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ATOMICITY OF TOLERANCE LATTICES

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1. By a tolerance $T$ on an algebra $\mathfrak{A} = (A, F)$ we mean a reflexive and symmetric binary relation on $A$ having the substitution property with respect to each operation $f \in F$, i.e. $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in T$ for every $f \in F$ whenever $\langle a_i, b_i \rangle \in T$ for $i = 1, \ldots, n$. Especially, every congruence on $\mathfrak{A}$ is a tolerance on $\mathfrak{A}$. However, there exist algebras with tolerances different from congruences (see [1], [3], [5], [7] and references therein). As proved in [1] and [4], the set $LT(\mathfrak{A})$ of all tolerances on $\mathfrak{A}$ constitutes an algebraic lattice with respect to the set inclusion. The identity relation $\Delta$ is the least element in $LT(\mathfrak{A})$. Basic properties of $LT(\mathfrak{A})$ are described in [4] and tolerances on special types of algebras are considered in [3], [5] and [7]. Although a congruence lattice $Con(\mathfrak{A})$ is not in general a sublattice of $LT(\mathfrak{A})$ (see [1], [5]), $LT(\mathfrak{A})$ has some properties analogous to $Con(\mathfrak{A})$. The purpose of this paper is to find some conditions for the atomicity of $LT(\mathfrak{A})$ when $\mathfrak{A}$ is a lattice or a join-semilattice. For distributive lattices the condition found here is necessary and sufficient. The results here generalize the corresponding results in [6].

2. As usual we do not make any difference between a lattice and its support. The lattice operations join and meet are denoted by $\vee$ and $\wedge$, respectively. If $a, b \in L$, we denote by $\Theta(a, b)$ and $T(a, b)$ the least congruence and tolerance, respectively, on $L$ containing the pair $\langle a, b \rangle$ (this definition is correct because of the completeness of $LT(L)$, [4]).

**Lemma 1.** Let $L$ be a lattice, $a, b \in L$, and $T \in LT(L)$. Then

1. $T(a, b) = T(a \wedge b, a \vee b)$ and

2. $T = \bigvee \{T(a, b) : \langle a, b \rangle \in T\}$ in $LT(L)$.

For the proof of (1), see Lemma 2 in [5], and for that of (2), see Theorem 15 in [4].

**Definition.** A lattice $L$ is said to be weakly atomic if for every pair $a, b$ ($a > b$) of elements of $L$ there exist two elements $u, v \in L$ such that $a \geq u > v \geq b$, where $u > v$ means "$u$ covers $v$".
By a quotient $a/b$ of $L$ ($a, b \in L$ and $a \geq b$) we mean a sublattice $\{ x : x \in L$ and $a \geq x \geq b \}$ of $L$. If $a > b$, the quotient $a/b$ is called prime. A quotient $a/b$ is an upper transpose of $c/d$ if $b \vee c = a$ and $b \wedge c = d$; $c/d$ is then called a lower transpose of $a/b$. A quotient $a/b$ is a transpose of $c/d$ if $a/b$ is either an upper or a lower transpose of $c/d$. A quotient $a/b$ is weakly projective into $c/d$ if there exists a finite sequence of quotients $a/b = x_0/y_0, x_1/y_1, \ldots, x_n/y_n = c/d$ such that each $x_{i-1}/y_{i-1}$ is a subquotient of a transpose of $x_i/y_i$. If every $x_{i-1}/y_{i-1}$ is a transpose of $x_i/y_i$, the quotients $a/b$ and $c/d$ are called projective.

**Lemma 2.** Let $a/b$ and $c/d$ be projective quotients in a lattice $L$. Then $T(a, b) = T(c, d)$.

**Proof.** Let $a/b = x_0/y_0, x_1/y_1, \ldots, x_n/y_n = c/d$ be a desired sequence of transposes. Suppose that e.g. $x_{i-1}/y_{i-1}$ is an upper transpose of $x_i/y_i$. Then

$\langle x_i, y_i \rangle = \langle x_i \wedge x_{i-1}, x_i \wedge y_{i-1} \rangle \in T(x_{i-1}, y_{i-1})$

and

$\langle x_{i-1}, y_{i-1} \rangle = \langle y_{i-1} \vee x_i, y_{i-1} \vee y_i \rangle \in T(x_i, y_i)$.

