = O(t^{-\beta}) \ (\beta > 1) \), then \( y(t) = O(t^{-\delta}) \), where \( \delta = \min(1/(\alpha - 1), \beta/\alpha, \beta - 1) \).
Remark 4. Assertion 1, d) and Assertion 1, b) (ii)) can be deduced from a more general result due to Hardy (see [13], Chap. V, Theorem 3, where \( \alpha, \beta \) are integers) via the transformation \( u^* = y \) if \( \alpha = r/s \) and \( z^m = t \) if \( \beta = n/m \).

**Theorem 4.** Suppose (3)_1, (4)_1 and \( f(t) \to 0 \) in \( V^* \) for \( t \to \infty \). Then \( u(t) \to 0 \) in \( H \) for \( t \to \infty \). Moreover, if \( f(t) \equiv 0 \) and \( 1 < p < 2 \) then \( u(t) = 0 \) for \( t \geq 2C_1|u_0|^{2-p}|C(2-p)} \) (\( C \) is from (3)_1 and \( C_1 \) is from (27)).

**Consequence of Theorem 4.** If (3)_1 (for \( 1 < p < 2 \)) holds then the converse problem

\[
\begin{align*}
  u'(t) + A(t)u(t) &= 0 \quad 0 < t < T, \\
  u(T) &= 0
\end{align*}
\]

has many different solutions for sufficiently big \( T \).

In the following theorems we assume that (10) is satisfied and \( u(t) \) is a solution of (1).

**Theorem 5.** Suppose (4)_2, (13)_2 and \( f(t) \to f_{\infty}, A(t)u_{\infty} \to A_{\infty}u_{\infty} \) in \( V^* \) for \( t \to \infty \). Then \( u(t) \to u_{\infty} \) in \( H \) for \( t \to \infty \).

**Theorem 6.** Let \( p = 2 \) and let (13)_1 hold.

i) If \( \|f(t) - f_{\infty}\|_* = O(t^{-\gamma}) \) and \( \|A(t)u_{\infty} - A_{\infty}u_{\infty}\|_* = O(t^{-\beta}) \), then

\[
\|u(t) - u_{\infty}\|^2 = O(t^{-\delta}).
\]

ii) If \( \|f(t) - f_{\infty}\|_* = O(e^{-\delta t}) \) and \( \|A(t)u_{\infty} - A_{\infty}u_{\infty}\|_* = O(e^{-\lambda t}) \) (\( \lambda > 0 \)), then

\[
\|u(t) - u_{\infty}\|^2 = O(e^{-\delta t}), \text{ where } \delta = \min(C_2, \lambda) \text{ and } C_2 \text{ is from (28)}.
\]

**Theorem 7.** Let \( p > 2 \) and let (13)_2 hold. If \( \|f(t) - f_{\infty}\|_* = O(t^{-\gamma}) \) and \( \|A(t)u_{\infty} - A_{\infty}u_{\infty}\|_* = O(t^{-\delta}) \), then

\[
\|u(t) - u_{\infty}\|^2 = O(t^{-\delta}), \text{ where}
\]

\[
\delta = \min\left(\frac{2}{p - 2}, \frac{2q}{p}, q\beta - 1\right).
\]

**Proof of Theorems 4—7.** From (17) we deduce the estimate

\[
\frac{d}{dt}\|u(t) - u_{\infty}\|^2 + \left(C - \frac{2p}{p}\right)\|u(t) - u_{\infty}\|^p \leq \frac{e^{-q}}{q}
\]

\[
\leq \frac{e^{-q}}{q}\left(\|f(t) - f_{\infty}\|_*^q + \|A(t)u_{\infty} - A_{\infty}u_{\infty}\|_*^q\right)
\]

for a.e. \( t > 0 \), since \( |u(t)|^2 \) is an absolutely continuous function in \( t \) and

