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INTEGRABILITY FOR THE DOBRAKOV INTEGRAL

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In [7], I. DOBRAKOV has constructed a very versatile and general method for integrating vector-valued functions with respect to vector-valued measures which generalizes the integral of BARTLE ([1]). As is the case for the Pettis integral and the integration of scalar functions with respect to vector measures, it is difficult to give criteria for the integrability of a given function. Such criteria have, however, recently been established for both the Pettis integral and the integral of scalar functions with respect to vector measures ([4], [13]). In this paper we give analogous such criteria for the Dobrakov integral. The methods employed show that the integral of Dobrakov is a "weak type" integral in the spirit of the Pettis integral or the integral of Lewis for scalar functions and vector measures ([11]).

Throughout Σ will denote a σ-algebra of subsets of a set S, X and Y will denote B-spaces and $L(X, Y)$ will denote the space of bounded linear operators from X into Y. Also $\mu : \Sigma \to L(X, Y)$ will denote a set function which is countably additive with respect to the strong operator topology of $L(X, Y)$ and it is assumed that μ has finite semi-variation $\hat{\mu}$ ($\hat{\mu}(E) = \sup \|\sum_{i=1}^{n} \mu(E_i)\, x_i\|$ where the supremum is taken over all measurable partitions $\{E_i\}$ of E and all $\|x_i\| \leqq 1$ ([7])). Though not as general, this set-up is simpler than the δ-ring formulation of [7] and makes the situation being analyized more transparent. If $y' \in Y'$, $y'\mu$ will denote the vector measure $E \to y'\, \mu(E)$. Each $y'\mu$ has bounded variation, $|y'\mu|$, and $\hat{\mu}(E) = \sup_{\|y'\| \leqq 1} |y'\mu|\,(E)$ ([6] Lemma 1). The measure μ is said to be *strongly bounded* (or $\hat{\mu}$ continuous), if there is a finite positive measure λ on Σ, called a *control measure for* μ, such that $\lim_{\lambda(E)\to 0} \hat{\mu}(E) = 0$ ([6] Lemma 2).

Throughout the paper, we use the integral of Dobrakov ([7], [8]) and refer the reader to his papers for the terminology and results.

We first define a *weak type* integral for X-valued functions with respect to the measure μ much in the spirit of the Pettis integral ([5]) and the integral of LEWIS ([11]). We then show that the integral of Dobrakov is just this weak integral.

Definition 1. A strongly measurable function $f : S \to X$ is *scalarly μ-integrable* if for each $y' \in Y'$ f is $y'\mu$ integrable. (Our terminology differs somewhat from Dobrakov ([7]); he calls such functions weakly integrable, but we reserve this term for later use.)

If f is scalarly μ-integrable, for each $E \in \Sigma$, $y' \to \int_E f \, dy'\mu$ defines an element of the algebraic dual of Y'. In the following Proposition we show that this actually determines an element of Y''. We denote this element somewhat ambiguously as $\int_E f \, d\mu$ and refer to the set function $E \to \int_E f \, d\mu$ as the indefinite integral of f with respect to μ; it will be clear from the context as to what type of integral we are referring.

Proposition 2. *If f is scalarly μ-integrable, $\int_E f \, d\mu \in Y''$ for each $E \in \Sigma$ and $\int f \, d\mu$ is countably additive with respect to the weak * topology of Y''.*

Proof. Let $\{f_n\}$ be a sequence of simple functions which converge pointwise to f with $\|f_n(\cdot)\| \leq \|f(\cdot)\|$. Each f_n is μ-integrable so $\int_E f_n \, d\mu \in Y$. But by the Dominated Convergence Theorem for the measure $y'\mu$ ([8] Th. 10, 17), $\langle y', \int_E f_n \, d\mu \rangle \to$ $\to \int_E f \, dy'\mu$ so that $\int_E f \, d\mu \in Y''$. The last statement follows from applying Theorem 3 of [7] to each $y'\mu$.

Definition 3. A scalarly μ-integrable function f is *weakly μ-integrable* if $\int_E f \, d\mu \in Y$ for each $E \in \Sigma$. (Here we are using the natural inbedding of Y in Y''.)