Hence $T(x_i, y_i) = T(x_{i-1}, y_{i-1})$. If $x_{i-1}/y_{i-1}$ is a lower transpose of $x_i/y_i$, the desired result is obtained dually. By $n$ steps of induction we obtain $T(a, b) = T(c, d)$.

**Definition.** A lattice has the projectivity property if, whenever a quotient $a/b$ is weakly projective into $c/d$, then $a/b$ and a subquotient of $c/d$ are projective.

**Lemma 3.** Let $a/b$ and $c/d$ be proper quotients of $L$. Then $\langle a, b \rangle \in \Theta(c, d)$ if and only if there exist $x_0, x_1, \ldots, x_n \in L$ such that $a = x_0 \geq x_1 \geq \ldots \geq x_n = b$ and $x_{i-1}/x_i$ is weakly projective into $c/d$.

The proof follows immediately from Theorem 10.2 in [6].

**Theorem 1.** If weakly atomic lattice $L$ has the projectivity property, then $LT(L)$ is atomic.

**Proof.** Let $p/q$ be a prime quotient. We shall show that $T(p, q)$ is an atom in $LT(L)$. If $a/b$ is a proper quotient of $L$ such that $T(a, b) \subseteq T(p, q)$, then $\langle a, b \rangle \in T(p, q) \subseteq \Theta(p, q)$. By Lemma 3, there exist $x_0, x_1, \ldots, x_n \in L$ such that $a = x_0 \geq x_1 \geq \ldots \geq x_n = b$ and $x_{i-1}/x_i$ is weakly projective into $p/q$. Since $L$ has the projectivity property and $p/q$ is prime, $x_{i-1}/x_i$ and $p/q$ are projective. Lemma 2 implies now that

$T(p, q) = T(x_{i-1}, x_i) \subseteq T(a, b) \subseteq T(p, q)$.

According to Lemma 1, this result implies that $T(p, q)$ is an atom of $LT(L)$. Further, let $T \in LT(L)$ and let $a/b$ be a proper quotient of $L$ such that $\langle a, b \rangle \in T$. The weak atomicity of $L$ implies existence of $p, q \in L$ with $a \geq p > q \geq b$, whence $p/q$ is prime.
Thus $T(p, q) \subseteq T(a, b) \subseteq T$. Since $T(p, q)$ is an atom of $LT(L)$, the later is at atomic lattice.

**Corollary.** For every modular weakly atomic lattice $L$, $LT(L)$ is atomic.

**Proof.** By Theorem 10.3 in [6], every modular lattice has the projectivity property. Thus Corollary is a direct consequence of Theorem 1.

**Lemma 4.** If $L$ is a distributive lattice, $a, b, c, d \in L$ and $a > b \geq c > d$, then $T(a, b) \land T(c, d) = \Delta$ in $LT(L)$.

**Proof.** Evidently, $T(a, b) \subseteq \Theta(a, b)$ and $T(c, d) \subseteq \Theta(c, d)$. Moreover, the meet in $LT(L)$ is equal to the set intersection as in $Con(L)$. Hence

$$T(a, b) \land T(c, d) \subseteq \Theta(a, b) \land \Theta(c, d) = \Delta$$

by 10.4 and the note following it in [6].

**Theorem 2.** Let $L$ be a distributive lattice and let $LT(L)$ be atomic. Then $L$ is a weakly atomic lattice.

**Proof.** Let $T$ be an atom of $LT(L)$. By Lemma 1, $T = T(p, q)$ for some $p > q$ in $L$. Suppose that $p/q$ is not prime, i.e. $p > x > q$ for some $x \in L$. Since $T(p, q)$ is an atom,

$$T(p, q) = T(p, x) = T(x, q)$$

which is a contradiction with Lemma 4. Thus $p/q$ is a prime quotient.