\[
\frac{d}{dt}|u(t)|^2 = 2(u'(t), u(t))
\]
holds for a.e. } t > 0. Due to the imbedding } V \subset H \text{ we have }
\tag{27} \quad |v| \leq C_1 \|v\| \quad \text{for all } \quad v \in V

and hence from (20) for a suitable } } \varepsilon > 0 \text{ we deduce }
\tag{28} \quad \frac{d}{dt} |u(t) - u_\infty|^2 \leq -C_2 (|u(t) - u_\infty|^2)^{p/2} +
\quad + C_3 \left( \|f(t) - f_\infty\|^q_\# + \|A(t) u_\infty - A_\infty u_\infty\|^q_\# \right),

where } C_2 = C_2(C, C_1, \varepsilon). \text{ In the case of Theorem 4 we obtain the estimate }
\quad \frac{d}{dt} |u(t)|^2 \leq -C_2 (|u(t)|^2)^{p/2} + C_3 \|f(t)\|^q_\#.

Thus, putting } z(t) = |u(t) - u_\infty|^2, \alpha = \frac{1}{p} \text{ and }
\quad \varphi(t) = C_3 \left( \|f(t) - f_\infty\|^q_\# + \|A(t) u_\infty - A_\infty u_\infty\|^q_\# \right)

we obtain the differential inequality
\tag{29} \quad z'(t) \leq -C_2 z(t) + \varphi(t)

where } z(t) \geq 0, \varphi(t) \geq 0 \text{ for } t > 0. \text{ Comparing any two solutions } y(t) \text{ of (25) and } \quad z(t) \text{ of (29) with } y(0) = z(0) \geq 0 \text{ we conclude that } z(t) \leq y(t) \text{ for all } t > 0. \text{ From this fact and Assertion 1 we successively obtain Theorems 4–7.}

**Theorem 8.** Let } A(t) \equiv A \text{ and let the assumptions of Remark 2 be satisfied. If } (9_1), (18) \text{ and } (13_3) \text{ hold then the estimate }
\quad \|u(t) - u_\infty\| = O(\|u(t) - u_\infty\|^{1/p} + \|f(t) - f_\infty\|^q_{\#})

takes place.

**Proof.** From (19) and (13_3) we deduce
\quad C \|u(t) - u_\infty\|^p \leq \left| \frac{du(t)}{dt} \right| |u(t) - u_\infty| + \|f(t) - f_\infty\|_\# \|u(t) - u_\infty\|.

Hence, using (20) and Young’s inequality, we obtain the required result.

**Remark 5.** In many applications it is more suitable to replace the assumptions
\quad \|A(t) u_\infty - A_\infty u_\infty\|_\# \to 0 \text{ for } t \to \infty \text{ and } \|A(t) u_\infty - A_\infty u_\infty\|_\# = O(\cdot) \text{ in Theorems 5, 6 and 7 by stronger assumptions }
\tag{30} \quad \|A(t) v - A_\infty v\|_\# \to 0 \text{ for } t \to \infty \text{ for all } v \in V

and
\tag{31} \quad \|A(t) v - A_\infty v\|_\# = O(\cdot) \text{ for an arbitrary } v \in V,
which can be directly verified. Then, in Theorems 5, 6 and 7 it suffices to assume the existence of the solution $u_{\infty}$ of (10), which is guaranteed by certain properties of $A_{\infty}$.

3

Let us consider nonlinear parabolic equations of the form

\begin{equation}
\frac{\partial u}{\partial t} + \sum_{|i| \leq k} (-1)^{|i|} D^i a_i(t, x, Du) = f(t, x)
\end{equation}

in the domain $Q = \Omega \times (0, \infty)$, where $\Omega$ is a bounded domain in $E^N$ ($N$-dimensional Euclidean space) with a Lipschitzian boundary $\partial \Omega$, $x \in \Omega$, $t > 0$, $i$ is a multiindex and $Du$ is the vector function $Du = (D^i u, |i| \leq k)$.

