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Ivan Chajda; Juhani Nieminen

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ATOMICITY OF TOLERANCE LATTICES

IVAN CHAJDA, Přerov and JUHANI NIEMINEN, Oulu

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1. By a *tolerance* T on an algebra $\mathfrak{A} = (A, F)$ we mean a reflexive and symmetric binary relation on A having the *substitution property* with respect to each operation $f \in F$, i.e. $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in T$ for every $f \in F$ whenever $\langle a_i, b_i \rangle \in T$ for $i = 1, \ldots, n$. Especially, every congruence on $\mathfrak{A}$ is a tolerance on $\mathfrak{A}$. However, there exist algebras with tolerances different from congruences (see [1], [3], [5], [7] and references therein). As proved in [1] and [4], the set $LT(\mathfrak{A})$ of all tolerances on $\mathfrak{A}$ constitutes an algebraic lattice with respect to the set inclusion. The identity relation Δ is the least element in $LT(\mathfrak{A})$. Basic properties of $LT(\mathfrak{A})$ are described in [4] and tolerances on special types of algebras are considered in [3], [5] and [7]. Although a congruence lattice $Con(\mathfrak{A})$ is not in general a sublattice of $LT(\mathfrak{A})$ (see [1], [5]), $LT(\mathfrak{A})$ has some properties analogous to $Con(\mathfrak{A})$. The purpose of this paper is to find some conditions for the atomicity of $LT(\mathfrak{A})$ when $\mathfrak{A}$ is a lattice or a join-semilattice. For distributive lattices the condition found here is necessary and sufficient. The results here generalize the corresponding results in [6].

2. As usual we do not make any difference between a lattice and its support. The lattice operations join and meet are denoted by $\vee$ and $\wedge$, respectively. If $a, b \in L$, we denote by $\Theta(a, b)$ and $T(a, b)$ the *least congruence* and *tolerance*, respectively, on L containing the pair $\langle a, b \rangle$ (this definition is correct because of the completeness of $LT(L)$, [4]).

Lemma 1. *Let L be a lattice, $a, b \in L$, and $T \in LT(L)$. Then*

(1) $T(a, b) = T(a \wedge b, a \vee b)$ *and*

(2) $T = \bigvee\{T(a, b)\,;\, \langle a, b \rangle \in T\}$ *in* $LT(L)$.

For the proof of (1), see Lemma 2 in [5], and for that of (2), see Theorem 15 in [4].

Definition. A lattice L is said to be *weakly atomic* if for every pair a, b $(a > b)$ of elements of L there exist two elements $u, v \in L$ such that $a \geqq u \succ v \geqq b$, where $u \succ v$ means "*u covers v*".

By a *quotient* a/b of L $(a, b \in L$ and $a \geqq b)$ we mean a sublattice $\{x\,;\, x \in L$ and $a \geqq x \geqq b\}$ of L. If $a \succ b$, the quotient a/b is called *prime*. A quotient a/b is an *upper transpose* of c/d if $b \vee c = a$ and $b \wedge c = d$; c/d is then called a *lower transpose* of a/b. A quotient a/b is a *transpose* of c/d if a/b is either an upper or a lower transpose of c/d. A quotient a/b is *weakly projective* into c/d if there exists a finite sequence of quotients $a/b = x_0/y_0, x_1/y_1, \ldots, x_n/y_n = c/d$ such that each x_{i-1}/y_{i-1} is a subquotient of a transpose of x_i/y_i. If every x_{i-1}/y_{i-1} is a transpose of x_i/y_i, the quotients a/b and c/d are called *projective.*

Lemma 2. *Let a/b and c/d be projective quotients in a lattice L. Then $T(a, b) = T(c, d)$.*