The definition of weak integrability is very analogous to the notion of Pettis integrability for vector functions and scalar measures ([5]) or the integral of scalar functions with respect to vector measures as treated in [10] or [11]. In Theorem 4 we show that Dobrakov s integral ([7]) is actually a weak-type integral in the sense of Definition 3.

Theorem 4. *Let $f : S \to X$ be scalarly μ-integrable. The following are equivalent*

(i) *f is weakly μ-integrable.*

(ii) *the indefinite integral of f with respect to μ is countably additive with respect to the norm.*

(iii) *f is Dobrakov-integrable with respect to μ.*

Proof. For (i) implies (ii) note from Proposition 2 that $\int f \, d\mu$ is weak* countably additive. Since $\int_F f \, d\mu \in Y$ for each $E \in \Sigma$, $\int f \, d\mu$ is actually countably additive with respect to the weak topology of Y and by the Orlicz-Pettis Theorem is norm countably additive ([9] IV.10.1).

For (ii) implies (iii) from Theorem 10 of [7] there exists an increasing sequence of measurable sets $\{F_k\}$ such that f is integrable over each F_k and $\int_{E \cap N} f \, d\mu = 0$ for each $E \in \Sigma$, where $N = S \setminus \bigcup F_k$. Set $f_n = C_{F_n} f$, where C_E denotes the characteristic function of the set E. By Theorem 16 of [7] it suffices to show $\{\int_E f_n \, d\mu\}$ converges for $E \in \Sigma$. Let $\nu(E) = \int_E f \, d\mu$. For $E \in \Sigma$ and $m > n$, we have $\|\int_E f_n \, d\mu - \int_E f_m \, d\mu\| = \|\nu(E \cap (F_m - F_n))\| \to 0$ as $n \to \infty$ by the norm countable additivity of ν.

That (iii) implies (i) has been noted by Dobrakov ([7], p. 533).

Remark 5. Compare (ii) with [5], II. 3.6 and [11], 2.6.

Since weak integrability and Dobrakov integrability are equivalent, henceforth we simply use the term integrable.

For the cases of integrating vector functions with respect to scalar measures and of integrating scalar functions with respect to vector measures, there are known and useful representations of the integrals as series ([1], [2], [5], [12]). In the next two theorems we present analogues results for the Dobrakov integral.

A function $g : S \to X$ is said to be *elementary* if $g = \sum_{k=1}^{\infty} C_{E_k} x_k$, where $x_k \in X$ and the $\{E_k\} \subseteq \Sigma$ are disjoint. Recall that a series $\sum x_m$ in a B-space X is *weakly unconditionally Cauchy* (w.u.c.) if $\sum |\langle x', x_m \rangle| < \infty$ for $x' \in X'$; the series determines an element of X'', denoted by $\sum_{n=1}^{\infty} x_m$, via $x' \to \sum_{n=1}^{\infty} \langle x', x_n \rangle$ and the series is weak* unconditionally covergent to $\sum_{n=1}^{\infty} x_n$.

Theorem 6. *Let $f : S \to X$ be scalarly μ-integrable.*

(i) *Then f has a decomposition*

$$f = h + g, \tag{1}$$

where h is bounded, strongly measurable and g is elementary. If $g = \sum C_{E_k} x_k$ with $\{E_k\} \subseteq \Sigma$ disjoint, the series $\sum_{k=1}^{\infty} \mu(E \cap E_k) x_k$ is w.u.c. for each $E \in \Sigma$ and

$$\int_E f \, d\mu = \int_E h \, d\mu + \sum_{k=1}^{\infty} \mu(E \cap E_k) x_k . \tag{2}$$

(ii) *If f is μ-integrable, f has a decomposition as in (1) which holds except in a set N with $\int_{E \cap N} f \, d\mu = 0$ for $E \in \Sigma$ with h bounded, μ-integrable and g elementary, μ-integrable. If $g = \sum C_{E_k} x_k$, then the series in (2) is unconditionally convergent in norm. In both (i) and (ii) above the bounded function h may be chosen to have arbitrarily small uniform norm.*

Proof. For (i) pick g elementary, $g = \sum C_{E_k} x_k$, such that $\|f(t) - g(t)\| < 1$ for all $t \in S$ ([5] II.3). Set $h = f - g$ so (1) holds. Note g is scalarly μ-integrable since the bounded function h is scalarly μ-integrable ([7], Th. 5). Then (2) follows by applying the Dominated Convergence Theorem to $y'\mu$.