The functions $a_i(t, x, \xi)$ $\xi \in E^d$ ($d = \text{card} \{i, |i| \leq k\}$) for $|i| \leq k$ are supposed to be real, defined for $0 \leq t < \infty$, $x \in \Omega$ and $|\xi| < \infty$, continuous in all the variables (it suffices to assume Caratheodory's conditions).

Let us consider the first initial — boundary value problem

\begin{equation}
u(x, 0) = u_0(x), \quad D^l u(x, t)|_{\partial \Omega \times (0, T)} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k - 1,
\end{equation}

where $D^l$ is the outward normal derivative of order $l$ with respect to $\partial \Omega$.

The functions $a_i(t, x, \xi)$ are supposed to satisfy the growth condition

\begin{equation}
|a_i(t, x, \xi)| \leq C(1 + |\xi|^{p-1}) \quad \text{for} \quad |i| \leq k,
\end{equation}

where $1 < p < \infty$. Let $W^k_p$ be the Sobolev space ($W^k_p = \{u \in L_p(\Omega); \quad D^i u \in L_p(\Omega)$ for $|i| \leq k\}$ with the norm $\|u\|_w = \sum_{|i| \leq k} \|D^i u\|_{L_p}$). By the duality form

\begin{equation}
(A(t) v, w) = \sum_{|i| \leq k} \int_{\Omega} D^i w a_i(t, x, Dv) \, dx \quad \text{for} \quad v, w \in W^k_p
\end{equation}

we define an (in general nonlinear) operator

\begin{equation}
A(t) : W^k_p \to W^{-k}_q \quad (W^{-k}_q \text{ is the dual space to } W^k_p),
\end{equation}

which is continuous and bounded because of Nemyckij's theorem;

\begin{equation}
a_{ij}(t, x, \xi) = \frac{\partial a_i(t, x, \xi)}{\partial \xi_j} \quad (|i|, |j| \leq k).
\end{equation}

Remark 6. Monotonicity and coerciveness of $A(t)$ is guaranteed by

\begin{equation}
\sum_{|i| \leq k} [a_i(t, x, \xi) - a_i(t, x, \eta)] (\xi_i - \eta_i) \geq 0,
\end{equation}

\begin{equation}
\sum_{|i| \leq k} a_i(t, x, \xi) \xi_i \geq C_1 |\xi|^p - C_2.
\end{equation}
Remark 7. Let $p \geq 2$. If the estimate
\begin{equation}
\sum_{|i| \leq k} a_{ij}(t, x, \xi) \eta_i \eta_j \geq C \sum_{|i| = k} |\xi_i|^{p-2} \eta_i^2
\end{equation}
holds for all $\xi, \eta \in E^d$ and $t > 0$, then $A(t)$ satisfies (13a) — see [12].

Remark 8. Let $p \geq 2$ and $a_i(t, x, \xi) = g_i(t, x) |\xi_i|^{p-2} \xi_i (|i| \leq k)$, where $g_i(t, x) \in C(\Omega) \cap L_\infty(\Omega) (|i| \leq k)$. If
\begin{equation}
g_i(t, x) \geq C > 0 \quad \text{for all } |i| = k, \quad g_i(t, x) \geq 0 \quad \text{for all } |i| < k
\end{equation}
then we can verify by elementary computation that the operator $A(t)$ generated by $a_i(t, x, \xi) (|i| \leq k)$ satisfies (13a).

Now, let $A(t), A$ be generated by $a_i(t, x, \xi), a_i(x, \xi) (|i| \leq k)$, respectively.

Assertion 2. Let $a_i(t, x, \xi), a_i(x, \xi)$ satisfy (34). If $a_i(t, x, \xi) \to a_i(x, \xi)$ with $t \to \infty$ for all fixed $|i| \leq k, x \in \Omega$ and $|\xi| < \infty$, then (30) holds with $A = A_\omega$.