Proof. Let $a/b = x_0/y_0, x_1/y_1, \ldots, x_n/y_n = c/d$ be a desired sequence of transposes. Suppose that e.g. x_{i-1}/y_{i-1} is an upper transpose of x_i/y_i. Then

$$\langle x_i, y_i\rangle = \langle x_i \wedge x_{i-1}, x_i \wedge y_{i-1}\rangle \in T(x_{i-1}, y_{i-1})$$

and

$$\langle x_{i-1}, y_{i-1}\rangle = \langle y_{i-1} \vee x_i, y_{i-1} \vee y_i\rangle \in T(x_i, y_i)\,.$$

Hence $T(x_i, y_i) = T(x_{i-1}, y_{i-1})$. If x_{i-1}/y_{i-1} is a lower transpose of x_i/y_i, the desired result is obtained dually. By n steps of induction we obtain $T(a, b) = T(c, d)$.

Definition. A lattice has the *projectivity property* if, whenever a quotient a/b is weakly projective into c/d, then a/b and a subquotient of c/d are projective.

Lemma 3. *Let a/b and c/d be proper quotients of L. Then $\langle a, b\rangle \in \Theta(c, d)$ if and only if there exist $x_0, x_1, \ldots, x_n \in L$ such that $a = x_0 \geqq x_1 \geqq \ldots \geqq x_n = b$ and x_{i-1}/x_i is weakly projective into c/d.*

The proof follows immediately from Theorem 10.2 in [6].

Theorem 1. *If weakly atomic lattice L has the projectivity property, then $LT(L)$ is atomic.*

Proof. Let p/q be a prime quotient. We shall show that $T(p, q)$ is an atom in $LT(L)$. If a/b is a proper quotient of L such that $T(a, b) \subseteq T(p, q)$, then $\langle a, b\rangle \in T(p, q) \subseteq \Theta(p, q)$. By Lemma 3, there exist $x_0, x_1, \ldots, x_n \in L$ such that $a = x_0 \geqq x_1 \geqq \ldots \geqq x_n = b$ and x_{i-1}/x_i is weakly projective into p/q. Since L has the projectivity property and p/q is prime, x_{i-1}/x_i and p/q are projective. Lemma 2 implies now that

$$T(p, q) = T(x_{i-1}, x_i) \subseteq T(a, b) \subseteq T(p, q)\,.$$

According to Lemma 1, this result implies that $T(p, q)$ is an atom of $LT(L)$. Further, let $T \in LT(L)$ and let a/b be a proper quotient of L such that $\langle a, b\rangle \in T$. The weak atomicity of L implies existence of $p, q \in L$ with $a \geqq p \succ q \geqq b$, whence p/q is prime.

Thus $T(p, q) \subseteq T(a, b) \subseteq T$. Since $T(p, q)$ is an atom of $LT(L)$, the later is at atomic lattice.

Corollary. *For every modular weakly atomic lattice L, $LT(L)$ is atomic.*

Proof. By Theorem 10.3 in [6], every modular lattice has the projectivity property. Thus Corollary is a direct consequence of Theorem 1.

Lemma 4. *If L is a distributive lattice, $a, b, c, d \in L$ and $a > b \geqq c > d$, then $T(a, b) \wedge T(c, d) = \Delta$ in $LT(L)$.*

Proof. Evidently, $T(a, b) \subseteq \Theta(a, b)$ and $T(c, d) \subseteq \Theta(c, d)$. Moreover, the meet in $LT(L)$ is equal to the set intersection as in $Con(L)$. Hence

$$T(a, b) \wedge T(c, d) \subseteq \Theta(a, b) \wedge \Theta(c, d) = \Delta$$

by 10.4 and the note following it in [6].

Theorem 2. *Let L be a distributive lattice and let $LT(L)$ be atomic. Then L is a weakly atomic lattice.*

Proof. Let T be an atom of $LT(L)$. By Lemma 1, $T = T(p, q)$ for some $p > q$ in L. Suppose that p/q is not prime, i.e. $p > x > q$ for some $x \in L$. Since $T(p, q)$ is an atom,

$$T(p, q) = T(p, x) = T(x, q)$$

which is a contradiction with Lemma 4. Thus p/q is a prime quotient.