For (ii), if f is μ-integrable, there is a sequence of simple functions $\{f_n\}$ converging μ-a.e. to f such that $\lim \int_E f_n \, d\mu = \int_E f \, d\mu$ uniformly for $E \in \Sigma$. By Theorem 1 of [7], there exist a set $N \in \Sigma$ and a disjoint sequence $\{A_k\}$ from Σ such that $\bigcup A_k = S \setminus N$, $\int_{E \cap N} f \, d\mu = 0$ for $E \in \Sigma$ and $\{f_n\}$ converges uniformly to f on each A_k. For each k there is an n_k such that $\|f(t) - f_{n_k}(t)\| < 1$ for $t \in A_k$ and

$$(3) \qquad \left\| \int_{E \cap A_k} (f - f_{n_k}) \, d\mu \right\| < 1/2^k \quad \text{for} \quad E \in \Sigma ,$$

where we may assume $n_{k+1} > n_k$. Set $g(t) = f_{n_k}(t)$ for $t \in A_k$ and $g(t) = 0$ for $t \in N$. Then g is an elementary function, and if $h : S \to X$ is defined by $h = f - g$ on $S \setminus N$ and $h = 0$ on N; then h is bounded and (1) holds. Note that h is actually μ-integrable since if we set $h_k(t) = f(t) - \sum_{j=1}^{k} f_{n_j}(t)$ for $t \in S \setminus N$ and $h_k = 0$ on N, then $\{h_k\}$ converges to h on $S \setminus N$ and the sequence $\{\int_E h_k \, d\mu\} = \{\sum_{j=1}^{k} \int_{E \cap A_j} (f - f_{n_j}) \, d\mu\}$ converges by (3) ([7], Th. 16). Thus, the elementary function g $(= f - h$ on $S \setminus N)$ is μ-integrable and the representation in (2) with the series being norm unconditionally convergent follows from (i) and Theorem 4.

The last statement in the Theorem is clear from the proof above.

Remark 7. The converse to (i) and (ii) in Theorem 6 obviously hold. It is also worthwhile recalling that unless the measure μ is strongly bounded, a bounded measurable function need not be μ-integrable ([7] Example 7′). This is the reason for the construction in (ii).

Using parts (i) and (ii) it is easy to drive Theorem 17 of [7].

Corollary 8. (Dobrakov). *Let Y contain no copy of c_0. Then every scalarly μ-integrable function is μ-integrable.*

Proof. Recall μ is strongly bounded ([7]* – Theorem) so every bounded measurable function is integrable ([7] Th. 5) and every w.u.c. series in Y is norm unconditionally convergent ([5] I. 4.5).

We next give a series representation for bounded functions which can be used in the representations (1) of Theorem 6.

Theorem 9. *Let $h : S \to X$ be bounded and strongly measurable. Then there exist $\{E_k\} \subseteq \Sigma$ and $\{x_k\} \subseteq X$ such that $h = \sum x_k C_{E_k}$, the series $\sum x_k C_{E_k}$ being absolutely convergent, with $\sum \|x_k\| \, |y'\mu| \, (E_k) < \infty$ for $y' \in Y$ and*

$$(4) \qquad \int_E h \, d\mu = \sum \mu(E_k \cap E) \, x_k \quad \textit{for} \quad E \in \Sigma ,$$

the series converging unconditionally in the weak topology of Y''. If h is μ-integrable, the $\{x_k\}$, $\{E_k\}$ may be chosen so the series in (4) is unconditionally convergent in the norm topology of Y.*

Proof. Set $g_0 = 0$ and use Theorem 6 (i) to choose for each n a scalarly μ-integrable elementary function g_n such that $\|g_n(\cdot) - h(\cdot)\| < 1/2^n$. Then $h = \sum_{n=0}^{\infty} (g_{n+1} - g_n)$, and if h is bounded by M, then for each k

$$\|\sum_0^k (g_{n+1} - g_n)\| \leqq \|g_1\| + \sum_1^k \|g_{n+1} - g_n\| \leqq$$

$$\leqq M + 1/2 + \sum_1^{\infty} 1/2^{n-1} = M + 5/2 .$$

Applying the Bounded Convergence Theorem to $y'\mu$ gives

$$\int_E hy \, \mathrm{d}'\mu = \sum_{n=0}^{\infty} \int_E (g_{n+1} - g_n) \, \mathrm{d}y'\mu . \tag{5}$$