Proof. We have
\begin{equation}
\|A(t) v - Av\|_* = \sup_{\|z\|_w \leq 1} |(A(t) v - Av, z)| \leq \sum_{|i| \leq k} \|a_i(t, x, Dv) - a_i(x, Dv)\|_{L_q},
\end{equation}
where $\|\cdot\|_*$ is the norm in $W_q^{-k}$. From (34) we deduce the estimate
\begin{equation}
|a_i(t, x, \xi) - a_i(x, \xi)| \leq C(1 + |\xi|^{p-1}) \quad \text{for } |i| \leq k.
\end{equation}
Since
\begin{equation}
|a_i(t, x, Dv) - a_i(x, Dv)|^t \leq C(1 + \sum_{|i| \leq k} |D^i v|^p)
\end{equation}
and $a_i(t, x, Dv) \to a_i(x, Dv)$ with $t \to \infty$ for all $x \in \Omega$, Lebesgue's convergence theorem and (39) yield the required result.

Using the estimate (39) we can estimate also the rate of convergence
\begin{equation}
\|A(t) v - Av\|_* \to 0 \quad \text{for } t \to \infty.
\end{equation}
If, e.g., $a_i(t, x, \xi) = g_i(t, x) a_i(x, \xi) (|i| \leq k)$, where $g_i(t, x)$ are continuous functions for $x \in \Omega, t \geq 0$, then we easily deduce
\begin{equation}
\|A(t) v - Av\|_* = O(\max_{|i| \leq k, x \in \Omega} |g_i(t, x) - 1|).
\end{equation}

Now let us consider a nonhomogeneous problem (32), (33'),
\begin{align}
(33') \quad u(x, 0) &= u_0(x, 0), \quad D_t^l u(x, t)|_{t \not\in (0, \infty)} = D_t^l u_0(x, t)|_{t \not\in (0, \infty)}, \quad l = 0, 1, \ldots, k - 1,
\end{align}

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where $u_0(x, t)$ is a sufficiently smooth function in $\Omega \times (0, \infty)$. Considering $u$ in the form $u = u_0 + z$ we can transform (32) (33') into a homogeneous problem (32*) (33*):

\[
(32*) \quad \frac{\partial z}{\partial t} + \sum_{|\ell| \leq k} (-1)^{|\ell|} D^\ell a^*_i(t, x, Dz) = f^*(x, t),
\]

\[
(33*) \quad z(x, 0) = 0, \quad D^\ell_t z(x, t)|_{\partial \Omega \times (0, \infty)} = 0, \quad l = 0, 1, \ldots, k - 1,
\]

where $a^*_i(t, x, Dz) = a_i(t, x, Du_0 + Dz) (|\ell| \leq k)$, $f^*(x, t) = f(x, t) - \partial u_0/\partial t$.

By means of $a_i(t, x, \xi) (|\ell| \leq k)$ we define the operator $A^*(t)$. If $a_i(t, x, \xi)$ satisfy (34), (35), (36), (37), respectively, then $A^*(t)$ has the corresponding properties as $A(t)$ — see Remarks 6, 7 and 8.

Let $u_0(x), u_0(x, t) W^4_\infty(\Omega)$ (for all $t > 0$). We shall assume

\[
(40) \quad u_0(x, t) \to u_0(x) \text{ in } W^4_\infty(\Omega) \text{ for } t \to \infty
\]

and

\[
(41) \quad \|u_0(x, t)\|_{W^4_\infty(\Omega)} \leq C \text{ for all } t > 0.
\]

By means of $a^*_i(x, \xi) (|\ell| \leq k)$ let us define a stationary operator $A^*$.

**Assertion 3.** Suppose $a_i(t, x, \xi)$ and $a_i(x, \xi) (|\ell| \leq k)$ satisfy (34) and

\[
(42) \quad a_i(t, x, \xi) \to a_i(x, \xi) \text{ with } t \to \infty
\]

for all fixed $x \in \Omega$ uniformly for $\xi$ from a bounded set in $E^d$.