Let a/b be a proper quotient of L. It follows from the atomicity of $LT(L)$ that there exists a prime quotient p/q in L such that $T(a, b) \supseteq T(p, q)$. Then $\langle p, q \rangle \in T(a, b) \subseteq \Theta(a, b)$, i.e. p/q is weakly projective into a/b by Lemma 3 because p/q is prime. Since L is distributive, it has the projectivity property (Theorem 10.3 in [6]). Accordingly, there exists a subquotient c/d of a/b such that p/q and c/d are projective. In a distributive lattice projective quotients are isomorphic and thus also c/d is a prime quotient. Hence $a \geqq c \succ d \geqq b$, whence L is weakly atomic.

Corollary. *Let L be a distributive lattice. The following conditions are equivalent:*

(1) *L is weakly atomic.*

(2) *$LT(L)$ is atomic.*

3. In this section we consider semilattices. Let S be a join-semilattice and $a, b \in S$. By a *part* (a, b) of S we mean the set $[a, a \vee b] \cup [b, a \vee b]$ of elements of S. Call (a, b) to be *proper* if $a \neq b$. Clearly every interval of S is a part of S. In [8, Thm. 1] the following lemma is proved:

Lemma 5. *Let* (a, b) *be a part and* Q *a binary relation of a join-semilattice* S *defined as follows*: $\langle x, y\rangle \in Q$ *if and only if one of the conditions* (i), (ii) *and* (iii) *holds, where*

(i) $x = y$;

(ii) $x = a \vee t$ *and* $y = b \vee t$ *for some* $t \in S$;

(iii) $x = a \vee t,\ y = a \vee b \vee t$ *or* $x = a \vee b \vee t,\ y = b \vee t$ *for some* $t \in S$. *Then* $Q = T(a, b)$.

Now we are ready to prove

Theorem 3. *Let* S *be a join-semilattice.* $T(a, b)$ *is an atom of* $LT(S)$ *if and only if* (a, b) *is the only proper part of* S *collapsed by* $T(a, b)$.

Proof. Let $T(a, b)$ be an atom of $LT(S)$ and assume that there is a proper part $(c, d) \neq (a, b)$ collapsed by $T(a, b)$. Then $\langle c, d\rangle \in T(a, b)$, whence $T(c, d) \subseteq \subseteq T(a, b)$. By Lemma 5, (c, d) is obtained from (a, b) by (ii) or (iii), whence $\langle a, b\rangle \notin \notin T(c, d)$ and thus $T(a, b) \neq T(c, d)$. As (c, d) is a proper part of S, $T(a, b)$ is not an atom of $LT(S)$, which is a contradiction.

Conversely, as $a \neq b$, $T(a, b) \neq \Delta$ and according to the assumption of the theorem, there is no $T \in LT(S)$ between $T(a, b)$ and Δ. Thus $T(a, b)$ is an atom of $LT(S)$.

It is evident that if $T(a, b)$ is an atom of $LT(S)$, the part (a, b) is an interval of S. Now it is easy to prove the following theorem:

Theorem 4. *Let* S *be a join-semilattice. The following conditions are equivalent*:

(1) $LT(S)$ *is atomic.*

(2) *For every proper interval* $[a, b]$ *of* S *there is an element* $t \in S$ *such that* $[a \vee t, b \vee t]$ *is prime and* $a \vee s = b \vee s$ *for every* $s > t$.

As is easily seen, each tree semilattice S where for each proper $[a, b]$ of S the element b covers some $c \in [a, b]$, has an atomic $LT(S)$.

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Authors' addresses: Ivan Chajda, třída LM 290, 750 00 Přerov, Czechoslovakia; Juhani Nieminen, University of Oulu, 90100 Oulu 10, Finland.