If $g_{n+1} - g_n = \sum_{k=1}^{\infty} a_{nk} C_{A_{nk}}$ with $\{A_{nk}\}_{k=1}^{\infty} \subseteq \Sigma$ disjoint and $a_{nk} \in X$, then $\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} a_{nk} C_{A_{nk}}$ is absolutely convergent and $\int_E h \, \mathrm{d}y'\mu = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} y'\mu(A_{nk}) \, a_{nk}$ from (5). Also $\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \|a_{nk}\| \, |y'\mu| \, (A_{nk}) < \infty$ since $\sum_n \int_S \|g_{n+1} - g_n\| \, \mathrm{d}|y'\mu| = \sum_n \sum_k |y'\mu| \, (A_{nk}) \, \|a_{nk}\| = \int_S \sum_n \|g_{n+1} - g_n\| \, \mathrm{d}|y'\mu| \leqq (M + 5/2) \, |y'\mu| \, (S)$. This establishes the first part of the Theorem.

For the last statement note that from Theorem 6 (ii) if h is μ-integrable the elementary functions g_n can be chosen to be μ-integrable and then (5) along with the Orlicz-Pettis Theorem gives the desired conclusion.

Theorems 6 and 9 can be combined to give a series representation for the integral of either a scalarly μ-integrable function or a μ-integrable function. Such representations are possible for the Pettis and Bochner integrals ([5] p. 55, [1], [2]) and the integral of D. Lewis ([12]). We leave it to the reader to formulate the appropriate statements.

If the measure μ is assumed to be strongly bounded, conditions for integrability analogous to these [4] and [13] can also be derived for the Dobrakov integral. Henceforth, we assume that μ is strongly bounded and λ is a control measure for μ, i.e., $\lim_{\lambda(E) \to 0} \hat{\mu}(E) = 0$.

Suppose that $f : S \to X$ is clearly μ-integrable. Then f induces a linear map F from Y' into $\mathrm{ca}(\Sigma)$, the space of all finite countably additive measures on Σ ([9] IV. 2.16), via $F(y') = f \, \mathrm{d}y'\mu$, where $f \, \mathrm{d}y'\mu$ is the measure $E \to \int_E f \, \mathrm{d}y'\mu$. (Compare with [4] Prop. 1 and [13].) It is easily checked that F has a closed graph and is, therefore, continuous.

For the principle theorem we require

Lemma 10. *Let* $\{y'_\alpha\}$ *be a bounded net in* Y' *which is weak* convergent to* 0. *Let* $g : S \to X$ *be strongly measurable with* $\|g(t)\| \leqq M$ *for* $t \in S$. *Then* $\int_S g \, \mathrm{d}y'_\alpha\mu \to \to 0$.

Proof. Suppose $\|y'_\alpha\| \leqq 1$. Note since $\langle y'_\alpha, \mu(E)\, x\rangle \to 0$ for each $E \in \Sigma$ $x \in X$, the conclusion holds for g a simple function. Let $\varepsilon > 0$. Pick $\delta > 0$ such that $\lambda(E) < \delta$ implies $\hat{\mu}(E) < \varepsilon$. Let $\{g_n\}$ be a sequence of simple functions such that $\{g_n\}$ converges to g pointwise with $\|g_n(t)\| \leqq M$. By Egoroff's Theorem there exist N and $E \in \Sigma$ such that $\|g_N(t) - g(t)\| < \varepsilon$ for $t \in S \setminus E$ and $\lambda(E) < \delta$. Then

$$\left\|\int_S g \, \mathrm{d}y'_\alpha\mu\right\| \leqq \left\|\int_S g_N \, \mathrm{d}y'_\alpha\mu\right\| + \left\|\int_{S\setminus E} (g - g_N)\, \mathrm{d}y'_\alpha\mu\right\| + \left\|\int_S (g - g_N)\, \mathrm{d}y'_\alpha\mu\right\| \leqq$$

$$\leqq \left\|\int_S g_N \, \mathrm{d}y'_\alpha\mu\right\| + \varepsilon\, \hat{\mu}(S) + 2\, M\varepsilon$$

and the first term in the right hand side of this inequality goes to 0 since g_N is simple.