If (34), (40) and (41) are satisfied then (30) holds with $A^*(t)$ and $A^*$.

**Proof.** Analogously as in the proof of Assertion 2 we have

\[
(43) \quad \|A^*(t) v - A^* v\|_* \leq \sum_{|\ell| \leq k} \|a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x) + Dv)\|_{L^q}
\]

and

\[
\|a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x, t) + Dv)\|_q \leq C(1 + \sum_{|\ell| \leq k} |D^\ell v|^p)
\]

because of (34) and (41). Thus, from (41), (42) and Lebesgue's convergence theorem we conclude

\[
(44) \quad (a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x, t) + Dv)) \to 0
\]

with $t \to \infty$ in $L^q(\Omega)$ for all $v \in W^p_\infty$.

Due to the theorem of Nemyckij (see [14]) and (40) we have

\[
(45) \quad a_i(x, Du_0(x, t) + Dv) \to a_i(x, Du_0(x) + Dv) \text{ with } t \to \infty \text{ in } L^q(\Omega)
\]

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for all \( v \in W_p^k \). The inequality
\[
|a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x) + Dv)| \leq \\
\leq |a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x, t) + Dv)| + \\
+ |a_i(x, Du_0(x, t) + Dv) - a_i(x, Du_0(x) + Dv)|
\]
together with (44), (45) and (43) implies the required result.

**Remark 9.** If \( a_i(t, x, \xi) \equiv a_i(x, \xi) \ (|i| \leq k) \) then Assertion 3 holds true if we assume \( u_0(x), u_0(x, t) \in W_p^k(\Omega) \) for all \( t > 0 \) and \( u_0(x, t) \to u_0(x) \) with \( t \to \infty \) in \( W_p^k \) instead of (40), (41).

Investigating (31) we can easily prove

**Assertion 4.** Let \( a_i(t, x, \xi) = g_i(x, t) a_i(x, \xi) |i| \leq k \), where \( g_i(x, t) \ (|i| \leq k) \) are continuous functions in \( \Omega \times (0, \infty) \). Suppose \( u_0(x), u_0(x, t) \in W_p^k(\Omega) \) and \( \|u_0(x, t)\|_W \leq C \) for all \( t > 0 \). If \( a_i(x, \xi) \) satisfy (34) and \( a_i(x, \xi) \) satisfy

\[
|a_i(x, \xi)| \leq C(1 + |\xi|^{p-2}) \quad \text{where} \quad p \geq 2,
\]
then the estimate
\[
\left\| A^*(t) v - A^* v \right\|_* = O\left( \max_{1 \leq k \leq k, x \in \Omega} |g_i(x, t) - 1| + \right. \\
+ \|v\|_W^{-2} \left\| u_0(x, t) - u_0(x) \right\|_W + \left\| u_0(x, t) - u_0(x) \right\|_W^{-1}
\]
takes place.

For the proof we use (43), the formula
\[
a_i(x, D u_0(x, t) + Dv) - a_i(x, Du_0(x) + Dv) = \\
= \int_0^1 \frac{d}{ds} a_i(x, D u_0(x) + s (Du_0(x, t) - u_0(x))) \, ds,
\]
the stimate (46) and Hölder’s inequality.