Let $a/b$ be a proper quotient of $L$. It follows from the atomicity of $LT(L)$ that there exists a prime quotient $p/q$ in $L$ such that $T(a, b) \supseteq T(p, q)$. Then $<p, q> \in T(a, b) \subseteq \Theta(a, b)$, i.e. $p/q$ is weakly projective into $a/b$ by Lemma 3 because $p/q$ is prime. Since $L$ is distributive, it has the projectivity property (Theorem 10.3 in [6]). Accordingly, there exists a subquotient $c/d$ of $a/b$ such that $p/q$ and $c/d$ are projective. In a distributive lattice projective quotients are isomorphic and thus also $c/d$ is a prime quotient. Hence $a \geq c > d \geq b$, whence $L$ is weakly atomic.

**Corollary.** Let $L$ be a distributive lattice. The following conditions are equivalent:

1. $L$ is weakly atomic.
2. $LT(L)$ is atomic.

3. In this section we consider semilattices. Let $S$ be a join-semilattice and $a, b \in S$. By a part $(a, b)$ of $S$ we mean the set $[a, a \vee b] \cup [b, a \vee b]$ of elements of $S$. Call $(a, b)$ to be proper if $a \neq b$. Clearly every interval of $S$ is a part of $S$. In [8, Thm. 1] the following lemma is proved:
Lemma 5. Let \((a, b)\) be a part and \(Q\) a binary relation of a join-semilattice \(S\) defined as follows: \(\langle x, y \rangle \in Q\) if and only if one of the conditions (i), (ii) and (iii) holds, where

(i) \(x = y\);
(ii) \(x = a \lor t\) and \(y = b \lor t\) for some \(t \in S\);
(iii) \(x = a \lor t\), \(y = a \lor b \lor t\) or \(x = a \lor b \lor t\), \(y = b \lor t\) for some \(t \in S\).

Then \(Q = T(a, b)\).

Now we are ready to prove

Theorem 3. Let \(S\) be a join-semilattice. \(T(a, b)\) is an atom of \(LT(S)\) if and only if \((a, b)\) is the only proper part of \(S\) collapsed by \(T(a, b)\).

Proof. Let \(T(a, b)\) be an atom of \(LT(S)\) and assume that there is a proper part \((c, d) \neq (a, b)\) collapsed by \(T(a, b)\). Then \(\langle c, d \rangle \in T(a, b)\), whence \(T(c, d) \subseteq T(a, b)\). By Lemma 5, \((c, d)\) is obtained from \((a, b)\) by (ii) or (iii), whence \(\langle a, b \rangle \notin T(c, d)\) and thus \(T(a, b) \neq T(c, d)\). As \((c, d)\) is a proper part of \(S\), \(T(a, b)\) is not an atom of \(LT(S)\), which is a contradiction.

Conversely, as \(a \neq b\), \(T(a, b) \neq \Delta\) and according to the assumption of the theorem, there is no \(T \in LT(S)\) between \(T(a, b)\) and \(\Delta\). Thus \(T(a, b)\) is an atom of \(LT(S)\).

It is evident that if \(T(a, b)\) is an atom of \(LT(S)\), the part \((a, b)\) is an interval of \(S\). Now it is easy to prove the following theorem:

Theorem 4. Let \(S\) be a join-semilattice. The following conditions are equivalent:

1. \(LT(S)\) is atomic.
2. For every proper interval \([a, b]\) of \(S\) there is an element \(t \in S\) such that:
   \([a \lor t, b \lor t]\) is prime and \(a \lor s = b \lor s\) for every \(s > t\).

As is easily seen, each tree semilattice \(S\) where for each proper \([a, b]\) of \(S\) the element \(b\) covers some \(c \in [a, b]\), has an atomic \(LT(S)\).

References


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