In Theorem 11 we use the bounded Y topology of Y', $b(Y')$ ([9], V. 5.3); recall that $b(Y')$ is the strongest topology on Y' which coincides with the weak* topology on balls about the origin. Also if the vector spaces E and F are in duality, we denote the weak topology on E from F by $\sigma(E, F)$. For $\lambda \in \mathrm{ca}(\Sigma)$, let $\mathrm{ca}\,(\Sigma, \lambda)$ denote the subspace of $\mathrm{ca}(\Sigma)$ consisting of all measures which are absolutely continuous with respect to λ.

Theorem 11. *The following are equivalent.*

(i) f *is* μ*-integrable*

(ii) F *is weakly compact*

(iii) $\lim_{\lambda(E)\to 0} \int_E f \, \mathrm{d}y'\mu = 0$ *uniformly for* $\|y'\| \leqq 1$

(iv) F *is* $b(Y') - \sigma(\mathrm{ca}(\Sigma, \lambda), L^\infty(\lambda))$ *continuous.*

Proof. (i) implies (ii): It is enough to show that $F(B)$ is conditionally weakly compact in $\mathrm{ca}(\Sigma, \lambda)$, where B is the unit ball of Y'. Let $\{y'_\delta\}$ be a net in B. There is a subnet, which we still denote by $\{y'_\delta\}$, which converges weak* to some $y' \in B$. For $E \in \Sigma$, $\langle y'_\delta, \int_E f \, \mathrm{d}\mu\rangle \to \langle y', \int_E f \, \mathrm{d}\mu\rangle$, i.e., $Fy'_\delta(E) \to Fy'(E)$. Since F is bounded, this implies that $Fy'_\delta \to Fy'$ weakly in $\mathrm{ca}(\Sigma, \lambda)$ ([9] IV. 9.5) and, therefore, weakly in $\mathrm{ca}(\Sigma)$.

Conditions (ii) and (iii) are equivalent by the familiar criteria for weak compactness in $\mathrm{ca}(\Sigma, \lambda)$ ([9] IV. 9.2).

For (ii) implies (iv) let $\{y'_\delta\}$ be a net in B which converges weak* to 0. Since F is weakly compact, there is a subnet, which we still denote by $\{y'_\delta\}$, and a $\nu \in \mathrm{ca}(\Sigma, \lambda)$ such that $F(y'_\delta) \to \nu$ weakly. In particular $\int_E f \, \mathrm{d}y'_\delta\mu \to \nu(E)$ for each $E \in \Sigma$. It suffices to show that $\nu = 0$. Let $A_n = \{t \in S : \|f(t)\| \leqq n\}$. By Lemma 10 $\lim \int_E f \, \mathrm{d}y'_\delta\mu = = 0 = \nu(E)$ for each $E \in \Sigma$, $E \subseteq A_n$. Since $\bigcup A_n = S$, this gives $\nu = 0$.

To show (iv) implies (i), it suffices to show $\int_E f \, d\mu \in Y$ for $E \in \Sigma$. Now F has range in $ca(\Sigma, \lambda)$ so consider F as a map from Y' into $ca(\Sigma, \lambda)$ and look at the adjoint, F', of F when Y' and $ca(\Sigma, \lambda)$ have the topologies in (iv). Recalling that the dual of Y' with $b(Y')$ is Y, F' is continuous from $L^\infty(\lambda)$ into Y when $L^\infty(\lambda)$ had $\sigma(L^\infty(\lambda), ca(\Sigma, \lambda))$ and Y has $\sigma(Y, Y')$. But $F'\phi \int_S \phi f \, d\mu$ for $\phi \in L^\infty(\lambda)$ so in particular if $\phi = C_E$, then $F'C_E = \int_E f \, d\mu \in Y$.

Remark 12. The proof of (iv) implies (i) also gives an alternate proof of Theorem 4 of [7]. Theorem 11 should be compared with Proposition 1 of [4] and Corollary 3 of [13].

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