**Remark 10.** Let \( p \geq 2 \). If \( a_i(t, x, \xi) \) satisfy (34), (37), \( u_0(x, t) \in W_p^k \) for all \( t > 0 \) and \( \partial u_0(x, t)/\partial t, f(x, t) \in L_q(0, T; W_q^{-k}) \) (for all \( T < \infty \) then there exists a unique solution \( u(x, t) \) of (32), (33) — see Remark 1 ((37) implies \( c_1 \) and \( d_1 \)). If \( u_0(x) \in W_p^k \), \( f(x) \in W_q^{-k} \) and \( a_i(x, \xi) \ (|i| \leq k) \) satisfy (34)–(36) then there exists a solution \( u(x) \) of the stationary problem

\[
\sum_{|i|=k} (-1)^{|i|} D^i a_i(x, Du) = f(x),
\]

\[
D^l u(x)|_{\partial \Omega} = D^l u_0(x)|_{\partial \Omega}, \quad l = 0, 1, \ldots, k - 1.
\]

If in (35) the sign > holds for \( \xi \neq \eta \), then the solution \( u(x) \) is unique.

Applying certain results of this section and § 2 we obtain

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Theorem 10. Let \( u(x, t) \) be a solution of (32), (33') and let \( u(x) \) be a solution of (47), (48). Let us assume (40), \( p \geq 2 \) and let \( a_i(t, x, \xi) \) \( (|i| \leq k) \) satisfy (34), (37).

i) Suppose that the assumptions of Assertion 3 or Assertion 4 are satisfied. If
\[
\frac{\partial u_0(x, t)}{\partial t}, \quad f(x, t) \in W_{-k}^k(\Omega) \quad \text{(for all } t > 0) \]
and
\[
\frac{\partial u_0(x, t)}{\partial t} \to 0, \quad f(x, t) \to f(x) \quad \text{in } W_{-k}^k \quad \text{for } t \to \infty, \]
then \( u(x, t) \to u(x) \) in \( L_2(\Omega) \) for \( t \to \infty \).

ii) Suppose \( a_i(t, x, \xi) \equiv a_i(x, \xi) \) \( (|i| \leq k) \), \( u_0(x, t) \equiv u_0(x) \) and \( d_2 \) (form Remark 2). We assume \( f \in C^1((0, \infty), L_2(\Omega)) \),
\[
\int_0^\infty \left\| \frac{\partial f(x, t)}{\partial t} \right\|_{L_2} \, dt < \infty \quad \text{and} \quad \int_0^\infty \left\| f(x, t) - f(x) \right\|_{L_2} \, dt < \infty.
\]
If in (35) the sign \( > \) holds for \( \xi \neq \eta \) and if \( p > pN/(N - kp) \), then \( u(x, t) \to u(x) \) in \( L_2(\Omega) \) for \( t \to \infty \).

iii) Suppose that the assumptions of ii) are satisfied. If \( p \geq 2 \) and if (37) holds, then \( u(x, t) \to u(x) \) in \( W_{-k}^k(\Omega) \) for \( t \to \infty \).

Assertion i) is a consequence of Theorem 5. Theorem 3 implies Assertions ii) and iii).

Applying other results of §§ 1, 2 we can deduce the corresponding results on stabilization of the solution of the initial-boundary value problems (32), (33') and (32), (33), respectively.

The above results can be applied to the following examples.

Example 1. Let \( u(x, t) \), \( u(x) \) be the solutions of the problems
\[
\frac{\partial u}{\partial t} + \sum_{|i| = k} (-1)^{|i|} D^i(g_0(x)) \left| D^i u \right|^{p-2} D^i u = 0, \\
u(x, 0) = u_0(x), \quad D^i u(x, t)|_{\partial \Omega} = D^i u_0(x)|_{\partial \Omega} \quad \text{for } t > 0, \quad l = 0, 1, \ldots, k - 1.
\]
We assume that \( u_0(x) \in W_{-k}^k \) \( (p > 1) \) and that \( g_i(x) \in C(\Omega) \) \( (|i| \leq k) \) satisfy (38). If \( 2 > p \geq p_0 \) then the identity \( u(x, t) \equiv 0 \) holds for \( t \geq 2 C_1(C(2 - p) C_2)^{-1} \).
\[
\left\| u_0(x) \right\|_{L_2}^{2-p}. \quad \text{The constants } C, C_1 \text{ are obtained from (38), (27), respectively, and } C_2 \text{ is obtained from the inequality } \sum_{|i| = k} \left\| D^i u \right\|_{L_p} \geq C_2 \left\| u \right\|_W \text{ for all } u \in W_{-k}^k(\Omega) \text{ (equivalence of norms in } W_{-k}^k). \]

Example 2. Let \( u(x, t) \), \( u(x) \) be the solutions of the problems
\[
\frac{\partial u}{\partial t} + \sum_{|i| \leq k} (-1)^{|i|} D^i(g_0(x, t)) \left| D^i u \right|^{p-2} D^i u = r(t) f(x),
\]

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\[ u(x, 0) = s(0) u_0(x), \quad D^l_t u(x, t)|_{\partial \Omega \times (0, \infty)} = s(t) D^l u_0(x)|_{\partial \Omega} \]

for \( t > 0, \quad l = 0, 1, \ldots, k - 1 \)

and

\[
\sum_{|I| \leq k} (-1)^{|I|} D^I (g_i(x)) D^I u|_{\partial \Omega} = D^I u_0(x)|_{\partial \Omega} \quad \text{for} \quad l = 0, 1, \ldots, k - 1,
\]

respectively. Suppose \( p \geq 2, (38), f(x) \in W^1_q, u_0(x) \in W^k_p, g_i(x, t) \in C(Q) \cap L_\infty(Q) \) and \( g_i(x) \in C(\bar{Q}) (|I| \leq k) \).

i) Let \( s'(t), r(t) \in L_q(\langle 0, T \rangle) \) for all \( T < \infty \).

If \( s(t) \rightarrow 1, s'(t) \rightarrow 0, r(t) \rightarrow 1 \) for \( t \rightarrow \infty \), and if \( g_i(x, t) \rightarrow g_i(x) (|I| \leq k) \) for \( x \in \Omega \) and \( t \rightarrow \infty \) then \( u(x, t) \rightarrow u(x) \) in \( L^2_2(\Omega) \) for \( t \rightarrow \infty \).

ii) Suppose \( g_i(x, t) \equiv g_i(x) (|I| \leq k) \), \( s(t) \equiv 1 \) (stationary case). If

\[
\int_0^\infty |r(t)| \, dt < \infty \quad \text{and} \quad \int_0^\infty |r(t) - 1| \, dt < \infty
\]

then \( u(x, t) \rightarrow u(x) \) in the norm of the space \( W^k_p \) for \( t \rightarrow \infty \).

Example 3. Let \( u(x, t) \) be the solution of the problem

\[
\frac{\partial u}{\partial t} - \Delta u + f(x, u) = 0,
\]

\[ u(x, 0) = u_0(x), \]

a) \( u(x, t)|_{\partial \Omega} = 0 \) \( t > 0 \); b) \( (\partial u(x, t)/\partial v)|_{\partial \Omega} = 0 \) \( t > 0 \) and let \( u(x) \) be the solution of the stationary problem

\[
-\Delta u + f(x, u) = 0
\]

a) \( u|_{\partial \Omega} = 0; \) b) \( \partial u/\partial v|_{\partial \Omega} = 0. \)

Let \( f(x, s) \) be a continuous function in all its variables. Assume

\[
(f(x, \xi) - f(x, \eta))(\xi - \eta) > 0 \quad \text{for all} \quad \xi, \eta \in E^1, \quad \xi \neq \eta
\]

and

\[
C_1 s \leq s f(x, s) \leq C_2 (1 + |s|^r) \quad \text{for} \quad |s| < \infty,
\]

where \( r \leq 2N(N - 1)^{-1} \) for \( N > 1 \) and \( r \) is arbitrary for \( N = 1 \). Then in the case of the boundary conditions a) or b) we have \( u(x, t) \rightarrow u(x) \) in \( W^1_2(\Omega) \) for \( t \rightarrow \infty \).

References


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